

Interpretable sets in dense o-minimal structures

Will Johnson

November 25, 2019

Abstract

We give an example of a dense o-minimal structure in which there is a definable quotient that cannot be eliminated, even after naming parameters. Equivalently, there is an interpretable set which cannot be put in parametrically definable bijection with any definable set. This gives a negative answer to a question of Eleftheriou, Peterzil, and Ramakrishnan. Additionally, we show that interpretable sets in dense o-minimal structures admit definable topologies which are “tame” in several ways: (a) they are Hausdorff, (b) every point has a neighborhood which is definably homeomorphic to a definable set, (c) definable functions are piecewise continuous, (d) definable subsets have finitely many definably connected components, and (e) the frontier of a definable subset has lower dimension than the subset itself.

1 Introduction

Let us say that a structure M has *parametric elimination of imaginaries* if given any M -definable set X and M -definable equivalence relation E on X , there is an M -definable map eliminating the quotient X/E . Replacing “ M -definable” with “0-definable” gives the usual notion of elimination of imaginaries, which is a stronger condition.

It is well-known that o-minimal expansions of ordered abelian groups have parametric elimination of imaginaries. When working with o-minimal structures, it is common to assume that the structure expands an ordered abelian group, or even an ordered field. This assumption simplifies life, and holds in most o-minimal structures arising in applications of o-minimality. Nevertheless, some o-minimal structures do not expand ordered abelian groups, and one can pose the following question:

2010 Mathematical Subject Classification: 03C64

Key words and phrases: o-minimality, interpretable sets, definable topologies.

The author would like to thank Kobi Peterzil for asking the question that prompted this paper, as well as Tom Scanlon for reviewing earlier versions of this paper that were part of the author’s PhD dissertation.

Parts of this material are based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE 1106400. Any opinion, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

Question 1.1. *Do all o-minimal structures have parametric elimination of imaginaries?*

This question was first asked by Eleftheriou, Peterzil, and Ramakrishnan in [6]. They gave a partial answer, proving that an o-minimal quotient X/E can be eliminated whenever it admits a definable group structure, as well as when $\dim(X/E) = 1$.

We answer Question 1.1 in the negative in §2. Specifically, we give an o-minimal expansion of (\mathbb{R}, \leq) in which there is a 0-definable quotient X/E which cannot be eliminated over any set of parameters.

A structure M has parametric elimination of imaginaries if every interpretable set in M can be put in definable bijection with a definable set. The negative answer to Question 1.1 therefore means that o-minimal structures can have exotic interpretable sets which are intrinsically different from definable sets.

O-minimality provides many tools for working with definable sets, and it is natural to wonder which of these tools can be generalized to interpretable sets. For example, Peterzil and Kamenkovich generalized the dimension and Euler characteristic machinery to interpretable sets in [4] and [3], respectively.

As a step in this direction, we show in §3 that interpretable sets X/E in dense o-minimal theories can be given nice definable topologies. More precisely, we show in Theorem 1.3 that the quotient topology on X/E is a Hausdorff definable topology, *provided* one first discards a set of low dimension from X .¹

Using this theorem, we show that interpretable sets admit Hausdorff definable topologies satisfying certain “tameness” properties, including the following:

- Every definable subset has finitely many definably connected components.
- Every definable map is continuous off a set of low dimension.

For a precise statement, see Theorem 1.5, which is proven in §4.

1.1 Notation and conventions

“Definable” will mean “definable with parameters,” and “ A -definable” will mean “definable with parameters from A ”. We will write “0-definable” as shorthand for “ \emptyset -definable.”

When talking about sets, a “definable set” means a definable subset of a power of the home sort, and an “interpretable set” means a definable set in T^{eq} . Outside of this distinction, we will always say “definable” instead of “interpretable.” For example, we will talk about definable subsets of interpretable sets, and definable maps between interpretable sets, rather than “interpretable subsets” or “interpretable maps”. We will say that a subset of an

¹Without this proviso, one can produce pathological examples such as the line with doubled origin. Indeed, if $X = \mathbb{R} \times \{0, 1\}$ and E is the equivalence relation generated by

$$(x, 0)E(x, 1) \text{ for } x \neq 0,$$

then the quotient X/E is the line with doubled origin.

interpretable set is “ind-definable” (over some parameters A) if it is a union of A -definable subsets.

A “definable quotient” is a pair X/E consisting of a definable set X and a definable subset $E \subseteq X \times X$ defining an equivalence relation on X . The quotient can be “eliminated” if one of the following equivalent conditions is true:

- There is a definable bijection between the interpretable set X/E and some definable set $Y \subseteq M^k$.
- There is some definable map $f : X \rightarrow M^k$ such that

$$xE x' \iff f(x) = f(x') \quad \forall x, x' \in X$$

“O-minimal” will mean dense o-minimal, i.e., we require o-minimal structures to expand dense linear orders without endpoints.

In an o-minimal structure, $\dim(X)$ will denote the standard o-minimal dimension of a definable or interpretable set X (see [4] for the interpretable case). The o-minimal rank of a finite tuple a over a set of parameters S will be denoted $\dim(a/S)$; this is the minimum of $\dim(X)$ for S -definable $X \ni a$. We will write \perp^b to denote thorn-forking independence, so $a \perp_C^b B$ means $\dim(a/BC) = \dim(a/C)$.

In a topological space, the interior, boundary, frontier, and closure of a set X will be denoted $\text{int}(X)$, $\text{bd}(X)$, ∂X , and \overline{X} . Thus

$$\begin{aligned} \text{bd}(X) &= \overline{X} \setminus \text{int}(X) \\ \partial X &= \overline{X} \setminus X \end{aligned}$$

An “embedding” will be a continuous map that is a homeomorphism onto its image.

If E is an equivalence relation on a set X , and $X' \subseteq X$, we will write X'/E to indicate $X'/(E \upharpoonright X')$.

A map $f : P_1 \rightarrow P_2$ between two posets will be called *order-preserving* if

$$x \leq y \implies f(x) \leq f(y)$$

and *order-reversing* if

$$x \leq y \implies f(x) \geq f(y).$$

(Usually the posets will be powersets with inclusion ordering.)

If X is a definable set in a structure M , then $\ulcorner X \urcorner$ will denote a canonical parameter for X , i.e., a finite tuple from M^{eq} fixed pointwise by exactly the automorphisms that fix X setwise. If r is a real number, $\lceil r \rceil$ and $\lfloor r \rfloor$ will denote the ceiling and floor of r , respectively.

If X is a definable or interpretable set in a structure M , a topology on X is “definable” if there is a definable family of subsets of X forming a basis of opens. This means that there is a definable relation $U \subseteq X \times M^k$ for which the sets

$$U_{\vec{a}} := \{x \in X \mid (x, \vec{a}) \in U\} \text{ for } \vec{a} \in M^k$$

form a basis for the topology. A “definable topological space” is an interpretable set together with a definable topology.

If X is a definable topological space in an o-minimal structure (M, \leq, \dots) , we will say that X is *Euclidean* at a point $x \in X$ if there is a definable homeomorphism between an open neighborhood of x in X and an open subset of M^k for some k . We will say that X is “locally Euclidean” if X is Euclidean at every $x \in X$. (Note that k might depend on x .)

1.2 Statement of results

Proposition 1.2. *There is a (dense) o-minimal structure M containing an interpretable set which cannot be put in M -definable bijection with any M -definable set.*

Theorem 1.3. *Fix an o-minimal structure M . Let $X \subseteq M^k$ be a definable set and E be a definable equivalence relation on X . Then we can write X as a disjoint union $X' \cup X_0$ satisfying the following conditions:*

1. X' is open in X
2. $\dim(X_0) < \dim(X)$ or $X_0 = \emptyset$.
3. The quotient topology on X'/E is definable, Hausdorff, and locally Euclidean
4. If X'' is any open subset of X' , the map of quotient spaces

$$X''/E \hookrightarrow X'/E$$

is continuous, and in fact an open embedding.

Condition 2 means that X' is “generic” in X in a certain sense. Condition 4 shows that the quotient topology is somewhat independent of the choice of X' : as long as we have chosen a sufficiently small generic open subset of X , the quotient topology will agree.

Definition 1.4. *A Hausdorff topology on an interpretable set Y is admissible if there is a definable surjection $f : X \twoheadrightarrow Y$ where X is a definable subset of M^n , such that f is a continuous open map with respect to the standard topology on X .*

The next result says that admissible locally Euclidean topologies exist, and share many properties with the standard topology on M^k .

Theorem 1.5. *Fix an o-minimal structure M .*

1. Every interpretable set can be endowed with an admissible locally Euclidean topology.
2. Admissible topologies are definable.
3. If Y is an admissible locally Euclidean topological space and D is a non-empty definable subset of Y , then

- (a) D has finitely many definably connected components.
 - (b) $\dim \partial D < \dim D$.
 - (c) There is a point $p \in D$ such that $\dim N \cap D = \dim D$ for every neighborhood N of p . In other words, the local dimension of D at p equals the global dimension of D .
4. If $f : Y \rightarrow Y'$ is a definable map between two admissible locally Euclidean topological spaces, then f is continuous on a dense open subset of Y . Moreover, Y can be written as a finite disjoint union of locally closed definable subsets, on which the restriction of f is continuous.

Note that there are other ways to put locally Euclidean definable topologies on interpretable sets, such as the discrete topology. However, the discrete topology fails to satisfy many of the conditions listed above, such as 3a, 3c, and 4.

2 A pathological quotient

In this section, we give an example of an o-minimal structure in which parametric elimination of imaginaries fails, namely

$$(\mathbb{R}, \leq, R)$$

where $R(x_0, \dots, x_5)$ is the 5-ary predicate holding if and only if

$$\frac{\cos(x_1 - x_0)}{\sin(x_1 - x_0)} - \frac{\cos(x_2 - x_0)}{\sin(x_2 - x_0)} = \frac{\cos(x_3 - x_0)}{\sin(x_3 - x_0)} - \frac{\cos(x_4 - x_0)}{\sin(x_4 - x_0)} \text{ and } \bigwedge_{i=1}^4 x_0 < x_i < x_0 + \pi.$$

2.1 A toy example

We first discuss the simplest example of an o-minimal theory which lacks elimination of imaginaries. Let $M = (\mathbb{R}, \leq, E)$, where $E(\vec{x})$ is the 4-ary relation

$$E(x_1, \dots, x_4) \iff x_1 - x_2 = x_3 - x_4.$$

The relation E defines an equivalence relation on the set $X := \mathbb{R}^2$. The quotient X/E cannot be 0-definably eliminated, and this can be seen using automorphisms. Suppose for the sake of contradiction that there is a 0-definable injection $X/E \hookrightarrow M^k$ for some k . Consider the automorphisms

$$\begin{aligned} \sigma_1(x) &:= x + 1 \\ \sigma_2(x) &:= 2x. \end{aligned}$$

of the structure M . Let $e \in X/E \subseteq M^{\text{eq}}$ be the E -equivalence class of $(0, 1) \in X$. Then one verifies easily that

$$\begin{aligned} \sigma_1(e) &= e \\ \sigma_2(e) &\neq e \end{aligned}$$

Let \vec{r} denote $f(e)$. Then e and \vec{r} are inter-definable over \emptyset , so

$$\begin{aligned}\sigma_1(\vec{r}) &= \vec{r} \\ \sigma_2(\vec{r}) &\neq \vec{r}\end{aligned}$$

However, \vec{r} is a tuple of elements from M . By inspection, every element of M fixed by σ_1 is fixed by σ_2 , yielding a contradiction.²

The structure M gives an example of an o-minimal theory which does not have elimination of imaginaries. Nevertheless, after naming two constants, this example has a strong form of elimination of imaginaries: every non-empty definable set X contains an $\ulcorner X \urcorner$ -definable point. So this is not yet an example of an o-minimal structure in which parametric elimination of imaginaries fails. However, this toy example will play a role in the construction below.

For future reference, we record the configuration that showed that a quotient was not eliminated:

Lemma 2.1. *Let M be a structure, A be a small set of parameters, and X/E be an A -definable quotient. Suppose there exist $\sigma_i \in \text{Aut}(M/A)$ for $i = 1, 2$ such that*

- *Every element of M fixed by σ_1 is fixed by σ_2*
- *Some element of X/E fixed by σ_1 is not fixed by σ_2 .*

Then there is no A -definable injection $X/E \hookrightarrow M^k$, so the quotient X/E cannot be eliminated over A .

2.2 Preliminaries

Let $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ be the real projective line. The group of linear fractional transformations $x \mapsto \frac{ax+b}{cx+d}$ acts transitively on \mathbb{RP}^1 , and the stabilizer of ∞ is exactly the group of affine transformations $x \mapsto ax + b$.

For $x_0, \dots, x_5 \in \mathbb{RP}^1$, let $P(x_0, \dots, x_4)$ indicate that

$$f(x_1) - f(x_2) = f(x_3) - f(x_4)$$

for any/every linear fractional transformation f mapping x_5 to ∞ . This is well-defined because f is determined up to an affine transformation, and affine transformations preserve the 4-ary relation $y_1 - y_2 = y_3 - y_4$.

Remark 2.2. *Any linear fractional transformation (and in particular, any affine transformation) preserves the predicate P .*

We will write $\cot \theta$ and $\tan \theta$ for the cotangent and tangent of the angle θ .

²In this toy example, σ_1 has no fixed points in M , and so σ_2 's only role is to rule out the possibility that \vec{r} is the tuple of length 0. Later, we will use the same argument in a more complicated situation where σ_1 has fixed points.

Remark 2.3. For fixed $\alpha \in \mathbb{R}$, there is a linear fractional transformation mapping $\tan x \mapsto \cot(x - \alpha)$, by the trigonometric angle-sum formulas. This transformation sends $\tan \alpha \mapsto \cot(\alpha - \alpha) = \infty$, and so

$$\begin{aligned} & P(\tan \alpha, \tan x_1, \dots, \tan x_4) \\ & \iff \cot(x_1 - \alpha) - \cot(x_2 - \alpha) = \cot(x_3 - \alpha) - \cot(x_4 - \alpha) \end{aligned}$$

We also record the trivial example

$$P(\infty, x_1, \dots, x_4) \iff x_1 - x_2 = x_3 - x_4 \tag{1}$$

2.3 Details of the construction

Let M be the structure $(\mathbb{R}, \leq, \iota, \tilde{P})$ where

- $\iota(x) = x + \pi$
- $\tilde{P}(x_0, x_1, \dots, x_4)$ holds if

$$P(\tan(x_0), \dots, \tan(x_4)) \wedge \bigwedge_{i=1}^4 x_0 < x_i < \iota(x_0).$$

By Remark 2.3, $\tilde{P}(x_0, \dots, x_4)$ holds if and only if $\{x_1, \dots, x_4\} \subseteq (x_0, x_0 + \pi)$ and

$$\cot(x_1 - x_0) - \cot(x_2 - x_0) = \cot(x_3 - x_0) - \cot(x_4 - x_0)$$

Let N be the structure $(\mathbb{Z} \times \mathbb{RP}^1, \leq, \iota, \tilde{P})$ where

- \leq is the lexicographic ordering on $\mathbb{Z} \times \mathbb{RP}^1$, where \mathbb{RP}^1 is ordered by putting $\infty > \mathbb{R}$.
- ι is the map $(n, x) \mapsto (n + 1, x)$
- $\tilde{P}(x_0, x_1, \dots, x_4)$ holds if

$$P(\pi(x_0), \dots, \pi(x_4)) \wedge \bigwedge_{i=1}^4 x_0 < x_i < \iota(x_0)$$

where $\pi : \mathbb{Z} \times \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ is the projection.

It is easy to verify that there is an isomorphism $M \xrightarrow{\sim} N$ given by

$$x \mapsto \left(\left[\frac{x}{\pi} + \frac{1}{2} \right], \tan x \right).$$

The map preserves \tilde{P} essentially because the following diagram commutes

$$\begin{array}{ccc} M & \longrightarrow & N \\ & \searrow \text{tan} & \downarrow \pi \\ & & \mathbb{RP}^1 \end{array} .$$

The two structures M and N are o-minimal, because M is a definable reduct of

$$(\mathbb{R}, \leq, +, \cdot, \sin \upharpoonright [0, \pi], \cos \upharpoonright [0, \pi]),$$

which is o-minimal by Gabrielov's theorem (see e.g. Theorem 4.6 in [1]).

For any $a \in M$, let

$$X_a = \{(x, y) : a < x < y < \iota(a)\} \subseteq M^2$$

and let E_a be the equivalence relation on X_a given by

$$\begin{aligned} (x, y)E_a(x', y') &\iff \tilde{P}(a, x, y, x', y') \\ &\iff \cot(x - a) - \cot(y - a) = \cot(x' - a) - \cot(y' - a) \end{aligned}$$

Via the isomorphism, the same definitions make sense in N .

Example 2.4. *In the structure N , consider the case $a = (n - 1, \infty)$. The open interval from a to $\iota(a) = (n, \infty)$ consists of points (n, x) with $x \in \mathbb{R}$. Abusing notation and identifying (n, x) with x , we have*

$$\begin{aligned} X_a &= \{(s, t) \in \mathbb{R}^2 : s < t\} \\ (s, t)E_a(s', t') &\iff s - t = s' - t' \end{aligned}$$

So X_a/E_a is the toy example of §2.1.

Lemma 2.5. *In the structures M and N , there are automorphisms τ_1, τ_2 such that*

1. *Every element of the home sort fixed by τ_1 is fixed by τ_2*
2. *The set of elements fixed by τ_1 is unbounded above*
3. *If a is fixed by τ_1 , then under the induced action on M^{eq} or N^{eq} , τ_1 fixes every element of X_a/E_a and τ_2 fixes no elements of X_a/E_a .*

Proof. By the isomorphism $M \cong N$, we only need to consider the case of N . In this case, let

$$\begin{aligned} \tau_1((n, x)) &= (n, x + 1) \\ \tau_2((n, x)) &= (n, 2x) \end{aligned}$$

These maps are indeed automorphisms; \tilde{P} is preserved because of Remark 2.2. The fixed points of τ_1 are exactly the points (n, ∞) , which are cofinal and fixed by τ_2 . For part 3, suppose $a = (n - 1, \infty)$. Under the identification of Example 2.4,

$$\begin{aligned} X_a &= \{(s, t) \in \mathbb{R}^2 : s < t\} \\ (s, t)E_a(s', t') &\iff s - t = s' - t' \\ \tau_1(s) &= s + 1 \\ \tau_2(s) &= 2s \end{aligned}$$

As in §2.1, τ_1 fixes X_a/E_a pointwise. In contrast, τ_2 moves every point, because

$$s < t \implies s - t \neq 2s - 2t$$

□

Now let M^* be an \aleph_1 -saturated ultrapower of M ; there are canonical extensions of τ_1 and τ_2 to M^* having the same first-order properties. In particular, the properties listed in Lemma 2.5 continue to hold.

The following lemma allows us to glue automorphisms across Dedekind cuts in M^* :

Lemma 2.6. *Let (Ξ^-, Ξ^+) be a Dedekind cut on M^* , meaning specifically that M^* is the disjoint union of Ξ^- and Ξ^+ , and $\Xi^- < \Xi^+$. Let ρ^+ and ρ^- be two automorphisms of M . Suppose that ρ^+, ρ^- , and ι each preserve the Dedekind cut (for example, $\iota(\Xi^-) = \Xi^-$). Then the map:*

$$\rho := (\rho^- \upharpoonright \Xi^-) \cup (\rho^+ \upharpoonright \Xi^+)$$

is an automorphism of M^ .*

Proof. By inspection, $\tilde{P}(x_0, \dots, x_4)$ cannot hold unless the x_i are within distance π of each other, in which case they must lie entirely on one side of the Dedekind cut. Consequently, the preservation of \tilde{P} by ρ can be checked on each side of the Dedekind cut in isolation. The preservation of \leq and ι by ρ are similar or easier. □

Let Ξ^\pm be the Dedekind cut just beyond the end of M , so Ξ^+ is the set of upper bounds of M in M^* . This Dedekind cut is fixed by ι , τ_1 and τ_2 , because each of these maps sends M to M setwise. By \aleph_1 -saturation, Ξ^+ is non-empty. For $i = 1, 2$, let σ_i be the automorphism obtained by gluing the identity map on Ξ^- with τ_i on Ξ^+ . So σ_i fixes Ξ^- pointwise, and agrees with τ_i on Ξ^+ . Thus,

1. The σ_i fix $M \subseteq \Xi^-$ pointwise.
2. Every element of the home sort fixed by σ_1 is fixed by σ_2 .
3. There is an element $a \in \Xi^+$ fixed by τ_1 and σ_1 , as the fixed points of τ_1 are cofinal.
4. For this element a , the maps σ_i and τ_i agree on X_a . Consequently σ_1 fixes every element of X_a/E_a , and σ_2 fixes no element of X_a/E_a .

By Lemma 2.1, it follows that the aM -definable quotient X_a/E_a is not aM -definably eliminated.

Now let X be the 0-definable set of all triples of real elements

$$\{(x, y, z) : x < y < z < \iota(x)\}$$

and let E be the 0-definable relation

$$\begin{aligned} (x, y, z)E(x', y', z') &\iff x = x' \wedge \tilde{P}(x, y, z, y', z') \\ &\iff x = x' \text{ and} \\ &\cot(z - x) - \cot(y - x) = \cot(z' - x') - \cot(y' - x') \end{aligned}$$

Then E is an equivalence relation on X . For any a , there is an a -definable injection $X_a \hookrightarrow X$ given by $(x, y) \mapsto (a, x, y)$, and this induces an a -definable injection $X_a/E_a \hookrightarrow X/E$.

Proposition 2.7. *In the structure M , the quotient X/E is not M -definably eliminated.*

Proof. Otherwise, there would be an M -definable injection from X/E into M^k . In the elementary extension M^* considered above, this would yield an M -definable injection from X/E into $(M^*)^k$. Above, we found an element $a \in M^*$ such that the aM -definable quotient X_a/E_a is not aM -definably eliminated. However, the composition

$$X_a/E_a \hookrightarrow X/E \hookrightarrow (M^*)^k$$

is an aM -definable injection that eliminates the quotient X_a/E_a , a contradiction. \square

3 Good quotient topologies

We next turn our attention to Theorem 1.3, which shows that quotient topologies on definable quotients are sometimes well-behaved. We begin by discussing the topological tools that will be used in the proof.

3.1 Definable topologies and definable compactness

Work inside a model-theoretic structure M . Recall that a topology on an interpretable set X is *definable* if some definable family of subsets of X constitutes a basis for the topology. Typical examples include:

1. The order topology on any ordered structure
2. The standard topology on M^n for any o-minimal structure M .
3. The valuation topology on any model of ACVF or p CF (p -adically closed fields).
4. The discrete topology on any structure

Remark 3.1. *Let X and Y be definable topological spaces.*

1. *The subspace topology on any definable subset of X is a definable topology.*
2. *The sum and products topologies on $X \coprod Y$ and $X \times Y$ are definable.*
3. *If D is a definable subset of X , then \overline{D} is definable.*
4. *As D ranges over a definable family of subsets of X , \overline{D} ranges over a definable family.*

In definable topological spaces, there are notions of “definable connectedness” and “definable compactness” behaving similarly to normal connectedness and compactness. Here we will only deal with definable compactness.³⁴

Say that a partial order (\leq, P) is downwards-directed if every finite non-empty subset of P has a lower bound, and upwards-directed if every finite non-empty subset of P has an upper bound. Recall that a topological space is compact if every downwards-directed family of non-empty closed sets has non-empty intersection.

Definition 3.2. *A definable topological space X is definably compact if $\bigcap \mathcal{F}$ is non-empty, for every definable family \mathcal{F} of non-empty closed subsets of X that is downwards-directed with respect to inclusion.*

More generally, a definable subset $D \subseteq X$ is said to be definably compact if the induced subspace topology on D is definably compact.

Example 3.3.

1. *The order topology on $(\mathbb{R}, <)$ is not definably compact due to the family of half-infinite intervals $[a, +\infty)$, which has empty intersection in spite of being a downwards directed family of closed non-empty sets.*
2. *In contrast, $[0, 1]$ is definably compact in $(\mathbb{R}, <)$, because it is compact.*
3. *The closed interval $[0, 1]$ is definably compact in (\mathbb{Q}, \leq) , because this is elementarily equivalent to the previous example.*
4. *The discrete topology on any pseudofinite or NSOP set is definably compact, because downwards-directed families of subsets must have minima. For example, the discrete topology on a pseudofinite field or an algebraically closed field is definably compact.*

³Definition 3.2 does not appear in the literature, except for some slides and unpublished notes of Fornasiero [2].

⁴There is an alternative notion of “definable compactness” in the o-minimal setting, due to Peterzil and Steinhorn [5]. The Peterzil-Steinhorn definition uses completable curves, and is primarily geared for the setting of “definable spaces.” In our terminology, Peterzil-Steinhorn definable spaces are definable topological spaces covered by finitely many open sets, each of which is homeomorphic to a definable subset of M^n with the induced subspace topology. Since exotic interpretable sets never admit such coverings, we do not use the Peterzil-Steinhorn theory. It is unclear whether our notion of definable compactness (Definition 3.2) agrees with Peterzil and Steinhorn’s definition, when restricted to definable spaces.

5. In \mathbb{Q}_p , the ring of integers \mathbb{Z}_p is definably compact in the valuation topology, because it is compact. More generally, the ring of integers in a p -adically closed field is definably compact in the valuation topology.
6. If K is a pseudofinite field, then the ring $K[[t]]$ is definably compact with respect to the valuation topology (i.e., the (t) -adic topology), because it is elementarily equivalent to an ultraproduct of the previous examples.
7. One can show that $\mathbb{C}[[t]]$ is definably compact in the valuation topology, using the fact that the residue field is a pure algebraically closed field.

We now verify that many of the familiar properties of compactness hold for definable compactness. (Fornasiero has independently made these observations in [2].)

Lemma 3.4. *Let $f : X \rightarrow Y$ be a definable continuous map between two definable topological spaces. Then $f(K)$ is definably compact for any definable compact set $K \subseteq X$.*

Proof. Replacing X and Y with K and $f(K)$, we may assume $K = X$ and f is surjective. Let \mathcal{F} be a downwards-directed definable family of non-empty closed subsets of Y . As f is surjective, $f^{-1}(F)$ is a non-empty closed subset of X for each $F \in \mathcal{F}$. Moreover, the map

$$F \mapsto f^{-1}(F)$$

is order-preserving, so the family

$$\{f^{-1}(F) : F \in \mathcal{F}\}$$

is downwards-directed. This family is a definable family, so by definable compactness on X , there is some $x_0 \in X$ such that

$$x_0 \in f^{-1}(F) \quad \forall F \in \mathcal{F}$$

or equivalently,

$$f(x_0) \in F \quad \forall F \in \mathcal{F}$$

Thus $\bigcap \mathcal{F}$ is non-empty, proving definable compactness of Y . □

Lemma 3.5.

1. If K is a definably compact definable topological space, and $F \subseteq K$ is a closed subset, then F is definably compact itself.
2. If K_1 and K_2 are definably compact, so is $K_1 \cup K_2$.

Proof. 1. Any downwards-directed definable family of closed non-empty subsets of F is also a downwards-directed definable family of closed non-empty subsets of K , so definable compactness directly transfers.

2. Let \mathcal{F} be a downwards-directed definable family of closed subsets of $K_1 \cup K_2$. Suppose $\bigcap \mathcal{F} = \emptyset$. We will show $\emptyset \in \mathcal{F}$.

If F is a closed definable subset of $K_1 \cup K_2$, then $F \cap K_1$ and $F \cap K_2$ are closed subsets of K_1 and K_2 . The maps

$$\begin{aligned} F &\mapsto F \cap K_1 \\ F &\mapsto F \cap K_2 \end{aligned}$$

are order-preserving, so the families

$$\begin{aligned} \mathcal{F}_1 &:= \{F \cap K_1 : F \in \mathcal{F}\} \\ \mathcal{F}_2 &:= \{F \cap K_2 : F \in \mathcal{F}\} \end{aligned}$$

are also downwards-directed definable families of closed sets. Note that

$$\bigcap \mathcal{F}_i \subseteq \bigcap \mathcal{F} = \emptyset$$

for $i = 1, 2$. Consequently $\emptyset \in \mathcal{F}_i$ for $i = 1, 2$, meaning that there are $F_1, F_2 \in \mathcal{F}$ such that

$$F_i \cap K_i = \emptyset$$

for $i = 1, 2$. By downward-directedness, there is some $F_3 \in \mathcal{F}$ such that $F_3 \subseteq F_1 \cap F_2$. Then

$$F_3 \cap (K_1 \cup K_2) = (F_3 \cap K_1) \cup (F_3 \cap K_2) \subseteq (F_1 \cap K_1) \cup (F_2 \cap K_2) = \emptyset \cup \emptyset = \emptyset.$$

So $\emptyset \in \mathcal{F}$. □

Say that a definable map $f : X \rightarrow Y$ of definable topological spaces is *definably closed* if $f(D)$ is closed for every closed definable subset $D \subseteq X$. This is a weaker condition than being a closed map: for example, in the structure (\mathbb{Q}, \leq) , the projection $\mathbb{Q} \times [0, 1] \rightarrow \mathbb{Q}$ is not closed⁵, but is definably closed (by Lemmas 3.6 and 3.9).

Lemma 3.6. *Let X and K be definable topological spaces, with K definably compact. Consider the product topology on $X \times K$ and let $\pi : X \times K \rightarrow X$ be the projection. Then π is definably closed.*

Proof. Suppose F is a closed subset of $X \times K$ and $x_0 \in X \setminus \pi(F)$. We will show $x_0 \notin \overline{\pi(F)}$, so that $\pi(F) = \overline{\pi(F)}$. For each open neighborhood N of x_0 , let

$$N^\dagger := \{k \in K : \text{there is an open neighborhood } U \text{ of } k \text{ such that } (N \times U) \cap F = \emptyset\}$$

⁵Take a sequence a_1, a_2, \dots of rational numbers in $[0, 1]$ converging to an irrational number. If $S = \{(1/n, a_n) : n \in \mathbb{N}\}$, then S is closed (as a subset of $\mathbb{Q} \times \mathbb{Q}$), but its projection onto the first coordinate is not closed.

Note that N^\dagger is open and map $N \mapsto N^\dagger$ is order-reversing. Let \mathcal{N} be a definable neighborhood basis of x_0 , and let

$$\mathcal{N}^\dagger := \{N^\dagger : N \in \mathcal{N}\}$$

Because \mathcal{N} is downwards-directed, \mathcal{N}^\dagger is upwards-directed.

Furthermore, $\bigcup \mathcal{N}^\dagger = K$. Indeed, if k is any element of K , then $(x_0, k) \notin F$, by choice of x_0 , so some open neighborhood $N \times U$ of (x_0, k) avoids F , as F is closed.

So \mathcal{N}^\dagger is an upwards-directed definable family of open subsets of K , whose union is all of K . By definable compactness, $K \in \mathcal{N}^\dagger$. So there is some $N \in \mathcal{N}$ with $N^\dagger = K$, implying that $(N \times K) \cap F = \emptyset$, and thus $N \cap \pi(F) = \emptyset$. Thus we have produced an open neighborhood N of x_0 disjoint from $\pi(F)$, showing that $x_0 \notin \overline{\pi(F)}$. As x_0 was an arbitrary point not in $\pi(F)$, it follows that $\pi(F)$ is closed. \square

Proposition 3.7. *Let X and Y be definably compact definable topological spaces. Then $X \times Y$ is definably compact.*

Proof. We may assume X and Y are non-empty. Let $\pi : X \times Y \rightarrow X$ denote the projection. Suppose \mathcal{F} is a downwards-directed definable family of non-empty closed subsets of $X \times Y$. For each $F \in \mathcal{F}$, the projection $\pi(F)$ is closed, by Lemma 3.6, and obviously non-empty. Furthermore, the map $F \mapsto \pi(F)$ is order-preserving. Consequently, the family

$$\{\pi(F) : F \in \mathcal{F}\}$$

is a downwards-directed definable family of closed non-empty subsets of X . By definable compactness of X , we may find some x_0 such that

$$x_0 \in \pi(F) \quad \forall F \in \mathcal{F}$$

Equivalently, $F \cap (\{x_0\} \times Y)$ is non-empty for every $F \in \mathcal{F}$. Note that $\{x_0\} \times Y$ is definably compact (as a subset of $X \times Y$) because it is definably homeomorphic to Y . The family

$$\{F \cap (\{x_0\} \times Y) : F \in \mathcal{F}\}$$

is a definable family of non-empty closed subsets of $\{x_0\} \times Y$, and it is downwards-directed because the map

$$F \mapsto F \cap (\{x_0\} \times Y)$$

is order-preserving. By definable compactness of $\{x_0\} \times Y$, we can find some (x_0, y_0) which is in every F , showing that $\bigcap \mathcal{F}$ is non-empty. \square

Lemma 3.8. *Let X be a definable topological space that is Hausdorff, and let K be a definably compact subset. Then K is closed.*

Proof. Otherwise, fix $x_0 \in \partial K$. Let \mathcal{N} be a definable neighborhood basis of x_0 . The family \mathcal{N} is downwards directed, and the map

$$N \mapsto \overline{N} \cap K$$

is order-preserving, so the family

$$\{\overline{N} \cap K : N \in \mathcal{N}\}$$

is a downwards-directed definable family of closed subsets of K . Furthermore, none of the sets $\overline{N} \cap K$ is empty, because $x \in \partial K$, so each N intersects K . By definable compactness, there is some x_1 such that

$$x_1 \in \overline{N} \cap K \quad \forall N \in \mathcal{N}$$

Then $x_1 \in K$, so $x_1 \neq x_0$. By the Hausdorff property, some open neighborhood N of x_0 satisfies $x_1 \notin \overline{N}$. Shrinking N a little, we may assume $N \in \mathcal{N}$, and obtain a contradiction. \square

Lemma 3.9. *If M is an o-minimal structure, then any closed interval $[c, d] \subset M^1$ is definably compact in the order topology.*

Proof. Let \mathcal{F} be a downwards-directed definable family of non-empty closed subsets of $[c, d]$. O-minimality ensures that $\max F$ exists for each $F \in \mathcal{F}$. Let

$$S = \{\max F : F \in \mathcal{F}\}$$

This is a definable subset of $[c, d]$, so $s_0 = \inf S$ exists. We claim that $s_0 \in F$ for all $F \in \mathcal{F}$.

Otherwise, by closedness of the F 's, there must be some open interval (a, b) around s_0 , and some $F_0 \in \mathcal{F}$, such that $(a, b) \cap F_0 = \emptyset$. Since s_0 is the infimum of S , it must be in the closure of S , so S must intersect (a, b) . In particular, there must be some $s_1 \in S \cap (a, b)$. By definition of S , there is some $F_1 \in \mathcal{F}$ such that $s_1 = \max F_1$. By downwards directedness, there is some $F_2 \in \mathcal{F}$ such that $F_2 \subseteq F_0 \cap F_1$. Then

$$F_2 \subseteq F_1 \subseteq (-\infty, s_1] \subseteq (-\infty, b)$$

because $s_1 = \max F_1$ and $s_1 < b$. Additionally,

$$F_2 \cap (a, b) \subseteq F_0 \cap (a, b) = \emptyset.$$

Combining these, we see that $F_2 \subseteq (-\infty, a]$. Consequently, $\max F_2 \leq a < s_0$, contradicting the choice of s_0 . \square

The next proposition shows that our definition of definable compactness agrees with the standard one in o-minimal structures.

Proposition 3.10. *Let $(M, <, \dots)$ be an o-minimal structure. In the standard topology on M^n the definably compact sets are exactly the closed bounded sets.*

Proof. Let $X \subseteq M^n$ be definable.

First suppose that X is closed and bounded. Then $X \subseteq [a, b]^n$ for some $a, b \in M$. By Lemma 3.9, $[a, b]$ is definably compact, and by Proposition 3.7, $[a, b]^n$ is definably compact. Finally, the closed subset X of $[a, b]^n$ is compact by Lemma 3.5(1).

Next suppose X is not bounded. Then for every $a \leq b$, the intersection

$$X \cap ((-\infty, a] \cup [b, +\infty))^n$$

is non-empty. The family of all such intersections is a definable downwards-directed family of closed non-empty subsets of X . However, its intersection is empty, so X is not definably compact.

Finally, suppose X is not closed. Then X fails to be definably compact by Lemma 3.8, because the standard topology on M^n is Hausdorff. \square

Lemma 3.11. *Let $f : X \rightarrow Y$ be a definable continuous map from a definable topological space X to a definable topological space Y . If X is definably compact, Y is Hausdorff, and f is injective, then f is a homeomorphism onto its image.*

Proof. Shrinking Y , we may assume f is a bijection. For any definable subset $D \subseteq X$, D is closed in X if and only if $f(D)$ is closed in Y . Indeed, if $f(D)$ is closed, then $D = f^{-1}(f(D))$ is closed by continuity, and conversely, if D is closed, then D is compact by Lemma 3.5(1), $f(D)$ is compact by Lemma 3.4, and $f(D)$ is closed by Lemma 3.8.

Equivalently, a definable subset $D \subseteq X$ is open in X if and only if $f(D)$ is open in Y . Because X and Y have definable bases of opens, this is enough to ensure that f is a homeomorphism. \square

3.2 Quotient topologies and open maps

Recall ([7] §6.4) that a surjective continuous map $f : X \rightarrow Y$ is an *identifying map* if

$$f^{-1}(U) \text{ is open} \implies U \text{ is open}$$

for all $U \subseteq Y$. For a fixed topological space X , the identifying maps out of X are exactly the maps of the form $X \twoheadrightarrow X/E$ where X/E has the quotient topology.

Note that surjective open maps are identifying. Say that an equivalence relation E on a topological space is an *open equivalence relation* if the quotient map $X \twoheadrightarrow X/E$ is an open map.

For D a subset of X , let D^E denote the union of E -equivalence classes intersecting D . We will call this the *E -closure* of D . An equivalence relation E is an open equivalence relation exactly if the E -closure of any open set is open.

We are interested in open equivalence relations because they ensure definability of the quotient topology, in a model-theoretic setting:

Lemma 3.12. *Let X be an interpretable set with a definable topology. Let E be a definable open equivalence relation on X . Then the quotient topology on X/E is a definable topology.*

Proof. Let $f : X \twoheadrightarrow X/E$ be the quotient map, which is a surjective open map. Note that the open subsets of X/E are exactly the sets of the form $f(U)$ for U an open in X . Let \mathcal{B} be a definable basis of opens for X . Then

$$\{f(B) : B \in \mathcal{B}\}$$

is a definable basis for the topology on X/E . \square

In the proof of Theorem 1.3, we will prove that certain properties hold generically, and then shrink to open sets on which these properties hold. Open equivalence relations help ensure that the topology does not change too much when we pass to open subsets:

Lemma 3.13. *Let X be a topological space and E be an open equivalence relation on X . Let X' be an open subset of X .*

1. *The restriction of E to X' is an open equivalence relation.*
2. *There are two topologies on X'/E , the subspace topology (as a subset of X/E) and the quotient topology (as a quotient of X'). These two topologies agree.*
3. *The map $X'/E \hookrightarrow X/E$ is an open embedding.*

Proof. View X'/E as topological space via the subspace topology. The map $f : X \rightarrow X/E$ is an open map, so $f(X') = X'/E$ is an open subset of X/E , proving (3).

The top and right maps in the following commutative diagram are open maps, so their composition is also an open map.

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ X'/E & \longrightarrow & X/E \end{array}$$

Because the diagonal is an open map and the bottom map $X'/E \hookrightarrow X/E$ is a continuous injection, it follows that the left map $X' \rightarrow X'/E$ is an open map. Open surjective maps are identifying maps, so X'/E has the quotient topology from X' , proving (2). Having shown that X'/E has the quotient topology, (1) means precisely that $X' \rightarrow X'/E$ is an open map, which we showed. \square

3.3 Proof of Theorem 1.3

In this section, we will work inside a fixed o-minimal structure M . If $X \subseteq Y$ is an inclusion of interpretable sets, we will say that X is a *full subset* of Y if $\dim(Y \setminus X) < \dim Y$.

We will prove the following refinement of Theorem 1.3:

Theorem 3.14. *Let $X \subseteq M^n$ be a definable set, and E be a definable equivalence relation on X . There is a definable full open subset X' of X such that if E' is the restriction of E to X' , then E' is an open equivalence relation on X' (in the sense of §3.2), and the quotient topology on X'/E' is Hausdorff and locally Euclidean.*

The requirement that X' is a full open subset of X is exactly equivalent to conditions 1 and 2 of Theorem 1.3. Lemma 3.12 ensures that the quotient topology X'/E' is definable, and Lemma 3.13 ensures that the final condition 4 of Theorem 1.3 holds.

For the proof of Theorem 3.14, we may assume that X and E are 0-definable, by naming parameters otherwise. We may also assume that the language is countable (by passing to a reduct otherwise).

In proving Theorem 3.14, we may replace M with an \aleph_1 -saturated elementary extension. The topological properties other than local Euclideanity are all expressible by first-order sentences. In \aleph_1 -saturated models, local Euclideanity implies *uniform local Euclideanity*, local Euclideanity witnessed by charts of bounded complexity. And then uniform local Euclideanity can be expressed as a disjunction of first-order sentences, so it descends from the elementary extension to the original structure.

Thus, in what follows, we will assume that the language is countable, and that the ambient o-minimal structure is \aleph_1 -saturated. For a 0-definable or 0-interpretable set D , we will say that an element $a \in D$ is *generic (in D)* if $\dim(a/\emptyset) = \dim D$.

The following lemma contains the main tricks we will use in the proof:

Lemma 3.15. *Let $X \subseteq M^n$ be a 0-definable set. Working inside the definable topological space X ,*

1. $\dim \partial D < \dim D$ for any non-empty definable set D , where the frontier is taken inside X .
2. Let P be a subset of X which is 0-definable or 0-ind-definable. Suppose that P contains every generic element of X . Then P contains a full open 0-definable subset X' of X .
3. Let S be any countable set, and let a be an element of X . The collection of definable open neighborhoods B of a such that

$$\lceil B \rceil \overset{b}{\downarrow} aS$$

form a neighborhood basis of a .

Proof. 1. The frontier of D within X is smaller than the frontier of D within the ambient space M^n , and for M^n this fact is [7] Theorem 4.1.8.

2. Note that $X \setminus P$ is type-definable over \emptyset and contains only elements of rank less than $\dim X$ over \emptyset . Thus

$$X \setminus P \subseteq D$$

for some 0-definable D with

$$\dim \overline{D} = \dim D < \dim X,$$

and then we can take $X' = X \setminus \overline{D}$.

3. We can take B of the form

$$X \cap \prod_{i=1}^n]b_i, c_i[$$

where the b_i and c_i are close to a but independent from everything in sight. □

We break the proof of Theorem 3.14 into three steps, which are the next three propositions.

Proposition 3.16. *Let $X \subseteq M^n$ be 0-definable and E be a 0-definable equivalence relation on X . There is a 0-definable full open subset $X' \subseteq X$ on which the restriction $E \upharpoonright X'$ is an open equivalence relation.*

Proof. Recall from §3.2 that for $S \subseteq X$, the E -closure of S , denoted S^E , is the union of all E -equivalence classes that intersect S .

Say that a point $a \in X$ is *nice* if for every $b \in \{a\}^E$, and every neighborhood B of b ,

$$a \in \text{int}(B^E).$$

Note that we could equivalently restrict to basic open neighborhoods, so “niceness” is definable.

Claim 3.17. *Every generic element of X is nice. That is, if $a \in X$ and $\dim(a/\emptyset) = \dim X$, then a is nice.*

Proof. Suppose otherwise. Let a be generic, b be another point such that aEb holds, and B be an open neighborhood of b such that $a \notin \text{int}(B^E)$. Shrinking B , we may assume by Lemma 3.15(3) that

$$\ulcorner B \urcorner \downarrow^b ab.$$

As aEb and $b \in B$, we see $a \in B^E$. But by assumption, $a \notin \text{int}(B^E)$, and so $a \in \partial(X \setminus B^E)$. Then

$$\dim(a/\ulcorner B \urcorner) \leq \dim \partial(X \setminus B^E) < \dim(X \setminus B^E) \leq \dim X = \dim(a/\emptyset),$$

contradicting the independence of a and $\ulcorner B \urcorner$. □

The set of nice points is a 0-definable subset of X . By Lemma 3.15(2), there is a 0-definable full open subset $X' \subseteq X$ consisting only of nice points. Let E' be the restriction of E to X' . Note that a subset of X' is open as a subset of X' if and only if it is open as a subset of X . So we can talk unambiguously about “open” sets.

We claim that E' is an open equivalence relation on X' . Otherwise, there is an open subset U of X' such that $U^{E'}$ is not open. Take $a \in U^{E'} \setminus \text{int}(U^{E'})$, and choose a point $b \in U$ such that $aE'b$ holds.

In X , we have two E -equivalent points a, b and an open neighborhood U of b . As a is nice, $a \in \text{int}(U^E)$, meaning that there is a neighborhood V of a in X such that every point of V is connected via E to a point in U . Shrinking V , we may assume $V \subseteq X'$. Then V and U are in X' , so every element of V is connected via E' to some element of U , meaning that $V \subseteq U^{E'}$. Now V witnesses that $a \in \text{int}(U^{E'})$, a contradiction. □

Proposition 3.18. *Let $X \subseteq M^n$ be 0-definable and E be a 0-definable open equivalence relation on X . There is a 0-definable full open subset $X' \subseteq X$ such that X'/E is Hausdorff.*

Here, the topology on X'/E is either the quotient topology from the subspace topology on X' , or the subspace topology from the quotient topology on X/E . These two topologies agree by Lemma 3.13.

Proof. Let $\pi : X \rightarrow X/E$ denote the quotient map.

Claim 3.19. *Let a and b be two generic elements of X (perhaps not jointly generic). If a and b are in different E -equivalence classes, then there exist basic open neighborhoods N_1 and N_2 around a and b , respectively, such that $\pi(N_1) \cap \pi(N_2) = \emptyset$.*

Proof. We claim that $a \notin \overline{\{b\}^E}$. Suppose otherwise. Then $a \in \partial(\{b\}^E)$. Let c be an element of $\{b\}^E$ of maximal rank over $\pi(b)$. Then

$$\begin{aligned} \dim(a/\emptyset) &\leq \dim(a\pi(b)/\emptyset) = \dim(a/\pi(b)) + \dim(\pi(b)/\emptyset) \\ &\leq \dim(\partial(\{b\}^E)) + \dim(\pi(b)/\emptyset) < \dim(\{b\}^E) + \dim(\pi(b)/\emptyset) \\ &= \dim(c/\pi(b)) + \dim(\pi(b)/\emptyset) = \dim(c\pi(b)/\emptyset) \\ &= \dim(c/\emptyset) \leq \dim X, \end{aligned}$$

contradicting the fact that a is generic.

So a is not in the closure of $\{b\}^E$, and therefore some open neighborhood N_1 of a is disjoint from $\{b\}^E$. Shrinking N_1 slightly, we may assume by Lemma 3.15(3) that $\ulcorner N_1 \urcorner$ is independent from b . Because b is then generic over $\ulcorner N_1 \urcorner$, we see that $b \notin \partial(N_1^E)$. Now by choice of N_1 , $b \notin N_1^E$. Therefore $b \notin \overline{N_1^E}$. So we can find an open neighborhood N_2 of b , disjoint from N_1^E . The fact that N_2 is disjoint from N_1^E means exactly that $\pi(N_1)$ and $\pi(N_2)$ are disjoint. \square

Let $\Sigma(x)$ be the partial type over \emptyset asserting that x is generic over \emptyset . Let D be the set of pairs $(x, y) \in X \times X$ such that either $\pi(x) = \pi(y)$ or there exist neighborhoods N_1 of x and N_2 of y such that $\pi(N_1)$ and $\pi(N_2)$ are disjoint. Note that D is 0-definable. By Claim 3.19,

$$\Sigma(x) \wedge \Sigma(y) \vdash (x, y) \in D$$

By compactness, there is some 0-definable set X' such that

$$\Sigma(x) \vdash x \in X'$$

and $X' \times X' \subseteq D$. Shrinking X' a little, we may assume X' is a full open subset of X , as in the proof of Lemma 3.15(2).

Let E' be the restriction of E to X' . By Lemma 3.13, X'/E' is an open subset of X/E , and E' is an open equivalence relation on X' . We claim that X'/E' is Hausdorff.

Let a_0, b_0 be two distinct elements of X'/E' , and let a and b be lifts of a_0 and b_0 to X' . By choice of X' , the pair (a, b) is in D . As $\pi(a) = a_0 \neq b_0 = \pi(b)$, a and b are not E -equivalent. By definition of D , there exist neighborhoods N_1 and N_2 in X , around a and b , such that $\pi(N_1)$ is disjoint from $\pi(N_2)$. Because $\pi : X \rightarrow X/E$ is an open map, $\pi(N_1)$ is a neighborhood of a_0 , and $\pi(N_2)$ is an open neighborhood of b_0 . Therefore, a_0 and b_0 can be separated by open neighborhoods in X/E , hence also in X'/E' . \square

Proposition 3.20. *Let $X \subseteq M^n$ be 0-definable and E be a 0-definable open equivalence relation on X , with X/E Hausdorff. Then there is a 0-definable full open subset $X' \subseteq X$ such that X'/E is locally Euclidean.*

In the proposition, note that the topologies on X/E and X'/E are definable, thanks to Lemma 3.12.

Proof. Let $\pi : X \rightarrow X/E$ be the quotient map.

Claim 3.21. *It suffices to show that X/E is Euclidean at $\pi(a)$ for every generic $a \in X$.*

Proof. The set of $a \in X$ such that local Euclideanity holds at $\pi(a)$ is *ind-definable* over \emptyset . By Lemma 3.15(2), if this set includes every generic of X , then there must be a 0-definable full open subset X' of X such that local Euclideanity holds at $\pi(a)$ for all $a \in X'$. By Lemma 3.12 the map of quotient spaces $X'/E \hookrightarrow X/E$ is an open embedding. Therefore, X'/E is also Euclidean at $\pi(a)$, for every $a \in X'$. In other words, X'/E is locally Euclidean. \square

So assume that $a \in X$ is generic. Let $e = \pi(a)$ be the image of a in X/E . We will show that X/E is Euclidean at e , i.e., that some neighborhood of e is definably homeomorphic to an open subset of M^k for some k .

Choose b such that $\text{tp}(a/e) = \text{tp}(b/e)$ and $a \perp_e^b b$. Note that $e = \pi(b)$.

Claim 3.22. *After re-ordering coordinates, we may write $b = b_1 b_2 b_3$ where*

- $b_3 \in \text{dcl}^{eq}(b_1 b_2)$
- $\dim(b_1 b_2 / \emptyset) = |b_1 b_2|$.
- $b_2 \in \text{dcl}^{eq}(b_1 e)$
- $\dim(b_1 / e) = |b_1|$.

Proof. In the pregeometry of definable closure over \emptyset , take $b_1 b_2$ to be a maximal independent subset of b . In the pregeometry of definable closure over e , take b_1 to be a maximal independent subset of $b_1 b_2$. \square

Let f and g be 0-definable functions such that

$$b_2 = f(b_1, e)$$

$$b_3 = g(b_1, b_2)$$

Let N be a countable model containing a, b , and let B be a closed box with b_2 in its interior, such that

$$\ulcorner B \urcorner \downarrow^b N$$

and such that B is contained in every N -definable open neighborhood of b_2 .

Because b_1, b_2 are generic in $M^{|b_1 b_2|}$, the function g is continuous on an open neighborhood of $b_1 b_2$. In particular, g is continuous on $\{b_1\} \times B$.

The set of x such that

$$x = f(b_1, \pi(b_1, x, g(b_1, x))) \quad (2)$$

contains b_2 , and is b_1 -definable. Since $b_1 b_2$ is generic over \emptyset , b_2 is generic over b_1 . Therefore, b_2 is in the interior of the set of x such that (2) holds. Consequently, (2) holds for $x \in B$.

Let $h : B \rightarrow X/E$ be the map given by

$$h(x) = \pi(b_1, x, g(b_1, x))$$

Then h is continuous on B (because g is continuous there, and π is continuous everywhere). Furthermore, (2) shows that h is injective. By Lemma 3.11, B is homeomorphic to $h(B)$. Consequently, $\text{int}(B)$ is homeomorphic to $h(\text{int}(B))$.

To complete the proof of local Euclideanity around $\pi(a)$, it suffices to show that $\pi(a)$ is in the interior of $h(\text{int}(B))$. The set of $x \in X$ such that

$$\pi(x) \in h(\text{int}(B))$$

is definable over $b_1 \ulcorner B \urcorner$, and contains a . It suffices to show that a is generic (in X) over $b_1 \ulcorner B \urcorner$.

To see this, note that

$$\dim(b_1/a) = \dim(b_1/e) = |b_1| = \dim(b_1/\emptyset).$$

So b_1 is independent from a . As $\ulcorner B \urcorner$ is independent from everything in N , the sequence

$$a, b_1, \ulcorner B \urcorner$$

is independent. Then $\dim(a/b_1 \ulcorner B \urcorner) = \dim(a/\emptyset) = \dim X$, and so a is generic over $b_1 \ulcorner B \urcorner$. \square

We now prove Theorem 3.14

Proof (of Theorem 3.14). As noted previously, we may assume the language is countable and the ambient model is \aleph_1 -saturated. By Proposition 3.16, we may find a 0-definable full open subset $X_1 \subseteq X$ such that the restriction $E_1 := E \upharpoonright X_1$ is an open equivalence relation. By Proposition 3.18 applied to X_1 and E_1 , there is a 0-definable full open subset $X_2 \subseteq X_1$ such that X_2/E_1 is Hausdorff. By Proposition 3.20 applied to X_2 and $E_1 \upharpoonright X_2$, there is a 0-definable full open subset $X_3 \subseteq X_2$ such that X_3/E_1 is locally Euclidean.

Take $X' = X_3$. The relations “full subset” and “open subset” are transitive, so X_3 is a full open subset of X . By Lemma 3.13,

- $E \upharpoonright X_3 = E_1 \upharpoonright X_3$ is an open equivalence relation on X_3 , because X_3 is open in X_1 .
- The inclusion $X_3/E \hookrightarrow X_2/E$ is an open embedding, and therefore X_3/E is Hausdorff.

\square

4 Tameness in the quotient topology

Say that a topology on an interpretable set Y is *admissible* if it is Hausdorff and there is a definable set $X \subseteq M^n$ and a surjective definable (continuous) open map $X \twoheadrightarrow Y$ where X has the subspace topology from $X \subseteq M^n$. Admissible topologies are definable by Lemma 3.12. The quotient topologies of Theorem 3.14 are admissible and locally Euclidean.

Remark 4.1. *If Y_1 and Y_2 are two interpretable sets with admissible topologies, then the disjoint union topological space $Y_1 \amalg Y_2$ is also admissible.*

We leave the proof as an exercise to the reader.

Proposition 4.2. *Every interpretable set admits an admissible locally Euclidean topology.*

Proof. If $f : X \rightarrow Y$ is a definable surjection from a definable set to an interpretable set, then Y admits an admissible locally Euclidean topology. We prove this by induction on $\dim(X)$. By Theorem 3.14, there is an open subset $X' \subseteq X$ such that the quotient topology on $f(X')$ is admissible and locally Euclidean, and such that $\dim(X \setminus X') < \dim(X)$. Let $Y' = f(X')$. Then

$$f^{-1}(Y \setminus Y') \subseteq X \setminus X'$$

Therefore, the inductive hypothesis can be applied to the surjection

$$f^{-1}(Y \setminus Y') \twoheadrightarrow Y \setminus Y',$$

showing that $Y \setminus Y'$ admits an admissible locally Euclidean topology. Taking the disjoint union of this topological space with the quotient topology on $Y' = f(X')$ gives an admissible topology on Y . \square

We now show that admissible locally Euclidean topologies have some tameness properties.

Proposition 4.3. *If Y is an interpretable set with an admissible topology, then the subspace topology on any definable subset of Y is also admissible.*

Proof. Let $Y' \subseteq Y$ be a definable subset. Let $f : X \twoheadrightarrow Y$ be the surjection witnessing that the topology on Y is admissible. Let $X' = f^{-1}(Y')$. Note that $f(U \cap X') = f(U) \cap Y'$ for any $U \subseteq X$. Therefore

$$\begin{aligned} \{U \subseteq Y' : U \text{ is open in } Y'\} &= \{U \cap Y' : U \text{ is open in } Y\} \\ &= \{f(U) \cap Y' : U \text{ is open in } X\} \\ &= \{f(U \cap X') : U \text{ is open in } X\} \\ &= \{f(U) : U \text{ is open in } X'\} \end{aligned}$$

It follows that the map $f \upharpoonright X'$ is an open map from X' to Y' . Subspaces of Hausdorff spaces are Hausdorff, so Y' is admissible. \square

Proposition 4.4. *If Y is an interpretable set with an admissible topology, then every definable subset of Y can be written as a finite union of definably connected sets.*

Proof. By Proposition 4.3, it suffices to show that Y itself can be written as a finite union of definably connected sets. Let $X \rightarrow Y$ be a map witnessing admissibility, with $X \subseteq M^n$. Then X has finitely many definably connected components by cell decomposition. The image of a definably connected set under a definable continuous map is definably connected, so Y also has finitely many definably connected components. \square

Lemma 4.5. *Assume \aleph_1 -saturation. Let Y be an interpretable set with an admissible topology, as witnessed by some map $f : X \rightarrow Y$. Let S be a countable set of parameters over which f, X, Y are defined, and T be a countable set. For any point p and any neighborhood N of p , there is a smaller neighborhood $N' \subseteq N$ of p such that*

$$\ulcorner N' \urcorner \downarrow_S^b pT$$

Proof. Let \tilde{p} be some point in X mapping to p . The set $f^{-1}(N)$ is an open neighborhood of \tilde{p} , because f is continuous. By Lemma 3.15(3) there is some smaller neighborhood $\tilde{p} \in U \subseteq f^{-1}(N)$ such that $\ulcorner U \urcorner \downarrow_S^b \tilde{p}TS$. Let $N' = f(U)$. This is a neighborhood of p because f is an open map. Furthermore,

$$\ulcorner U \urcorner \downarrow_S^b \tilde{p}TS \implies \ulcorner U \urcorner \downarrow_S^b \tilde{p}T \implies \ulcorner N' \urcorner \downarrow_S^b pT$$

because N' is defined from U and p is defined from \tilde{p} . \square

If X is an interpretable set with a definable topology, it makes sense to talk about the “local dimension” $\dim_p X$ of X at any point $p \in X$. Namely, the local dimension is the minimum of $\dim(N)$ as N ranges over neighborhoods of p in X . We can also talk about the local dimension $\dim_D p$ of a definable subset $D \subseteq X$ at a point $p \in D$. Specifically,

$$\dim_p D := \min_N \dim(N \cap D) \quad N \text{ a neighborhood of } p \text{ in } X.$$

This is the same as the local dimension at p within the subspace topology on D .

Proposition 4.6. *Let Y be an interpretable set with an admissible topology. If D is any definable subset of Y , then*

$$\dim(D) = \max_{p \in D} \dim_p(D).$$

Proof. By Proposition 4.3 we may assume $D = Y$. The maximum of the local dimensions is certainly at most $\dim(Y)$, so we only need to show that there is some point $p \in Y$ at which $\dim_p(Y) = \dim(Y)$. Because of the definability of dimension,

$$\{p \in Y : \dim_p(Y) = k\}$$

is definable for each k , in particular for $k = \dim(Y)$. Therefore we may pass to an \aleph_1 -saturated elementary extension. Let S be a finite set of parameters over which Y is defined, and let $p \in Y$ be a point such that $\dim(p/S) = \dim(Y)$. We claim that the local dimension of Y at p is $\dim(Y)$. Let $N \subseteq Y$ be any neighborhood of p ; we will show that $\dim(N) = \dim(Y)$. By Lemma 4.5, there is a smaller neighborhood N' of p such that

$$\begin{array}{c} \text{b} \\ \lrcorner N' \lrcorner \downarrow p \\ S \end{array}$$

Therefore, $\dim(p/\lrcorner N' \lrcorner S) = \dim(p/S) = \dim(Y)$. Because p lies in N' ,

$$\dim(N') \geq \dim(p/\lrcorner N' \lrcorner S) = \dim(p/S) = \dim(Y).$$

On the other hand, $Y \supseteq N \supseteq N'$, so

$$\dim(Y) \geq \dim(N) \geq \dim(N').$$

Therefore the inequalities are equalities and $\dim(N) = \dim(Y)$. As N was an arbitrarily small neighborhood of p , it follows that the local dimension $\dim_p(Y)$ agrees with $\dim(Y)$. \square

Proposition 4.7. *Let Y be an interpretable set with an admissible locally Euclidean topology.*

1. *If D is any definable subset of Y , then $\dim \partial D < \dim D$.*
2. *If D is any definable subset of Y , then $\dim \overline{D} = \dim D$ and $\dim \text{bd}(D) < \dim Y$.*
3. *Assuming saturation: if D is a type-definable subset of Y of dimension d , then D is contained in a definable closed set of dimension d .*

Proof.

1. Let $k = \dim \partial D$. By Proposition 4.6, there is a point $x \in \partial D$ such that $\dim_x(\partial D) = k$. Let U be an open neighborhood of x which is definably homeomorphic to an open subset of M^n for some n . Transferring the situation along the homeomorphism, and using the analogous fact for definable sets (= Lemma 3.15(1) or [7] Theorem 4.1.8), we see that $\dim(D \cap U) > k$.
2. These bounds follow because $\overline{D} = D \cup \partial D$, and $\text{bd}(D) = \partial D \cup \partial(Y \setminus D)$.
3. By general properties of dimension, $D \subseteq D'$ for some definable subset D' of dimension d . Then $D \subseteq \overline{D'}$ and $\dim \overline{D'} = d$.

\square

Using Lemma 4.5 and Proposition 4.7, one can transfer facts about “generic behavior” from the definable setting to the admissible interpretable setting. We give two examples:

- Definable subsets are Euclidean at generic points

- Definable functions are continuous at generic points in their domain.

Remark 4.8. *If D is any definable subset of M^n , then there is a definable full open subset $D' \subseteq D$ such that D' is locally Euclidean as a subspace of M^n . (Here, we mean that D' is open in D , not open in M^n .)*

Proof. Write D as a disjoint union of cells $\bigcup_{i=1}^k C_i$ by cell decomposition. Each cell C_i is locally Euclidean in isolation. Take D' to be $D \setminus \bigcup_{i=1}^k \partial C_i$, where the closure and frontier are with respect to the topology on M^n . This is open as a subset of D , and a full open subset by standard dimension bounds. Every point p in D' is in the closure of exactly one C_i , so the Euclideanity of C_i at p implies the Euclideanity of D' at p . \square

Lemma 4.9. *Let Y be an interpretable set with admissible locally Euclidean topology. If D is any definable subset of Y , then there is a definable full open subset $D' \subseteq D$ such that the subspace topology on D' is locally Euclidean.*

Proof. Without loss of generality, we may assume the ambient model is sufficiently saturated, and that Y and D are 0-definable.

Claim 4.10. *If $a \in D$ is generic in D , then D is locally Euclidean at a .*

Proof. By local Euclideanity of Y , there is an open neighborhood $a \in U \subseteq Y$ definably homeomorphic to an open in some M^n . By Lemma 4.5, we may shrink U and assume that $a \in \text{b}^\ulcorner U \urcorner$. Let $\iota : U \hookrightarrow M^n$ be the definable open embedding. Moving ι by an automorphism fixing $\ulcorner U \urcorner$, we may assume $a \in \text{b}^\ulcorner U \urcorner \ulcorner \iota \urcorner$.

Therefore we can name $\ulcorner U \urcorner$ and $\ulcorner \iota \urcorner$ as constants, and assume that U and ι are 0-definable, without losing the fact that a is generic. Now because a is generic in $U \cap D$, the image $\iota(a)$ is generic in $\iota(U \cap D)$. By Remark 4.8, $\iota(U \cap D)$ is Euclidean at $\iota(a)$. Transferring things back along ι^{-1} , we see that D is Euclidean at a , proving the claim. \square

Now let D'' be the locally Euclidean locus of D . Then D'' is an ind-definable subset of D , i.e., $D \setminus D''$ is a type-definable set. By the claim, $D \setminus D''$ has lower dimension than D . Thus by Proposition 4.7(3) there is a definable closed set F containing $D \setminus D''$, with $\dim F < \dim D$. Take $D' = D \setminus F$. Then D' is a full open subset of D , and $D' \subseteq D''$. \square

We recall another basic fact about o-minimality:

Remark 4.11. *Let U and U' be 0-definable open subsets of powers of M , and let f be a 0-definable partial map from U to U' . Suppose a is generic in U and that f is defined at a . Then f is defined and continuous on an open neighborhood of a .*

Proposition 4.12. *Let Y and Y' be two interpretable sets with admissible locally Euclidean topologies, and f be a definable map from Y to Y' . Then Y can be written as a finite disjoint union of definable locally closed sets, on which the restriction of f is continuous. More generally, this holds when Y is a definable subspace of an admissible locally Euclidean space.*

Proof. By induction on $\dim(Y)$ it suffices to show that f is continuous on a full open subset of Y . By Lemma 4.9, we may assume Y is locally Euclidean. By Proposition 4.7, the interior of any full subset is a full subset of Y , so it suffices to show that the continuous locus of f is a full subset of Y .

Without loss of generality, we may assume that everything is defined over \emptyset and that the ambient model is \aleph_1 -saturated. It suffices to show that f is continuous at generic points of Y . Fix some generic $y \in Y$. By local Euclideanity, there are open neighborhoods U of y and U' of $f(y)$ admitting open embeddings into powers of M . By Lemma 4.5 we may shrink U and U' in such a way that

$$\ulcorner U \urcorner \downarrow^{\text{p}} y \text{ and } \ulcorner U' \urcorner \downarrow^{\text{p}} y \ulcorner U \urcorner$$

Thus $\ulcorner U \urcorner \ulcorner U' \urcorner \downarrow^{\text{p}} y$. Let ι and ι' be definable open embeddings from U and U' into powers of M . Moving ι and ι' by an automorphism over $\ulcorner U \urcorner \ulcorner U' \urcorner$, we may assume that

$$\ulcorner \iota \urcorner \ulcorner \iota' \urcorner \downarrow^{\text{p}} y$$

and so

$$\ulcorner \iota \urcorner \ulcorner \iota' \urcorner \ulcorner U \urcorner \ulcorner U' \urcorner \downarrow^{\text{p}} y$$

By naming constants, we may assume that U, U', ι, ι' are all 0-definable, and y is still generic in Y .

Now $f \upharpoonright (U \cap f^{-1}(U'))$ is a partial function from U to U' , defined at y . Transferring things along the open embeddings ι, ι' , we reduce to the case where U and U' are open subsets of powers of M , reducing to the situation of Remark 4.11 above. \square

This completes the proof of Theorem 1.5 in the introduction, which is merely a compilation of Proposition 4.2, Lemma 3.12, Propositions 4.4, 4.7, 4.6, and 4.12.

We close with a few open questions:

Question 4.13. *Do Propositions 4.7 and 4.12 hold without the local Euclidean assumption?*

Question 4.14. *For “definable spaces” in the sense of Peterzil and Steinhorn [5], is our definition of “definable compactness” (Definition 3.2) equivalent to Peterzil and Steinhorn’s definition using completable curves?*

References

- [1] Jan Denef and Lou van den Dries. p -adic and real subanalytic sets. *The Annals of Mathematics*, 128(1):79–138, July 1988.
- [2] Antongiulio Fornasiero. Definable compactness for topological structures. Unpublished, 2015.

- [3] Sofya Kamenkovich and Ya'acov Peterzil. Euler characteristic of imaginaries in o-minimal structures. *Mathematical Logic Quarterly*, 63(5):376–383, December 2017.
- [4] Ya'acov Peterzil. Constructing a group-interval in o-minimal structures. *Journal of Pure and Applied Algebra*, 94:85–100, 1994.
- [5] Ya'acov Peterzil and Charles Steinhorn. Definable compactness and definable subgroups of o-minimal groups. *J. London Math. Soc.*, 59:769–786, 1999.
- [6] Janak Ramakrishnan, Ya'acov Peterzil, and Pantelis Eleftheriou. Interpretable groups are definable. *Journal of Mathematical Logic*, 14(1), June 2014.
- [7] Lou van den Dries. *Tame Topology and O-minimal Structures*. Number 248 in London Mathematical Society Lecture Note Series. Cambridge University Press, 1998.