

Games and Ramsey-like cardinals

DAN SAATTRUP NIELSEN AND PHILIP WELCH

October 31, 2018

ABSTRACT. We generalise the α -Ramsey cardinals introduced in Holy and Schlicht (2018) for cardinals α to arbitrary ordinals α , and answer several questions posed in that paper. In particular, we show that α -Ramseys are downwards absolute to the core model K for all α of uncountable cofinality, that strategic ω -Ramsey cardinals are equiconsistent with remarkable cardinals and that strategic α -Ramsey cardinals are equiconsistent with measurable cardinals for all $\alpha > \omega$. We also show that the n -Ramseys satisfy indescribability properties and use them to provide a game-theoretic characterisation of completely ineffable cardinals, as well as establishing further connections between the α -Ramsey cardinals and the Ramsey-like cardinals introduced in Gitman (2011), Feng (1990) and Sharpe and Welch (2011).¹

1 Introduction

Most of the large cardinals above measurable cardinals can be characterised as the critical points of elementary embeddings $j : V \rightarrow \mathcal{M}$, where the strength of the large cardinal notion in question is increased by requiring more closure of the target model \mathcal{M} and more properties of the embedding j . In analogy, Ramsey-like cardinals were introduced in Gitman (2011) and Gitman and Welch (2011) to be a natural weakening of this concept, being roughly cardinals κ that can be characterised as critical points of elementary embeddings $j : \mathcal{M} \rightarrow \mathcal{N}$ between κ -sized ZFC^- -models \mathcal{M} and \mathcal{N} . Here we then increase our consistency strength by requiring more closure of the domain model \mathcal{M} and more properties of the embedding j .

Implicit work in Mitchell (1979) and Donder et al. (1981) shows that Ramsey cardinals are precisely of this type, in which the derived measure from j is both weakly amenable and countably complete.² The question is then how many of the well-known large cardinals can be characterised in this fashion? Gitman (2011) introduced various Ramsey-like cardinals, whose definitions we will recall in the next section, and recently Holy and Schlicht (2018) have introduced a new family of cardinals, called (strategic) α -Ramsey cardinals, which have the added feature of having a game-theoretic definition.

In Holy and Schlicht (2018) the (strategic) α -Ramseys were considered for α being an infinite cardinal, and in this paper we will expand this definition to any ordinal α . Section 3 will cover the finite case which

¹2010 *Mathematics Subject Classification*. 03E35, 03E45, 03E55.

Keywords and phrases. Ramsey-like cardinals, large cardinals, games, weakly compact cardinals, ineffable cardinals, completely ineffable cardinals, remarkable cardinals, virtually measurable cardinals, measurable cardinals, core model.

²For a proof of this result see Theorem 1.3 of Gitman (2011).

allows us to characterise ineffable-type cardinals and show indescribability properties of these cardinals — these arguments are based on arguments in Abramson et al. (1977).

Section 4 contains the countable case in which we establish that strategic ω -Ramseys are equiconsistent with Schindler’s *remarkable cardinals*, and use this to show that strategic ω -Ramseys are of strictly stronger consistency strength than the ω -Ramseys. We will also consider a hierarchy between ω -Ramsey cardinals and Ramsey cardinals called (ω, α) -Ramsey cardinals, which we will show interleaves with the α -iterable cardinals introduced in Gitman (2011), and lastly show that $(\omega+1)$ -Ramseys are Ramsey limits of Ramseys and that strategic $(\omega+1)$ -Ramseys are equiconsistent with a measurable cardinal.

In section 5 we investigate how the strongly Ramsey and super Ramsey cardinals introduced in Gitman (2011) relate to the α -Ramsey cardinals and show that these latter cardinals are downwards absolute to the core model K . The last part of this section is dedicated to showing a tight correspondence between strategic α -Ramsey cardinals and the α -very Ramsey cardinals introduced in Sharpe and Welch (2011), leading to the result that strategic ω_1 -Ramsey cardinals are measurable in the core model K below a Woodin cardinal. Section 6 contains an overview of open problems concerning these Ramsey-like cardinals.

The last section includes two diagrams, showing the relations between all the Ramsey-like cardinals considered in this paper, both in terms of consistency strength and direct implication. A solid line means that the (consistency or direct) implication is “strict”, in the sense that no proof exists for the implication in the opposite direction, and a dashed line means that we do not know whether the implication is strict or not.

2 Setting the scene

In this section we will recall a handful of definitions concerning Ramsey-like cardinals, as well as define the α -Ramsey cardinals for arbitrary ordinals α . We start out with the models and measures that we are going to consider.

DEFINITION 2.1. For a cardinal κ , a **weak κ -model** is a set \mathcal{M} of size κ satisfying that $\kappa + 1 \subseteq \mathcal{M}$ and $(\mathcal{M}, \in) \models \text{ZFC}^-$. If furthermore $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$, \mathcal{M} is a **κ -model**.³ \dashv

Recall that μ is an **\mathcal{M} -measure** if $(\mathcal{M}, \in, \mu) \models \ulcorner \mu \text{ is a } \kappa\text{-complete ultrafilter on } \kappa \urcorner$.

DEFINITION 2.2. Let \mathcal{M} be a weak κ -model and μ an \mathcal{M} -measure. Then μ is

- **weakly amenable** if $x \cap \mu \in \mathcal{M}$ for every $x \in \mathcal{M}$ with \mathcal{M} -cardinality κ ;
- **countably complete** if $\bigcap \vec{X} \neq \emptyset$ for every ω -sequence $\vec{X} \in {}^\omega \mu$;
- **\mathcal{M} -normal** if $(\mathcal{M}, \in, \mu) \models \forall \vec{X} \in {}^\kappa \mu : \Delta \vec{X} \in \mu$;
- **genuine** if $|\Delta \vec{X}| = \kappa$ for every κ -sequence $\vec{X} \in {}^\kappa \mu$;
- **normal** if $\Delta \vec{X}$ is stationary in κ for every κ -sequence $\vec{X} \in {}^\kappa \mu$;
- **0-good**, or simply **good**, if it has a well-founded ultrapower;
- **α -good** for $\alpha > 0$ if it is weakly amenable and has α -many well-founded iterates.

³Note that our (weak) κ -models do not have to be transitive, in contrast to the models considered in Gitman (2011) and Gitman and Welch (2011). Not requiring the models to be transitive was introduced in Holy and Schlicht (2018).

⊢

Note that a genuine \mathcal{M} -measure is \mathcal{M} -normal and countably complete, and a countably complete weakly amenable \mathcal{M} -measure is α -good for all ordinals α . We'll use the fact shown in Holy and Schlicht (2018) that an \mathcal{M} -measure μ is normal iff $\triangle \vec{X}$ is stationary for some enumeration $\vec{X} = \langle X_\alpha \mid \alpha < \kappa \rangle$ of μ . We are also going to use the following alternative characterisation of weak amenability.

PROPOSITION 2.3 (Folklore). *Let \mathcal{M} be a weak κ -model, μ an \mathcal{M} -measure and $j : \mathcal{M} \rightarrow \mathcal{N}$ the associated ultrapower embedding. Then μ is weakly amenable if and only if j is κ -powerset preserving, meaning that $\mathcal{M} \cap \mathcal{P}(\kappa) = \mathcal{N} \cap \mathcal{P}(\kappa)$.*

⊢

The α -Ramsey cardinals in Holy and Schlicht (2018) are based upon the following game.⁴

DEFINITION 2.4 (Holy-Schlicht). For an uncountable cardinal $\kappa = \kappa^{<\kappa}$, a limit ordinal $\gamma \leq \kappa$ and a regular cardinal $\theta > \kappa$ define the game $wfG_\gamma^\theta(\kappa)$ of length γ as follows.

$$\begin{array}{ccccccc} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \mathcal{M}_2 & \dots \\ \text{II} & & \mu_0 & & \mu_1 & & \mu_2 & \dots \end{array}$$

Here $\mathcal{M}_\alpha < H_\theta$ is a κ -model and μ_α is a filter for all $\alpha < \gamma$, such that μ_α is an \mathcal{M}_α -measure, the \mathcal{M}_α 's and μ_α 's are \subseteq -increasing and $\langle \mathcal{M}_\xi \mid \xi < \alpha \rangle, \langle \mu_\xi \mid \xi < \alpha \rangle \in \mathcal{M}_\alpha$ for every $\alpha < \gamma$. Letting $\mu := \bigcup_{\alpha < \gamma} \mu_\alpha$ and $\mathcal{M} := \bigcup_{\alpha < \gamma} \mathcal{M}_\alpha$, player II wins iff μ is an \mathcal{M} -normal good \mathcal{M} -measure.

⊢

Recall that two games G_1 and G_2 are **equivalent** if player I has a winning strategy in G_1 iff they have one in G_2 , and player II has a winning strategy in G_1 iff they have one in G_2 . Holy and Schlicht (2018) showed that the games $wfG_\gamma^{\theta_0}(\kappa)$ and $wfG_\gamma^{\theta_1}(\kappa)$ are equivalent for any γ with $\text{cof } \gamma \neq \omega$ and any regular $\theta_0, \theta_1 > \kappa$. We will be working with a variant of the $wfG_\gamma(\kappa)$ games in which we require less of player I but more of player II. It will turn out that this change of game is innocuous, as Proposition 2.6 will show that they are equivalent.

DEFINITION 2.5 (Holy-Schlicht-N.). Let $\kappa = \kappa^{<\kappa}$ be an uncountable cardinal, $\gamma \leq \kappa$ and ζ ordinals and $\theta > \kappa$ a regular cardinal. Then define the following game $\mathcal{G}_\gamma^\theta(\kappa, \zeta)$ with $(\gamma+1)$ -many rounds:

$$\begin{array}{ccccccc} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \dots & \mathcal{M}_\gamma \\ \text{II} & & \mu_0 & & \mu_1 & & \dots & \mu_\gamma \end{array}$$

Here $\mathcal{M}_\alpha < H_\theta$ is a weak κ -model for every $\alpha \leq \gamma$, μ_α is a normal \mathcal{M}_α -measure for $\alpha < \gamma$, μ_γ is an \mathcal{M}_γ -normal good \mathcal{M}_γ -measure and the \mathcal{M}_α 's and μ_α 's are \subseteq -increasing. For limit ordinals $\alpha \leq \gamma$ we furthermore

⁴Unless otherwise stated, every game considered will be a game with perfect information between two players I and II. For a formal framework modelling these games, see e.g. Kanamori (2008).

require that $\mathcal{M}_\alpha = \bigcup_{\xi < \alpha} \mathcal{M}_\xi$, $\mu_\alpha = \bigcup_{\xi < \alpha} \mu_\xi$ and that μ_α is ζ -good. Player II wins iff they could continue to play throughout all $(\gamma+1)$ -many rounds. \dashv

For convenience we will write $\mathcal{G}_\gamma^\theta(\kappa)$ for the game $\mathcal{G}_\gamma^\theta(\kappa, 0)$, and $\mathcal{G}_\gamma(\kappa)$ for $\mathcal{G}_\gamma^\theta(\kappa)$ whenever $\text{cof } \gamma \neq \omega$, as again the existence of winning strategies in these games doesn't depend upon a specific θ . Note that we assume that $\kappa = \kappa^{<\kappa}$ is uncountable in the definition of the games that we're considering, so this is a standing assumption throughout the paper, whenever any one of the above two games are considered.

PROPOSITION 2.6 (Holy-Schlicht-N.). *$\mathcal{G}_\gamma^\theta(\kappa)$, $\mathcal{G}_\gamma^\theta(\kappa, 1)$ and $wfG_\gamma^\theta(\kappa)$ are all equivalent for all limit ordinals $\gamma \leq \kappa$, and $\mathcal{G}_\gamma^\theta(\kappa, \zeta)$ is equivalent to $\mathcal{G}_\gamma^\theta(\kappa)$ whenever $\text{cof } \gamma > \omega$ and $\zeta \in \text{On}$.*

PROOF. We start by showing the latter statement, so assume that $\text{cof } \gamma > \omega$. Consider now the auxilliary game, call it \mathcal{G} , which is exactly like $\mathcal{G}_\gamma^\theta(\kappa, 0)$, but where we also require that ${}^\omega\mathcal{M}_\alpha \subseteq \mathcal{M}_{\alpha+1}$ and $\langle \mathcal{M}_\xi \mid \xi \leq \alpha \rangle, \langle \mu_\xi \mid \xi \leq \alpha \rangle \in \mathcal{M}_{\alpha+1}$ for every $\alpha < \gamma$.

Claim 2.6.1. \mathcal{G} is equivalent to $\mathcal{G}_\gamma^\theta(\kappa)$.

PROOF OF CLAIM. If player I has a winning strategy in \mathcal{G} then they also have one in $\mathcal{G}_\gamma^\theta(\kappa)$, by doing exactly the same. Analogously, if player II has a winning strategy in $\mathcal{G}_\gamma^\theta(\kappa)$ then they also have one in \mathcal{G} . If player I has a winning strategy σ in $\mathcal{G}_\gamma^\theta(\kappa)$ then we can construct a winning strategy σ' in \mathcal{G} , which is defined as follows. Fix some $\alpha \leq \gamma$ and, writing $\vec{\mathcal{M}}_\xi := \langle \mathcal{M}_\xi \mid \xi \leq \alpha \rangle$ and $\vec{\mu}_\xi := \langle \mu_\xi \mid \xi \leq \alpha \rangle$, we set

$$\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha \rangle) := \text{Hull}^{H_\theta}(\sigma(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha \rangle) \cup {}^\omega\mathcal{M}_\alpha \cup \{\vec{\mathcal{M}}_\xi, \vec{\mu}_\xi\}),$$

i.e. that we're simply throwing in the sequences into our models and making sure that we're still an elementary substructure of H_θ . This new strategy σ' is clearly winning. Assuming now that τ is a winning strategy for player II in \mathcal{G} , we define a winning strategy τ' for player II in $\mathcal{G}_\gamma^\theta(\kappa)$ by letting $\tau'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha \rangle)$ be the result of throwing in the appropriate sequences into the models \mathcal{M}_ξ , applying τ to get a measure, and intersecting that measure with \mathcal{M}_α to get an \mathcal{M}_α -measure. \dashv

Now, letting \mathcal{M}_γ be the final model of a play of \mathcal{G} , $\text{cof } \gamma > \omega$ implies that any ω -sequence $\vec{X} \in \mathcal{M}_\gamma$ really is a sequence of elements from some \mathcal{M}_ξ for $\xi < \gamma$, so that $\vec{X} \in \mathcal{M}_{\xi+1}$ by definition of \mathcal{G} , making \mathcal{M}_γ closed under ω -sequences and thus also μ_γ countably complete. Since γ is a limit ordinal and the models contain the previous measures and models as elements, the proof of e.g. Theorem 5.6 in Holy and Schlicht (2018) shows that μ_γ is also weakly amenable, making it ζ -good for all ordinals ζ .

Now we deal with the first statement, so fix a limit ordinal γ . Firstly $\mathcal{G}_\gamma^\theta(\kappa)$ is equivalent to $\mathcal{G}_\gamma^\theta(\kappa, 1)$ as above, since both are equivalent to the auxilliary game \mathcal{G} when γ is a limit ordinal. So it remains to show that $\mathcal{G}_\gamma^\theta(\kappa)$ is equivalent to $wfG_\gamma^\theta(\kappa)$. If player I has a winning strategy σ in $wfG_\gamma^\theta(\kappa)$ then define a winning

strategy σ' for player I in $\mathcal{G}_\gamma^\theta(\kappa)$ as

$$\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi \leq \alpha \rangle) := \sigma(\langle \mathcal{M}_0, \mu_0 \rangle \frown \langle \mathcal{M}_{\xi+1}, \mu_{\xi+1} \mid \xi + 1 \leq \alpha \rangle)$$

and for limit ordinals $\alpha \leq \gamma$ set $\sigma'(\langle \mathcal{M}_\xi, \mu_\xi \mid \xi < \alpha \rangle) := \bigcup_{\xi < \alpha} \mathcal{M}_\xi$; i.e. they simply follow the same strategy as in $wfG_\gamma^\theta(\kappa)$ but plugs in unions at limit stages. Likewise, if player II had a winning strategy in $\mathcal{G}_\gamma^\theta(\kappa)$ then they also have a winning strategy in $wfG_\gamma^\theta(\kappa)$, this time just by skipping the limit steps in $\mathcal{G}_\gamma^\theta(\kappa)$.

Now assume that player I has a winning strategy σ in $\mathcal{G}_\gamma^\theta(\kappa)$ and that player I *doesn't* have a winning strategy in $wfG_\gamma^\theta(\kappa)$. Then define a strategy σ' for player I in $wfG_\gamma^\theta(\kappa)$ as follows. Let $s = \langle \mathcal{M}_\alpha, \mu_\alpha \mid \alpha \leq \eta \rangle$ be a partial play of $wfG_\gamma^\theta(\kappa)$ and let s' be the modified version of s in which we have 'inserted' unions at limit steps, just as in the above paragraph. We can assume that every μ_α in s' is good and \mathcal{M}_α -normal as otherwise player II has already lost and player I can play anything. Now, we want to show that s' is a valid partial play of $\mathcal{G}_\gamma^\theta(\kappa)$. All the models in s are κ -models, so in particular weak κ -models.

Claim 2.6.2. Every μ_α in s' is normal.

PROOF OF CLAIM. Assume without loss of generality that $\alpha = \eta$. Let player I play any legal response \mathcal{M} to s in $wfG_\gamma^\theta(\kappa)$ (such a response always exists). If player II can't respond then player I has a winning strategy by simply following $s \frown \langle \mathcal{M} \rangle$, \nexists , so player II *does* have a response μ to $s \frown \mathcal{M}$. But now the rules of $wfG_\gamma^\theta(\kappa)$ ensures that $\mu_\eta \in \mathcal{M}$, so since

$$(\mathcal{M}, \in, \mu) \models \forall \vec{X} \in {}^\kappa \mu : \ulcorner \Delta \vec{X} \text{ is stationary in } \kappa \urcorner,$$

we then also get that $\mathcal{M} \models \ulcorner \Delta \mu_\eta \text{ is stationary in } \kappa \urcorner$ since $\mu_\eta \subseteq \mu$, so elementarity of \mathcal{M} in H_θ implies that $\Delta \mu_\eta$ really is stationary in κ , making μ_η normal. \dashv

This makes s' a valid partial play of $\mathcal{G}_\gamma^\theta(\kappa)$, so we may form the weak κ -model $\tilde{\mathcal{M}}_\eta := \sigma(s')$. Now let $\mathcal{M}_\eta < H_\theta$ be a κ -model with $\tilde{\mathcal{M}}_\eta \subseteq \mathcal{M}_\eta$ and $s \in \mathcal{M}_\eta$ and set $\sigma'(s) := \mathcal{M}_\eta$. This defines the strategy σ' for player I in $wfG_\gamma^\theta(\kappa)$, which is winning since the winning condition for the two games is the same for γ a limit.⁵

Next, assume that player II has a winning strategy τ in $wfG_\gamma^\theta(\kappa)$. We recursively define a strategy $\tilde{\tau}$ for player II in $\mathcal{G}_\gamma^\theta(\kappa)$ as follows. If $\tilde{\mathcal{M}}_0$ is the first move by player I in $\mathcal{G}_\gamma^\theta(\kappa)$, let $\mathcal{M}_0 < H_\theta$ be a κ -model with $\tilde{\mathcal{M}}_0 \subseteq \mathcal{M}_0$, making \mathcal{M}_0 a valid move for player I in $wfG_\gamma^\theta(\kappa)$. Write $\mu_0 := \tau(\langle \mathcal{M}_0 \rangle)$ and then set $\tilde{\tau}(\langle \tilde{\mathcal{M}}_0 \rangle)$ to be $\tilde{\mu}_0 := \mu_0 \cap \tilde{\mathcal{M}}_0$, which again is normal by the same trick as above, making $\tilde{\mu}_0$ a legal move for player II in $\mathcal{G}_\gamma^\theta(\kappa)$. Successor stages $\alpha + 1$ in the construction are analogous, but we also make sure that $\langle \mathcal{M}_\xi \mid \xi < \alpha + 1 \rangle, \langle \mu_\xi \mid \xi < \alpha + 1 \rangle \in \mathcal{M}_{\alpha+1}$. At limit stages τ outputs unions, as is required by the rules of $\mathcal{G}_\gamma^\theta(\kappa)$. Since the union of all the μ_α 's is good as τ is winning, $\tilde{\mu}_\gamma := \bigcup_{\alpha < \gamma} \tilde{\mu}_\alpha$ is good as well, making $\tilde{\tau}$

⁵More precisely, that σ is winning in $\mathcal{G}_\gamma^\theta(\kappa)$ means that there's a sequence $\langle f_n : \kappa \rightarrow \kappa \mid n < \omega \rangle$ with the f_n 's all being elements of the last model $\tilde{\mathcal{M}}_\gamma$, witnessing the illfoundedness of the ultrapower. But then all these functions will also be elements of the union of the \mathcal{M}_α 's, since we ensured that $\mathcal{M}_\alpha \supseteq \tilde{\mathcal{M}}_\alpha$ in the construction above, making the ultrapower of $\bigcup_{\alpha < \gamma} \mathcal{M}_\alpha$ by $\bigcup_{\alpha < \gamma} \mu_\alpha$ illfounded as well.

winning and we are done. ■

We now arrive at the definitions of the cardinals we will be considering. They were in Holy and Schlicht (2018) only defined for γ being a cardinal, but given the above result we generalise it to all ordinals γ .

DEFINITION 2.7. Let κ be a cardinal and $\gamma \leq \kappa$ an ordinal. Then κ is **γ -Ramsey** if player I does not have a winning strategy in $\mathcal{G}_\gamma^\theta(\kappa)$ for all regular $\theta > \kappa$. We furthermore say that κ is **strategic γ -Ramsey** if player II *does* have a winning strategy in $\mathcal{G}_\gamma^\theta(\kappa)$ for all regular $\theta > \kappa$. Define **(strategic) genuine γ -Ramseys** and **(strategic) normal γ -Ramseys** analogously, but where we require the last measure μ_γ to be genuine and normal, respectively. —

DEFINITION 2.8 (N.). A cardinal κ is **$<\gamma$ -Ramsey** if it is α -Ramsey for every $\alpha < \gamma$, **almost fully Ramsey** if it is $<\kappa$ -Ramsey and **fully Ramsey** if it is κ -Ramsey. Further, say that κ is **coherent $<\gamma$ -Ramsey** if it's strategic α -Ramsey for every $\alpha < \gamma$ and that there exists a choice of winning strategies τ_α in $\mathcal{G}_\alpha(\kappa)$ for player II satisfying that $\tau_\alpha \subseteq \tau_\beta$ whenever $\alpha < \beta$. In other words, there is a single strategy τ for player II in $\mathcal{G}_\gamma(\kappa)$ such that τ is a winning strategy for player II in $\mathcal{G}_\alpha(\kappa)$ for every $\alpha < \gamma$.⁶ —

This is not the original definition of (strategic) γ -Ramsey cardinals however, as this involved elementary embeddings between weak κ -models – but as the following theorem of Holy and Schlicht (2018) shows, the two definitions coincide whenever γ is a regular cardinal.

THEOREM 2.9 (Holy-Schlicht). *For regular cardinals λ , a cardinal κ is λ -Ramsey iff for arbitrarily large $\theta > \kappa$ and every $A \subseteq \kappa$ there is a weak κ -model $\mathcal{M} < H_\theta$ with $\mathcal{M}^{<\lambda} \subseteq \mathcal{M}$ and $A \in \mathcal{M}$ with an \mathcal{M} -normal 1-good \mathcal{M} -measure μ on κ .* —

3 The finite case

In this section we are going to consider properties of the n -Ramsey cardinals for finite n . Note in particular that the $\mathcal{G}_n^\theta(\kappa)$ games are determined, making the “strategic” adjective superfluous in this case. We further note that the θ 's are also dispensable in this finite case:

PROPOSITION 3.1 (N.). *Let $\kappa < \theta$ be regular cardinals and $n < \omega$. Then player II has a winning strategy in $\mathcal{G}_n^\theta(\kappa)$ iff they have a winning strategy in the game $\mathcal{G}_n(\kappa)$, which is defined as $\mathcal{G}_n^\theta(\kappa)$ except that we don't require that $\mathcal{M}_n < H_\theta$.*

PROOF. \Leftarrow is clear, so assume that II has a winning strategy τ in $\mathcal{G}_n^\theta(\kappa)$. Whenever player I plays \mathcal{M}_k in $\mathcal{G}_n(\kappa)$ for $k \leq n$ then define $\mathcal{M}_k^* := \text{Hull}^{H_\theta}(\mathcal{P})$ where $\mathcal{P} \cong \mathcal{M}_k$ is the transitive collapse of \mathcal{M}_k , and play \mathcal{M}_k^* in $\mathcal{G}_n^\theta(\kappa)$. Let μ_k be the τ -responses to the \mathcal{M}_k^* 's and let player II play the μ_k 's in $\mathcal{G}_n(\kappa)$ as well.

⁶Note that, with this terminology, “coherent” is a stronger notion than “strategic”. We could've called the cardinals *coherent strategic $<\gamma$ -Ramseys*, but we opted for brevity instead.

Assume that this new strategy isn't winning for player II in $\mathcal{G}_n(\kappa)$, so that $\text{Ult}(\mathcal{M}_n, \mu_n)$ is illfounded. This is witnessed by some ω -sequence $\vec{f} := \langle f_k \mid k < \omega \rangle$ of $f_k \in {}^\kappa o(\mathcal{M}_n) \cap \mathcal{M}_n$ with $X_k := \{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_n$ for all $k < \omega$. Let $\nu \gg \kappa$, $\mathcal{H} := \text{cHull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$ be the transitive collapse of the Skolem hull $\text{Hull}^{H_\nu}(\mathcal{M}_n \cup \{\vec{f}, \mathcal{M}_n, \mu_n\})$, and $\pi : \mathcal{H} \rightarrow H_\nu$ be the uncollapse; write $\bar{x} := \pi^{-1}(x)$ for all $x \in \text{ran } \pi$.

Now $\bar{A} = A$ for every $A \in \mathcal{P}(\kappa) \cap \mathcal{M}_n$ and thus also $\bar{\mu}_n = \mu_n$. But now the \vec{f}_k 's witness that $\text{Ult}(\bar{\mathcal{M}}_n, \mu_n)$ is illfounded and thus also that $\text{Ult}(\mathcal{M}_n^*, \mu_n)$ is illfounded since $\mathcal{M}_n^* = \text{Hull}^{H_\theta}(\bar{\mathcal{M}}_n)$, contradicting that τ is winning. ■

For this reason we'll work with the $\mathcal{G}_n(\kappa)$ games throughout this section. Since we don't have to deal with the θ 's anymore we note that n -Ramseyness can now be described using a Π_{2n+2}^1 -formula and normal n -Ramseyness using a Π_{2n+3}^1 -formula.

We already have the following characterisations, as proven in Abramson et al. (1977).

THEOREM 3.2 (Abramson et al.). *Let $\kappa = \kappa^{<\kappa}$ be a cardinal. Then*

- (i) κ is weakly compact if and only if it is 0-Ramsey;
- (ii) κ is weakly ineffable if and only if it is genuine 0-Ramsey;
- (iii) κ is ineffable if and only if it is normal 0-Ramsey.

PROOF. This is mostly a matter of changing terminology from Abramson et al. (1977) to the current game-theoretic one, so we only show (i). Theorem 1.1.3 in Abramson et al. (1977) shows that κ is weakly compact if and only if every κ -sized collection of subsets of κ is measured by a $<\kappa$ -complete measure, in the sense that every $<\kappa$ -sequence (in V) of measure one sets has non-empty intersection.

For the \Rightarrow direction we can let player II respond to any \mathcal{M}_0 by first getting the $<\kappa$ -complete \mathcal{M}_0 -measure ν_0 on κ from the above-mentioned result, forming the (well-founded) ultrapower $\pi : \mathcal{M}_0 \rightarrow \text{Ult}(\mathcal{M}_0, \nu)$ and then playing the derived measure of π , which is \mathcal{M}_0 -normal and good. For \Leftarrow , if $X \subseteq \mathcal{P}(\kappa)$ has size κ then, using that $\kappa = \kappa^{<\kappa}$, we can find a κ -model $\mathcal{M}_0 < H_\theta$ with $X \subseteq \mathcal{M}_0$. Letting player I play \mathcal{M}_0 in $\mathcal{G}_0(\kappa)$ we get some \mathcal{M}_0 -normal good \mathcal{M}_0 -measure μ_0 on κ . Since \mathcal{M}_0 is closed under $<\kappa$ -sequences we get that μ_0 is $<\kappa$ -complete. ■

Indescribability

In this section we aim to prove that n -Ramseys are Π_{2n+1}^1 -indescribable and that normal n -Ramseys are Π_{2n+2}^1 -indescribable, which will also establish that the hierarchy of alternating n -Ramseys and normal n -Ramseys forms a strict hierarchy. Recall the following definition.

DEFINITION 3.3. A cardinal κ is Π_n^1 -**indescribable** if whenever $\varphi(v)$ is a Π_n formula, $X \subseteq V_\kappa$ and $V_{\kappa+1} \models \varphi[X]$, then there is an $\alpha < \kappa$ such that $V_{\alpha+1} \models \varphi[X \cap V_\alpha]$. ↯

Our first indescribability result is then the following, where the $n = 0$ case is inspired by the proof of weakly compact cardinals being Π_1^1 -indescribable — see Abramson et al. (1977).

THEOREM 3.4 (N). *Every n -Ramsey κ is Π_{2n+1}^1 -indescribable for $n < \omega$.*

PROOF. Let κ be n -Ramsey and assume that it is not Π_{2n+1}^1 -indescribable, witnessed by a Π_{2n+1} -formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$, meaning that $V_{\kappa+1} \models \varphi[X]$ and, for every $\alpha < \kappa$, $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$. We will deal with the $(2n+1)$ -many quantifiers occurring in φ in $(n+1)$ -many steps. We will here describe the first two steps with the remaining steps following the same pattern.

First step. Write $\varphi(v) \equiv \forall v_1 \psi(v, v_1)$ for a Σ_{2n} -formula $\psi(v, v_1)$. As we are assuming that $V_{\alpha+1} \models \neg\varphi[X \cap V_\alpha]$ holds for every $\alpha < \kappa$, we can pick witnesses $A_\alpha^{(0)} \subseteq V_\alpha$ to the outermost existential quantifier in $\neg\varphi[X \cap V_\alpha]$.

Let \mathcal{M}_0 be a weak κ -model such that $V_\kappa \subseteq \mathcal{M}_0$ and $\vec{A}^{(0)}, X \in \mathcal{M}_0$. Fix a good \mathcal{M}_0 -normal \mathcal{M}_0 -measure μ_0 on κ , using the 0-Ramseyess of κ . Form $\mathcal{A}^{(0)} := [\vec{A}^{(0)}]_{\mu_0} \in \text{Ult}(\mathcal{M}_0, \mu_0)$, where we without loss of generality may assume that the ultrapower is transitive. \mathcal{M}_0 -normality of μ_0 implies that $\mathcal{A}^{(0)} \subseteq V_\kappa$, so that we have that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$. Now Łoś' Lemma, \mathcal{M}_0 -normality of μ_0 and $V_\kappa \subseteq \mathcal{M}_0$ also ensures that

$$\text{Ult}(\mathcal{M}_0, \mu_0) \models \ulcorner V_{\kappa+1} \models \neg\psi[X, \mathcal{A}^{(0)}] \urcorner. \quad (1)$$

This finishes the first step. Note that if $n = 0$ then $\neg\psi$ would be a Δ_0 -formula, so that (1) would be absolute to the true $V_{\kappa+1}$, yielding a contradiction. If $n > 0$ we cannot yet conclude this however, but that is what we are aiming for in the remaining steps.

Second step. Write $\psi(v, v_1) \equiv \exists v_2 \forall v_3 \chi(v, v_1, v_2, v_3)$ for a $\Sigma_{2(n-1)}$ -formula $\chi(v, v_1, v_2, v_3)$. Since we have established that $V_{\kappa+1} \models \psi[X, \mathcal{A}^{(0)}]$ we can pick some $B^{(0)} \subseteq V_\kappa$ such that

$$V_{\kappa+1} \models \forall v_3 \chi[X, \mathcal{A}^{(0)}, B^{(0)}, v_3] \quad (2)$$

which then also means that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \exists v_3 \neg\chi[X \cap V_\alpha, A_\alpha^{(0)}, B^{(0)} \cap V_\alpha, v_3]. \quad (3)$$

Fix witnesses $A_\alpha^{(1)} \subseteq V_\alpha$ to the existential quantifier in (3) and define the sets

$$S_\alpha^{(0)} := \{\xi < \kappa \mid A_\xi^{(0)} \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha\}$$

for every $\alpha < \kappa$ and note that $S_\alpha^{(0)} \in \mu_0$ for every $\alpha < \kappa$, since $V_\kappa \subseteq \mathcal{M}_0$ ensures that $\mathcal{A}^{(0)} \cap V_\alpha \in \mathcal{M}_0$ and \mathcal{M}_0 -normality of μ_0 then implies that $S_\alpha^{(0)} \in \mu_0$ is equivalent to

$$\text{Ult}(\mathcal{M}_0, \mu_0) \models \mathcal{A}^{(0)} \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha,$$

which is clearly the case. Now let $\mathcal{M}_1 \supseteq \mathcal{M}_0$ be a weak κ -model such that $\mathcal{A}^{(0)}, \vec{A}^{(1)}, \vec{S}^{(0)}, B^{(0)} \in \mathcal{M}_1$. Let $\mu_1 \supseteq \mu_0$ be an \mathcal{M}_1 -normal \mathcal{M}_1 -measure on κ , using the 1-Ramseyness of κ , so that \mathcal{M}_1 -normality of μ_1 yields that $\triangle \vec{S}^{(0)} \in \mu_1$. Observe that $\xi \in \triangle \vec{S}^{(0)}$ if and only if $A_\xi^{(0)} \cap V_\alpha = \mathcal{A}^{(0)} \cap V_\alpha$ for every $\alpha < \xi$, so if ξ is a limit ordinal then it holds that $A_\xi^{(0)} = \mathcal{A}^{(0)} \cap V_\xi$. Now, as before, form $\mathcal{A}^{(1)} := [\vec{A}^{(1)}]_{\mu_1} \in \text{Ult}(\mathcal{M}_1, \mu_1)$, so that (2) implies that

$$V_{\kappa+1} \models \chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}]$$

and the definition of the $A_\alpha^{(1)}$'s along with (3) gives that, for every $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg \chi[X \cap V_\alpha, A_\alpha^{(0)}, B^{(0)} \cap V_\alpha, A_\alpha^{(1)}].$$

Now this, paired with the above observation regarding $\triangle \vec{S}^{(0)}$, means that for every $\alpha \in \triangle \vec{S}^{(0)} \cap \text{Lim}$ we have that

$$V_{\alpha+1} \models \neg \chi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, A_\alpha^{(1)}],$$

so that \mathcal{M}_1 -normality of μ_1 and Łoś' lemma implies that

$$\text{Ult}(\mathcal{M}_1, \mu_1) \models \ulcorner V_{\kappa+1} \models \neg \chi[X, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}] \urcorner.$$

This finishes the second step. Continue in this way for a total of $(n+1)$ -many steps, ending with a Δ_0 -formula $\phi(v, v_1, \dots, v_{2n+1})$ such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \tag{4}$$

and that $\text{Ult}(\mathcal{M}_n, \mu_n) \models \ulcorner V_{\kappa+1} \models \neg \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n)}] \urcorner$. But now absoluteness of $\neg \phi$ means that $V_{\kappa+1} \models \neg \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n)}]$, contradicting (4). \blacksquare

Note that this is optimal, as n -Ramseyness can be described by a Π_{2n+2}^1 -formula. As a corollary we then immediately get the following.

COROLLARY 3.5 (N.). *Every $<\omega$ -Ramsey cardinal is Δ_0^2 -indescribable.* \dashv

The second indescribability result concerns the normal n -Ramseys, where the $n = 0$ case here is inspired by the proof of ineffable cardinals being Π_2^1 -indescribable — see Abramson et al. (1977).

THEOREM 3.6 (N.). *Every normal n -Ramsey κ is Π_{2n+2}^1 -indescribable for $n < \omega$.*

Before we commence with the proof, note that we cannot simply do the same thing as we did in the proof of Theorem 3.4, as we would end up with a Π_1^1 statement in an ultrapower, and as Π_1^1 statements are not

upwards absolute in general we would not be able to get our contradiction.

PROOF. Let κ be normal n -Ramsey and assume that it is not Π_{2n+2}^1 -indescribable, witnessed by a Π_{2n+2} -formula $\varphi(v)$ and a subset $X \subseteq V_\kappa$. Use that κ is n -Ramsey to perform the same $n+1$ steps as in the proof of Theorem 3.4. This gives us a Σ_1 -formula $\phi(v, v_1, \dots, v_{2n+1})$ along with sequences $\langle \mathcal{A}^{(0)}, \dots, \mathcal{A}^{(n)} \rangle$, $\langle B^{(0)}, \dots, B^{(n-1)} \rangle$ and a play $\langle \mathcal{M}_k, \mu_k \mid k \leq n \rangle$ of $\mathcal{G}_n(\kappa)$ in which player II wins and μ_n is normal, such that

$$V_{\kappa+1} \models \phi[X, \mathcal{A}^{(0)}, B^{(0)}, \dots, \mathcal{A}^{(n-1)}, B^{(n-1)}, \mathcal{A}^{(n)}] \quad (1)$$

and, for μ_n -many $\alpha < \kappa$,

$$V_{\alpha+1} \models \neg \phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \dots, \mathcal{A}^{(n-1)} \cap V_\alpha, B^{(n-1)} \cap V_\alpha, \mathcal{A}_\alpha^{(n)}].$$

Now form $S_\alpha^{(n)} \in \mu_n$ as in the proof of Theorem 3.4. The main difference now is that we do not know if $\vec{S}^{(n)} \in \mathcal{M}_n$ (in the proof of Theorem 3.4 we only ensured that $\vec{S}^{(k)} \in \mathcal{M}_{k+1}$ for every $k < n$ and we only defined $\vec{S}^{(k)}$ for $k < n$), but we can now use normality⁷ of μ_n to ensure that we *do* have that $\triangle \vec{S}^{(n)}$ is stationary in κ . This means that we get a stationary set $S \subseteq \kappa$ such that for every $\alpha \in S$ it holds that

$$V_{\alpha+1} \models \neg \phi[X \cap V_\alpha, \mathcal{A}^{(0)} \cap V_\alpha, B^{(0)} \cap V_\alpha, \dots, B^{(n-1)} \cap V_\alpha, \mathcal{A}^{(n)} \cap V_\alpha]. \quad (2)$$

Now note that since κ is inaccessible it is Σ_1^1 -indescribable, meaning that we can reflect (1). Furthermore, Lemma 3.4.3 of Abramson et al. (1977) shows that the set of reflection points of Σ_1^1 -formulas is in fact club, so intersecting this club with S we get a $\zeta \in S$ satisfying that

$$V_{\zeta+1} \models \phi[X \cap V_\zeta, \mathcal{A}^{(0)} \cap V_\zeta, B^{(0)} \cap V_\zeta, \dots, B^{(n-1)} \cap V_\zeta, \mathcal{A}^{(n)} \cap V_\zeta],$$

contradicting (2). ■

Note that this is optimal as well, since normal n -Ramsey can be described by a Π_{2n+3}^1 -formula. In particular this then means that every $(n+1)$ -Ramsey is a normal n -Ramsey stationary limit of normal n -Ramseys, and every normal n -Ramsey is an n -Ramsey stationary limit of n -Ramseys, making the hierarchy of alternating n -Ramseys and normal n -Ramseys a strict hierarchy.

Downwards absoluteness to L

The following proof is basically the proof of Theorem 4.1.1 in Abramson et al. (1977).

⁷Recall that this is stronger than just requiring it to be \mathcal{M}_n -normal — we don't require $\vec{S}^{(n)} \in \mathcal{M}_n$.

THEOREM 3.7 (N.). *Genuine- and normal n -Ramseys are downwards absolute to L , for every $n < \omega$.*

PROOF. Assume first that $n = 0$ and that κ is a genuine 0-Ramsey cardinal. Let $\mathcal{M} \in L$ be a weak κ -model — we want to find a genuine \mathcal{M} -measure inside L . By assumption we *can* find such a measure μ in V ; we will show that in fact $\mu \in L$. Fix any enumeration $\langle A_\xi \mid \xi < \kappa \rangle \in L$ of $\mathcal{P}(\kappa) \cap \mathcal{M}$. It then clearly suffices to show that $T \in L$, where $T := \{\alpha < \kappa \mid A_\xi \in \mu\}$.

Claim 3.7.1. $T \cap \alpha \in L$ for any $\alpha < \kappa$.

PROOF OF CLAIM. Let \vec{B} be the μ -positive part of \vec{A} , meaning that $B_\xi := A_\xi$ if $A_\xi \in \mu$ and $B_\xi := \neg A_\xi$ if $A_\xi \notin \mu$. As μ is genuine we get that $\triangle \vec{B}$ has size κ , so we can pick $\delta \in \triangle \vec{B}$ with $\delta > \alpha$. Then $T \cap \alpha = \{\xi < \alpha \mid \delta \in A_\xi\}$, which can be constructed within L . \dashv

But now Lemma 4.1.2 in Abramson et al. (1977) shows that there is a Π_1 formula $\varphi(v)$ such that, given any non-zero ordinal ζ , $V_{\zeta+1} \models \varphi[A]$ if and only if ζ is a regular cardinal and A is a non-constructible subset of ζ . If we therefore assume that $T \notin L$ then $V_{\kappa+1} \models \varphi[T]$, which by Π_1^1 -indescribability of κ means that there exists some $\alpha < \kappa$ such that $V_{\alpha+1} \models \varphi[T \cap V_\alpha]$, i.e. that $T \cap \alpha \notin L$, contradicting the claim. Therefore $\mu \in L$. It is still genuine in L as $(\triangle \mu)^L = \triangle \mu$, and if μ was normal then that is still true in L as clubs in L are still clubs in V . The cases where κ is a genuine- or normal n -Ramsey cardinal is analogous. ■

Since $(n+1)$ -Ramseys are normal n -Ramseys we then immediately get the following.

COROLLARY 3.8 (N.). *Every $(n+1)$ -Ramsey is normal n -Ramsey in L , for every $n < \omega$. In particular, $<\omega$ -Ramseys are downwards absolute to L .* \dashv

Complete ineffability

In this section we provide a characterisation of the *completely ineffable* cardinals in terms of the α -Ramseys. To arrive at such a characterisation, we need a slight strengthening of the $<\omega$ -Ramsey cardinals, namely the *coherent $<\omega$ -Ramseys* as defined in 2.8. Note that a coherent $<\omega$ -Ramsey is precisely a cardinal satisfying the ω -filter property, as defined in Holy and Schlicht (2018).

The following theorem shows that assuming coherency does yield a strictly stronger large cardinal notion. The idea of its proof is very closely related to the proof of Theorem 3.6 (the indescribability of normal n -Ramseys), but the main difference is that we want everything to occur locally inside our weak κ -models.

THEOREM 3.9 (N.). *Every coherent $<\omega$ -Ramsey is a stationary limit of $<\omega$ -Ramseys.*

PROOF. Let κ be coherent $<\omega$ -Ramsey. Let $\theta \gg \kappa$ be regular and let $\mathcal{M}_0 < H_\theta$ be a weak κ -model with $V_\kappa \subseteq \mathcal{M}_0$. Let then player I play arbitrarily while player II plays according to her coherent winning strategies in $\mathcal{G}_n(\kappa)$, yielding a weak κ -model $\mathcal{M} < H_\theta$ with an \mathcal{M} -normal \mathcal{M} -measure $\mu := \bigcup_{n < \omega} \mu_n$ on κ .

Assume towards a contradiction that $X := \{\xi < \kappa \mid \xi \text{ is } <\omega\text{-Ramsey}\} \notin \mu$. Since $X = \bigcap \vec{X}$ and $\vec{X} \in \mathcal{M}$, where $X_n := \{\xi < \kappa \mid \xi \text{ is } n\text{-Ramsey}\}$, we must have by \mathcal{M} -normality of μ that $\neg X_k \in \mu$ for some $k < \omega$. Note that $\neg X_k \in \mathcal{M}_0$ by elementarity, so that $\neg X_k \in \mu_0$ as well. Perform the $k + 1$ steps as in the proof of Theorem 3.6 with $\varphi(\xi)$ being ' ξ is k -Ramsey', so that we get a weak κ -model $\mathcal{M}_{k+1} < H_\theta$, an \mathcal{M}_{k+1} -normal \mathcal{M}_{k+1} -measure $\tilde{\mu}_{k+1}$ on κ , a Σ_1 -formula $\varphi(v, v_1, v_2, \dots, v_{2k+1})$ and sequences $\langle \mathcal{A}^{(0)}, \dots, \mathcal{A}^{(k)} \rangle$ and $\langle B^{(0)}, \dots, B^{(k-1)} \rangle$ such that

$$V_{\kappa+1} \models \varphi[\kappa, \mathcal{A}^{(0)}, B^{(0)}, \mathcal{A}^{(1)}, B^{(1)}, \dots, \mathcal{A}^{(k-1)}, B^{(k-1)}, \mathcal{A}^{(k)}] \quad (2)$$

and there is a $Y \in \tilde{\mu}_{k+1}$ with $Y \subseteq \neg X_k$ such that given any $\xi \in Y$,

$$V_{\xi+1} \models \neg \varphi[\xi, A_\xi^{(0)}, B^{(0)} \cap V_\xi, A_\xi^{(1)}, B^{(1)} \cap V_\xi, \dots, A_\xi^{(k-1)}, B^{(k-1)} \cap V_\xi, A_\xi^{(k)}], \quad (3)$$

where $\mathcal{A}^{(i)} = [\vec{A}^{(i)}]_{\mu_i} \in \text{Ult}(\mathcal{M}_i, \mu_i)$ as in the proof of Theorem 3.4.

Since κ in particular is Σ_1^1 -indescribable, Lemma 3.4.3 of Abramson et al. (1977) implies that we get a club $C \subseteq \kappa$ of reflection points of (2). Let $\mathcal{M}_{k+2} \supseteq \mathcal{M}_{k+1}$ be a weak κ -model with $\mathcal{A}^{(k)} \in \mathcal{M}_{k+2}$, where the above $(n + 1)$ -steps ensured that the $B^{(i)}$'s and the remaining $\mathcal{A}^{(i)}$'s are all elements of \mathcal{M}_{k+1} . In particular, as C is a definable subset in the $\mathcal{A}^{(i)}$'s and $B^{(i)}$'s we also get that $C \in \mathcal{M}_{k+2}$. Letting $\tilde{\mu}_{k+2}$ be the associated measure on κ , \mathcal{M}_{k+2} -normality of $\tilde{\mu}_{k+2}$ ensures that $C \in \tilde{\mu}_{k+2}$. Now define, for every $\alpha < \kappa$,

$$S_\alpha := \{\xi \in Y \mid \forall i \leq k : \mathcal{A}^{(i)} \cap V_\alpha = A_\xi^{(i)} \cap V_\alpha\}$$

and note that $S_\alpha \in \tilde{\mu}_{k+2}$ for every $\alpha < \kappa$. Write $\vec{S} := \langle S_\alpha \mid \alpha < \kappa \rangle$ and note that since \vec{S} is definable it is an element of \mathcal{M}_{k+2} as well. Then \mathcal{M}_{k+2} -normality of $\tilde{\mu}_{k+2}$ ensures that $\Delta \vec{S} \in \tilde{\mu}_{k+2}$, so that $C \cap \Delta \vec{S} \in \tilde{\mu}_{k+2}$ as well. But letting $\zeta \in C \cap \Delta \vec{S}$ we see, as in the proof of Theorem 3.4, that

$$V_{\zeta+1} \models \varphi[\zeta, A_\zeta^{(0)}, B^{(0)} \cap V_\zeta, A_\zeta^{(1)}, B^{(1)} \cap V_\zeta, \dots, A_\zeta^{(k)}]$$

since $\Delta \vec{S} \subseteq Y$, contradicting (3). Hence $X \in \mu$, and since $\mathcal{M} < H_\theta$ we have that \mathcal{M} is correct about stationary subsets of κ , meaning that κ is a stationary limit of $<\omega$ -Ramseys. \blacksquare

Now, having established the strength of this large cardinal notion, we move towards complete ineffability. We recall the following definitions.

DEFINITION 3.10. A collection $R \subseteq \mathcal{P}(\kappa)$ is a **stationary class** if

- (i) $R \neq \emptyset$;
- (ii) every $A \in R$ is stationary in κ ;
- (iii) if $A \in R$ and $B \supseteq A$ then $B \in R$.

—

DEFINITION 3.11. A cardinal κ is **completely ineffable** if there is a stationary class R such that for every $A \in R$ and $f : [A]^2 \rightarrow 2$ there is an $H \in R$ homogeneous for f . \dashv

We then arrive at the following characterisation, influenced by the proof of Theorem 1.3.4 in Abramson et al. (1977).

THEOREM 3.12 (N). A cardinal κ is completely ineffable if and only if it is coherent $<\omega$ -Ramsey.

PROOF. (\Leftarrow): Assume κ is coherent $<\omega$ -Ramsey, witnessed by strategies $\langle \tau_n \mid n < \omega \rangle$. Let $f : [\kappa]^2 \rightarrow 2$ be arbitrary and form the sequence $\langle A_\alpha^f \mid \alpha < \kappa \rangle$ as

$$A_\alpha^f := \{\beta > \alpha \mid f(\{\alpha, \beta\}) = 0\}.$$

Let \mathcal{M}_f be a transitive weak κ -model with $\vec{A}^f \in \mathcal{M}_f$, and let μ_f be the associated \mathcal{M}_f -measure on κ given by τ_0 .⁸ 1-Ramseyess of κ ensures that μ_f is normal, meaning $\Delta\mu_f$ is stationary in κ . Define a new sequence \vec{B}^f as the μ_f -positive part of \vec{A}^f .⁹ Then $B_\alpha^f \in \mu_f$ for all $\alpha < \kappa$, so that normality of μ_f implies that $\Delta\vec{B}^f$ is stationary.

Let now \mathcal{M}'_f be a new transitive weak κ -model with $\mathcal{M}_f \subseteq \mathcal{M}'_f$ and $\mu_f \in \mathcal{M}'_f$, and use τ_1 to get an \mathcal{M}'_f -measure $\mu'_f \supseteq \mu_f$ on κ . Then $\Delta\vec{B}^f \cap \{\xi < \kappa \mid A_\xi^f \in \mu_f\}$ and $\Delta\vec{B}^f \cap \{\xi < \kappa \mid A_\xi^f \notin \mu_f\}$ are both elements of \mathcal{M}'_f , so one of them is in μ'_f ; set H_f to be that one. Note that H_f is now both stationary in κ and homogeneous for f .

Now let $g : [H_f]^2 \rightarrow 2$ be arbitrary and again form

$$A_\alpha^g := \{\beta \in H_f \mid \beta > \alpha \wedge g(\{\alpha, \beta\}) = 0\}$$

for $\alpha \in H_f$. Let $\mathcal{M}_{f,g} \supseteq \mathcal{M}'_f$ be a transitive weak κ -model with $\vec{A}^g \in \mathcal{M}_{f,g}$ and use τ_2 to get an $\mathcal{M}_{f,g}$ -measure $\mu_{f,g} \supseteq \mu'_f$ on κ . As before we then get a stationary $H_{f,g} \in \mu'_{f,g}$ which is homogeneous for g . We can continue in this fashion since $\tau_n \subseteq \tau_{n+1}$ for all $n < \omega$. Define then

$$R := \{A \subseteq \kappa \mid \exists \vec{f} : H_{\vec{f}} \subseteq A\},$$

where the \vec{f} 's range over finite sequences of functions as above; i.e. $f_0 : [\kappa]^2 \rightarrow 2$ and $f_{k+1} : [H_{f_k}]^2 \rightarrow 2$ for $k < \omega$. This is clearly a stationary class which satisfies that whenever $A \in R$ and $g : [A]^2 \rightarrow 2$, we can find $H \in R$ which is homogeneous for f . Indeed, if we let \vec{f} be such that $H_{\vec{f}} \subseteq A$, which exists as $A \in R$, then we can simply let $H := H_{\vec{f},g}$. This shows that κ is completely ineffable.

(\Rightarrow): Now assume that κ is completely ineffable and let R be the corresponding stationary class. We show that κ is n -Ramsey for all $n < \omega$ by induction, where we inductively make sure that the resulting strategies are coherent as well. Let player I in $\mathcal{G}_0(\kappa)$ play \mathcal{M}_0 and enumerate $\mathcal{P}(\kappa) \cap \mathcal{M}_0$ as $\vec{A}^0 \langle A_\alpha^0 \mid \alpha < \kappa \rangle$ such

⁸Technically we would have to require that $\mathcal{M}_f < H_\theta$ for some regular $\theta > \kappa$ to be able to use τ_0 , but note that we could simply get a measure on $\text{Hull}^{H_\theta}(\mathcal{M}_f)$ and restrict it to \mathcal{M}_f . We will use this throughout the proof.

⁹The μ -positive part was defined in Claim 3.7.1.

that $A_\xi^0 \subseteq A_\zeta^0$ implies $\xi \leq \zeta$. For $\alpha < \kappa$ define sequences $r_\alpha : \alpha \rightarrow 2$ as $r_\alpha(\xi) = 1$ iff $\alpha \in A_\xi^0$. Let $<_{\text{lex}}^\alpha$ be the lexicographical ordering on ${}^\alpha 2$. Define now a colouring $f : [\kappa]^2 \rightarrow 2$ as

$$f(\{\alpha, \beta\}) := \begin{cases} 0 & \text{if } r_{\min(\alpha, \beta)} <_{\text{lex}}^{\min(\alpha, \beta)} r_{\max(\alpha, \beta)} \upharpoonright \min(\alpha, \beta) \\ 1 & \text{otherwise} \end{cases}$$

Let $H_0 \in R$ be homogeneous for f , using that κ is completely ineffable. For $\alpha < \kappa$ consider now the sequence $\langle r_\xi \upharpoonright \alpha \mid \xi \in H_0 \wedge \xi > \alpha \rangle$, which is of length κ so there is an $\eta \in [\alpha, \kappa)$ satisfying that $r_\beta \upharpoonright \alpha = r_\gamma \upharpoonright \alpha$ for every $\beta, \gamma \in H_0$ with $\eta \leq \beta < \gamma$. Define $g : \kappa \rightarrow \kappa$ as $g(\alpha)$ being the least such η , which is then a continuous non-decreasing cofinal function, making the set of fixed points of g club in κ – call this club C .

Since H_0 is stationary we can pick some $\zeta \in C \cap H_0$. As $\zeta \in C$ we get $g(\zeta) = \zeta$, meaning that $r_\beta \upharpoonright \zeta = r_\gamma \upharpoonright \zeta$ holds for every $\beta, \gamma \in H_0$ with $\zeta \leq \beta < \gamma$. As ζ is also a member of H_0 we can let $\beta := \zeta$, so that $r_\zeta = r_\gamma \upharpoonright \zeta$ holds for every $\gamma \in H_0, \gamma > \zeta$. Now, by definition of r_α we get that for every $\alpha, \gamma \in H_0 \cap C$ with $\alpha \leq \gamma$ and $\xi < \alpha, \alpha \in A_\xi^0$ iff $\gamma \in A_\xi^0$. Define thus the \mathcal{M}_0 -measure μ_0 on κ as

$$\begin{aligned} \mu_0(A_\xi^0) = 1 & \quad \text{iff} \quad (\forall \beta \in H_0 \cap C)(\beta > \xi \rightarrow \beta \in A_\xi^0) \\ & \quad \text{iff} \quad (\exists \beta \in H_0 \cap C)(\beta > \xi \wedge \beta \in A_\xi^0), \end{aligned}$$

where the last equivalence is due to the above-mentioned property of $H_0 \cap C$. Note that the choice of enumeration implies that μ_0 is indeed a filter. Letting $\vec{B} = \langle B_\alpha \mid \alpha < \kappa \rangle$ be the μ_0 -positive part of \vec{A}^0 , it is also simple to check that $H_0 \cap C \subseteq \triangle \vec{B}$, making μ_0 normal and hence also both \mathcal{M}_0 -normal and good, showing that κ is 0-Ramsey.

Assume now that κ is n -Ramsey and let $\langle \mathcal{M}_0, \mu_0, \dots, \mathcal{M}_n, \mu_n, \mathcal{M}_{n+1} \rangle$ be a partial play of $\mathcal{G}_{n+1}(\kappa)$. Again enumerate $\mathcal{P}(\kappa) \cap \mathcal{M}_{n+1}$ as $\vec{A}^{n+1} = \langle A_\xi^{n+1} \mid \xi < \kappa \rangle$, again satisfying that $\xi \leq \zeta$ whenever $A_\xi^{n+1} \subseteq A_\zeta^{n+1}$, but also such that given any $\xi < \kappa$ there are $\zeta, \zeta' \in (\xi, \kappa)$ satisfying that $A_\zeta^{n+1} \in \mathcal{P}(\kappa) \cap \mathcal{M}_n$ and $A_{\zeta'}^{n+1} \in (\mathcal{P}(\kappa) \cap \mathcal{M}_{n+1}) - \mathcal{M}_n$. The plan now is to do the same thing as before, but we also have to check that the resulting measure extends the previous ones.

Let $H_n \in R$ and C be club in κ such that $H_n \cap C \subseteq \triangle \mu_n$, which exist by our inductive assumption. For $\alpha < \kappa$ define $r_\alpha : \alpha \rightarrow 2$ as $r_\alpha(\xi) = 1$ iff $\alpha \in A_\xi^{n+1}$, and define a colouring $f : [H_n]^2 \rightarrow 2$ as

$$f(\{\alpha, \beta\}) := \begin{cases} 0 & \text{if } r_{\min(\alpha, \beta)} <_{\text{lex}}^{\min(\alpha, \beta)} r_{\max(\alpha, \beta)} \upharpoonright \min(\alpha, \beta) \\ 1 & \text{otherwise} \end{cases}$$

As $H_n \in R$ there is an $H_{n+1} \in R$ homogeneous for f . Just as before, define $g : \kappa \rightarrow \kappa$ as $g(\alpha)$ being the least $\eta \in [\alpha, \kappa)$ such that $r_\beta \upharpoonright \alpha = r_\gamma \upharpoonright \alpha$ for every $\beta, \gamma \in H_{n+1}$ with $\eta \leq \beta < \gamma$, and let D be the club of fixed points of g . As above we get that given any $\alpha, \gamma \in H_{n+1} \cap D$ with $\alpha \leq \gamma$ and $\xi < \alpha, \alpha \in A_\xi^{n+1}$ iff $\gamma \in A_\xi^{n+1}$. Define then the \mathcal{M}_{n+1} -measure μ_{n+1} on κ as

$$\begin{aligned} \mu_{n+1}(A_\xi^{n+1}) = 1 & \quad \text{iff} \quad (\forall \beta \in H_{n+1} \cap D \cap C)(\beta > \xi \rightarrow \beta \in A_\xi^{n+1}) \\ & \quad \text{iff} \quad (\exists \beta \in H_{n+1} \cap D \cap C)(\beta > \xi \wedge \beta \in A_\xi^{n+1}). \end{aligned}$$

Then $H_{n+1} \cap D \cap C \subseteq \triangle \mu_{n+1}$, making μ_{n+1} normal, \mathcal{M}_{n+1} -normal and good, just as before. It remains to show that $\mu_n \subseteq \mu_{n+1}$. Let thus $A \in \mu_n$ be given, and say $A = A_\xi^{n+1} = A_\eta^n$, where \vec{A}^n was the enumeration of $\mathcal{P}(\kappa) \cap \mathcal{M}_n$ used at the n 'th stage. Then by definition of μ_n we get that for every $\beta \in H_n \cap C$ with $\beta > \eta$, $\beta \in A_\eta^n$. We need to show that

$$(\exists \beta \in H_{n+1} \cap D \cap C)(\beta > \xi \wedge \beta \in A_\xi^{n+1})$$

holds. But here we can simply pick a $\beta > \max(\xi, \eta)$ with $\beta \in H_{n+1} \cap D \cap C \subseteq H_n \cap C$. This shows that $\mu_n \subseteq \mu_{n+1}$, making κ $(n+1)$ -Ramsey and thus inductively also coherent $<\omega$ -Ramsey. \blacksquare

4 The countable case

This section covers the (strategic) γ -Ramsey cardinals whenever γ has countable cofinality. This case is special because, as mentioned in Section 2, we cannot ensure that the final measure is countably complete and so the existence of winning strategies in the $\mathcal{G}_\gamma^\theta(\kappa)$ *might* depend on θ , in contrast with the uncountable cofinality case; see e.g. Question 6.3.

[Strategic] ω -Ramsey cardinals

We now move to the strategic ω -Ramsey cardinals and their relationship to the (non-strategic) ω -Ramseys. For this we define a new addition to the family of *virtual cardinals* from Gitman and Schindler (2015), the *virtually measurable cardinals*.

DEFINITION 4.1. A cardinal κ is **virtually measurable** if for every regular $\nu > \kappa$ there exists a transitive M and a forcing \mathbb{P} such that, in $V^\mathbb{P}$, there exists an elementary embedding $j : H_\nu^V \rightarrow M$ with $\text{crit } j = \kappa$. \dashv

We'll need the following well-known lemmata; see Lemma 7.1 in Holy and Schlicht (2018) and Lemma 3.1 in Gitman and Schindler (2015) for their proofs.

LEMMA 4.2 (Ancient Kunen Lemma). *Let $M \models \text{ZFC}^-$ and $j : M \rightarrow N$ an elementary embedding with critical point κ such that $\kappa + 1 \subseteq M \subseteq N$. Assume that $X \in M$ has M -cardinality κ . Then $j \restriction X \in N$. \dashv*

LEMMA 4.3 (Absoluteness of embeddings on countable structures). *Let M be a countable first-order structure and $j : M \rightarrow N$ an elementary embedding. If W is a transitive (set or class) model of (some sufficiently large fragment of) ZFC such that M is countable in W and $N \in W$, then for any finite subset of M , W has some elementary embedding $j^* : M \rightarrow N$, which agrees with j on that subset. Moreover, if both M and N are transitive \in -structures and j has a critical point, we can also assume that $\text{crit}(j^*) = \text{crit}(j)$. \dashv*

THEOREM 4.4 (Schindler-N.). *Every virtually measurable cardinal is strategic ω -Ramsey, and every strategic ω -Ramsey cardinal is virtually measurable in L .*

PROOF. Let κ be virtually measurable and fix a regular $\nu > \kappa$, a transitive M , a poset \mathbb{P} and, in $V^\mathbb{P}$, an elementary embedding $\pi : H_\nu^V \rightarrow M$ with $\text{crit } \pi = \kappa$. Fix a name $\dot{\mu}$ and a \mathbb{P} -condition p such that¹⁰

$$p \Vdash \dot{\mu} \text{ is a 1-good } \check{H}_\nu\text{-normal } \check{H}_\nu\text{-measure}^1$$

We now define a strategy σ for player II in $\mathcal{G}_\omega^\nu(\kappa)$ as follows. Whenever player I plays a weak κ -model $M_n < H_\nu^V$, player II fixes $p_n \in \mathbb{P}$, an M_n -measure μ_n and a function $\pi_n : M_n \rightarrow V$ such that $p_0 \leq p$, $p_n \leq p_k$ for every $k \leq n$ and that

$$p_n \Vdash \dot{\mu} \cap \check{M}_n = \check{\mu}_n \wedge \check{\pi}_n = \dot{\pi} \restriction \check{M}_n^1. \quad (1)$$

Note that by the Ancient Kunen Lemma 4.2 we get that $\pi \restriction M_n \in M \subseteq V$, so such π_n always exist in V . The μ_n 's also always exist in V , by weak amenability of μ . Player II responds to M_n with μ_n . It's clear that the μ_n 's are legal moves for player II, so it remains to show that $\mu_\omega := \bigcup_{n < \omega} \mu_n$ is good. Assume it's not, so that we have a sequence $\langle g_n \mid n < \omega \rangle$ of functions $g_n : \kappa \rightarrow M_\omega := \bigcup_{n < \omega} M_n$ such that $g_n \in M_\omega$ and

$$X_{n+1} := \{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu_\omega \quad (2)$$

for every $n < \omega$. Without loss of generality we can assume that $g_n, X_n \in M_n$. Then (2) implies that $p_{n+1} \Vdash \dot{\pi}(\check{g}_{n+1})(\check{\kappa}) < \dot{\pi}(\check{g}_n)(\check{\kappa})^1$, but by (1) this also means that

$$p_{n+1} \Vdash \check{\pi}_{n+1}(\check{g}_{n+1})(\check{\kappa}) < \check{\pi}_n(\check{g}_n)(\check{\kappa})^1, \quad (3)$$

so defining, in V , the ordinals $\alpha_n := \pi_n(g_n)(\kappa)$, (3) implies that $\alpha_{n+1} < \alpha_n$ for all $n < \omega$, \downarrow . So μ_ω is good, making σ a winning strategy and thus also making κ strategic ω -Ramsey since ν was arbitrary.

Next, let κ be strategic ω -Ramsey and fix a winning strategy σ for player II in $\mathcal{G}_\omega^\nu(\kappa)$ for a regular $\nu > \kappa$. Let $g \subseteq \text{Col}(\omega, H_\nu^L)$ be V -generic and in $V[g]$ fix an elementary chain $\langle L_{\kappa_n} \mid n < \omega \rangle$ of weak κ -models $L_{\kappa_n} < H_\nu^L$ such that $H_\nu^L \subseteq \bigcup_{n < \omega} L_{\kappa_n}$, using that ν is regular and has countable cofinality in $V[g]$. Player II follows σ , resulting in a H_ν^L -normal H_ν^L -measure μ on κ .

Claim 4.4.1. $\text{Ult}(H_\nu^L, \mu)$ is well-founded.

PROOF OF CLAIM. Assume for a contradiction that $\text{Ult}(H_\nu^L, \mu)$ is illfounded, witnessed by a sequence $\langle g_n \mid n < \omega \rangle$ of functions $g_n : \kappa \rightarrow \nu$ such that $g_n \in H_\nu^L$ and $\{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu$. Now, in V , define a tree \mathcal{T} of triples (f, M_f, μ_f) such that $f : \kappa \rightarrow \nu$, M_f is a weak κ -model, μ_f is an M_f -measure on κ and letting $f_0 <_{\mathcal{T}} \dots <_{\mathcal{T}} f_n = f$ be the \mathcal{T} -predecessors of f ,

- $\langle M_{f_0}, \mu_{f_0}, \dots, M_{f_n}, \mu_{f_n} \rangle$ is a partial play of $\mathcal{G}_\omega^\nu(\kappa)$ in which player II follows σ ; and

¹⁰Recall that an M -measure μ is 1-good if it's weakly amenable and $\text{Ult}(M, \mu)$ is well-founded.

- $\{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_{k+1}$ for every $k < n$.

Now, the g_n 's induce a cofinal branch through \mathcal{T} in $V[g]$, so by absoluteness of well-foundedness there's a cofinal branch b through \mathcal{T} in V as well. But b now gives us a play of $\mathcal{G}_\omega^\nu(\kappa)$ where player II is following σ but player I wins, a contradiction. Thus $\text{Ult}(H_\nu^L, \mu)$ is well-founded. \dashv

Let $j : H_\nu^L \rightarrow \text{Ult}(H_\nu^L, \mu) \cong M$ be the ultrapower embedding followed by the transitive collapse, so that $M = L_\alpha$ for some α by elementarity. Let now $h \subseteq \text{Col}(\omega, \kappa^{+L})^L$ be L -generic, so that H_ν^L is countable in $L[h]$ and (trivially) $M \in L[h]$. By Lemma 4.3 we then get that there's an elementary embedding $j^* : H_\nu^L \rightarrow M$ in $L[h]$ with critical point κ . Since we also have that $M \in L$ and as ν was arbitrary, this makes κ virtually measurable in L . \blacksquare

We get the following immediate corollary.

COROLLARY 4.5 (Schindler-N.). *Strategic ω -Ramseys are downwards absolute to L , and the existence of a strategic ω -Ramsey cardinal is equiconsistent with the existence of a virtually measurable cardinal. Further, in L the two notions are equivalent.* \dashv

Note also that the proof of Theorem 4.4 shows that whenever κ is strategic ω -Ramsey then for every regular $\nu > \kappa$ there's a generic extension in which there exists a weakly amenable H_ν^V -normal H_ν -measure on κ .

We end this section with a result showing precisely where in the large cardinal hierarchy the strategic ω -Ramsey cardinals and ω -Ramsey cardinals lie, namely that strategic ω -Ramseys are equiconsistent with *remarkables* and ω -Ramseys are strictly below. Theorem 4.8 of Gitman and Welch (2011) showed that 2-iterables are limits of remarkables, and our Propositions 2.6 and 4.13 shows that ω -Ramseys are limits of 1-iterables, so that the strategic ω -Ramseys and the ω -Ramseys both lie strictly between the 2-iterables and 1-iterables. It was shown in Holy and Schlicht (2018) that ω -Ramseys are consistent with $V = L$. Remarkable cardinals were introduced by Schindler (2000), and Gitman and Schindler (2015) showed the following two equivalent formulations.

DEFINITION 4.6. A cardinal κ is **remarkable** if one of the two equivalent properties hold:

- For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with $H_\lambda^V \subseteq M$ and a forcing poset \mathbb{P} , such that in $V^\mathbb{P}$ there's an elementary embedding $\pi : H_\nu^V \rightarrow M$ with critical point κ and $\pi(\kappa) > \lambda$;
- For all $\lambda > \kappa$ there exist $\nu > \lambda$, a transitive set M with ${}^\lambda M \subseteq M$ and a forcing poset \mathbb{P} , such that in $V^\mathbb{P}$ there's an elementary embedding $\pi : H_\nu^V \rightarrow M$ with critical point κ and $\pi(\kappa) > \lambda$.

\dashv

THEOREM 4.7 (N.). *Let κ be a virtually measurable cardinal. Then either κ is either remarkable in L or $L_\kappa \models$ ‘there is a proper class of virtually measurables’. In particular, the two notions are equiconsistent.*

PROOF. Virtually measurables are downwards absolute to L by Lemma 4.3, so we may assume $V = L$. Assume κ is not remarkable. This means that there exists some $\lambda > \kappa$ such that for every $\nu > \lambda$, transitive M with $H_\lambda^V \subseteq M$ and forcing poset \mathbb{P} it holds that, in $V^\mathbb{P}$, there’s no elementary embedding $\pi : H_\nu^V \rightarrow M$ with $\text{crit } \pi = \kappa$ and $\pi(\kappa) > \lambda$.

Fix $\nu := \lambda^+$ and use that κ is virtually ν -measurable to fix a transitive M and a forcing poset \mathbb{P} such that, in $V^\mathbb{P}$, there’s an elementary $\pi : H_\nu^V \rightarrow M$. Note that because $M \models V = L$ and M is transitive, $M = L_\alpha$ for some $\alpha \geq \nu$, so that $H_\nu^V = L_\nu \subseteq M$. This means that $\pi(\kappa) \leq \lambda < \nu$ since we’re assuming that κ isn’t remarkable. Then by restricting the generic embedding to H_κ^V we get that $H_\kappa^V < H_{\pi(\kappa)}^M = H_{\pi(\kappa)}^V$, using that $\pi(\kappa) < \nu$ and $H_\nu^V = H_\nu^M$ by the above.

Note that $\pi(\kappa)$ is a cardinal in H_ν^V since $\pi(\kappa) < \nu$, and as $H_\nu^V <_1 V$ we get that $\pi(\kappa)$ is a cardinal. But then, again using that $H_{\pi(\kappa)}^V <_1 V$, κ is virtually measurable in $H_{\pi(\kappa)}^V$ since being virtually measurable is Π_2 . This means that for every $\xi < \kappa$ it holds that

$$H_{\pi(\kappa)}^V \models \exists \alpha > \xi : \text{‘}\alpha \text{ is virtually measurable’},$$

implying that $H_\kappa^V \models$ ‘There is a proper class of virtually measurables’. ■

Now Theorem 4.7 and Corollary 4.5 yield the following immediate corollary.

COROLLARY 4.8 (Schindler-N.). *Let κ be strategic ω -Ramsey. Then either κ is remarkable in L or otherwise $L_\kappa \models$ ‘there is a proper class of strategic ω -Ramseys’. In particular, the two notions are equiconsistent. \dashv*

Now, using these results we show that the strategic ω -Ramseys have strictly stronger consistency strength than the ω -Ramseys.

THEOREM 4.9 (N.). *Remarkable cardinals are strategic ω -Ramsey limits of ω -Ramsey cardinals.*

PROOF. Let κ be remarkable. Using property (ii) in the definition of remarkability above we can find a transitive M closed under 2^κ -sequences and a generic elementary embedding $\pi : H_\nu^V \rightarrow M$ for some $\nu > 2^\kappa$. We will show that κ is ω -Ramsey in M . Note that remarkables are clearly virtually measurable, and thus by Theorem 4.4 also strategic ω -Ramsey; let τ_θ be the winning strategy for player II in $\mathcal{G}_\omega^\theta(\kappa)$ for all regular $\theta > \kappa$.

In M we fix some regular $\theta > \kappa$ and let σ be some strategy for player I in $\mathcal{G}_\omega^\theta(\kappa)^M$. Since M is closed under 2^κ -sequences it means that $\mathcal{P}(\mathcal{P}(\kappa)) \subseteq M$ and thus that M contains all possible filters on κ . We let player II follow τ , which produces a play $\sigma * \tau$ in which player II wins. But all player II’s moves are in $\mathcal{P}(\mathcal{P}(\kappa))$ and hence in M , and as M is furthermore closed under ω -sequences, $\sigma * \tau \in M$. This means that M sees that σ is not winning, so κ is ω -Ramsey in M .

This also implies that κ is a limit of ω -Ramseys in H_ν . But as κ is remarkable it holds that $H_\kappa <_2 V$, in analogy with the same property for strong and supercompacts, and as being ω -Ramsey is a Π_2 -notion this means that κ is a limit of ω -Ramseys. ■

This immediately yields the following corollary.

COROLLARY 4.10 (Schindler-N.). *If κ is a strategic ω -Ramsey cardinal then*

$$L_\kappa \models \text{'there is a proper class of } \omega\text{-Ramseys'}. \quad \dashv$$

(ω, α) -Ramsey cardinals

A natural generalisation of the γ -Ramsey definition is to require more iterability of the last measure. Of course, by Proposition 2.6 we have that $\mathcal{G}_\gamma(\kappa, \zeta)$ is equivalent to $\mathcal{G}_\gamma(\kappa)$ when $\text{cof } \gamma > \omega$ so the next definition is only interesting whenever $\text{cof } \gamma = \omega$.

DEFINITION 4.11 (N.). Let α, β be ordinals. Then a cardinal κ is (α, β) -Ramsey if player I does not have a winning strategy in $\mathcal{G}_\alpha^\theta(\kappa, \beta)$ for all regular $\theta > \kappa$.¹¹ \dashv

DEFINITION 4.12 (Gitman). A cardinal κ is α -iterable if for every $A \subseteq \kappa$ there exists a *transitive* weak κ -model \mathcal{M} with $A \in \mathcal{M}$ and an α -good \mathcal{M} -measure μ on \mathcal{M} . \dashv

PROPOSITION 4.13. *If $\beta > 0$ then every (α, β) -Ramsey is a β -iterable stationary limit of β -iterables.*

PROOF. Let (\mathcal{M}, \in, μ) be a result of a play of $\mathcal{G}_\alpha^{\kappa^+}(\kappa, \beta)$ in which player II won. Then the transitive collapse of (\mathcal{M}, \in, μ) witnesses that κ is β -iterable, since μ is β -good by definition of $\mathcal{G}_\alpha^{\kappa^+}(\kappa, \beta)$.

That κ is β -iterable is reflected to some H_θ , so let now (\mathcal{N}, \in, ν) be a result of a play of $\mathcal{G}_\alpha^\theta(\kappa, \beta)$ in which player II won. Then $\mathcal{N} < H_\theta$, so that κ is also β -iterable in \mathcal{N} . Since being β -iterable is witnessed by a subset of κ and $\beta > 0$ implies¹² that we get a κ -powerset preserving $j : \mathcal{N} \rightarrow \mathcal{P}$, \mathcal{P} also thinks that κ is β -iterable, making κ a stationary limit of β -iterables by elementarity. ■

We now move towards Theorem 4.17 which gives an upper consistency bound for the (ω, α) -Ramseys. We first recall a few definitions and a folklore lemma.

DEFINITION 4.14. For an infinite ordinal α , a cardinal κ is α -Erdős for $\alpha \leq \kappa$ if given any club $C \subseteq \kappa$ and regressive $c : [C]^{<\omega} \rightarrow \kappa$ there is a set $H \in [C]^\alpha$ homogeneous for c ; i.e. that $|c''[H]^n| \leq 1$ holds for every $n < \omega$. \dashv

¹¹Note that an α -Ramsey cardinal is the same as an $(\alpha, 0)$ -Ramsey cardinal.

¹²Recall that β -good for $\beta > 0$ in particular implies weak amenability.

DEFINITION 4.15. A set of indiscernibles I for a structure $\mathcal{M} = (M, \in, A)$ is **remarkable** if $I - \iota$ is a set of indiscernibles for $(M, \in, A, \langle \xi \mid \xi < \iota \rangle)$ for every $\iota \in I$. \dashv

LEMMA 4.16 (Folklore). *Let κ be α -Erdős where $\alpha \in [\omega, \kappa]$ and let $C \subseteq \kappa$ be club. Then any structure \mathcal{M} in a countable language \mathcal{L} with $\kappa + 1 \subseteq \mathcal{M}$ has a remarkable set of indiscernibles $I \in [C]^\alpha$.*

PROOF. Let $\langle \varphi_n \mid n < \omega \rangle$ enumerate all \mathcal{L} -formulas and define $c : [C]^{<\omega} \rightarrow \kappa$ as follows. For an increasing sequence $\alpha_1 < \dots < \alpha_{2n} \in C$ let

$$c(\{\alpha_1, \dots, \alpha_{2n}\}) := \text{the least } \lambda < \alpha_1 \text{ such that } \exists \delta_1 < \dots < \delta_k \exists m < \omega : \lambda = \langle m, \delta_1, \dots, \delta_k \rangle \wedge \\ \mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \dots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \dots, \alpha_{2n}]$$

if such a λ exists, and $c(s) = 0$ otherwise. Clearly c is regressive, so since κ is α -Erdős we get a homogeneous $I \in [C]^\alpha$ for c ; i.e. that $|c''[I]^n| \leq 1$ for every $n < \omega$. Then $c(\{\alpha_1, \dots, \alpha_{2n}\}) = 0$ for every $\alpha_1, \dots, \alpha_{2n} \in I$, as otherwise there exists an $m < \omega$ and $\delta_1 < \dots < \delta_k$ such that for any $\alpha_1 < \dots < \alpha_{2n} \in I$,

$$\mathcal{M} \not\models \varphi_m[\vec{\delta}, \alpha_1, \dots, \alpha_n] \leftrightarrow \varphi_m[\vec{\delta}, \alpha_{n+1}, \dots, \alpha_{2n}]. \quad (\dagger)$$

But then simply pick $\alpha_1 < \dots < \alpha_{2n} < \alpha'_1 < \dots < \alpha'_{2n}$ so that both $\{\alpha_1, \dots, \alpha_{2n}\}$ and $\{\alpha'_1, \dots, \alpha'_{2n}\}$ witness (\dagger) ; then either $\{\alpha_1, \dots, \alpha_n, \alpha'_1, \alpha'_n\}$ or $\{\alpha_1, \dots, \alpha_n, \alpha'_{n+1}, \dots, \alpha'_{2n}\}$ also witnesses that (\dagger) fails, \nexists . ■

THEOREM 4.17 (N.). *Let $\alpha \in [\omega, \omega_1]$ be additively closed. Then any α -Erdős cardinal is a limit of (ω, α) -Ramsey cardinals.*

PROOF. Let κ be α -Erdős, $\theta > \kappa$ a regular cardinal and $\beta < \kappa$ any ordinal. Use the above Lemma 4.16 to get a set of remarkable indiscernibles $I \in [\kappa]^\alpha$ for the structure $(H_\theta, \in, \langle \xi \mid \xi < \beta \rangle)$, and let $\iota \in I$ be the least indiscernible in I . We will show that player I has no winning strategy in $\mathcal{G}_\omega^\theta(\iota, \alpha)$, so by the proof of Theorem 5.5(d) in Holy and Schlicht (2018) it suffices to find a weak ι -model $\mathcal{M} < H_\theta$ and an α -good \mathcal{M} -measure on ι . Define

$$\mathcal{M} := \text{Hull}^{H_\theta}(\iota \cup I) < H_\theta$$

and let $\pi : I \rightarrow I$ be the right-shift map. Since I is remarkable, $I (= I - \iota)$ is a set of indiscernibles for the structure $(H_\theta, \in, \langle \xi \mid \xi < \iota \rangle)$, so that π induces an elementary embedding $j : \mathcal{M} \rightarrow \mathcal{M}$ with $\text{crit } j = \iota$, given as

$$j(\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \dots, \iota_{i_k}]) := \tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+1}, \dots, \iota_{i_k+1}],$$

with $\vec{\xi} \subseteq \iota$. Since j is trivially ι -powerset preserving we get that $\mathcal{M} < H_\theta$ is a weak ι -model satisfying ZFC^- with a 1-good \mathcal{M} -measure μ_j on ι . Furthermore, as we can linearly iterate \mathcal{M} simply by applying j we get

an α -iteration of \mathcal{M} since there are α -many indiscernibles. Note that at limit stages $\gamma < \alpha$ our iteration sends $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0}, \dots, \iota_{i_k}]$ to $\tau^{\mathcal{M}}[\vec{\xi}, \iota_{i_0+\gamma}, \dots, \iota_{i_k+\gamma}]$ so here we are using that α is additively closed.

This shows that player I has no winning strategy in $\mathcal{G}_\omega^\theta(\iota, \alpha)$. Since $\iota > \beta$ and $\beta < \kappa$ was arbitrary, κ is a limit of η such that player I has no winning strategy in $\mathcal{G}_\omega^\theta(\eta, \alpha)$. If we repeat this procedure for all regular $\theta > \kappa$ we get by the pigeon hole principle that κ is a limit of (ω, α) -Ramsey cardinals. ■

As Theorem 4.5 in Gitman and Schindler (2015) shows that $(\alpha+1)$ -iterable cardinals have α -Erdős cardinals below them for $\alpha \geq \omega$ additively closed, this shows that the (ω, α) -Ramseys form a strict hierarchy. Further, as α -Erdős cardinals are consistent with $V = L$ when $\alpha < \omega_1^L$ and ω_1 -iterable cardinals aren't consistent with $V = L$, we also get that (ω, α) -Ramsey cardinals are consistent with $V = L$ if $\alpha < \omega_1^L$ and that they aren't if $\alpha = \omega_1$.

[Strategic] $(\omega+1)$ -Ramsey cardinals

The next step is then to consider $(\omega+1)$ -Ramseys, which turn out to cause a considerable jump in consistency strength. We first need the following result which is implicit in Mitchell (1979) and in the proof of Lemma 1.3 in Donder et al. (1981) — see also Dodd (1982) and Gitman (2011).

THEOREM 4.18 (Dodd, Mitchell). *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is an element of a weak κ -model \mathcal{M} such that there exists a weakly amenable countably complete \mathcal{M} -measure on κ .* \dashv

The following theorem then supplies us with a lower bound for the strength of the $(\omega+1)$ -Ramsey cardinals. It should be noted that a better lower bound will be shown in Theorem 5.9, but we include this Ramsey lower bound as well for completeness.

THEOREM 4.19 (N.). *Every $(\omega+1)$ -Ramsey cardinal is a Ramsey limit of Ramseys.*

PROOF. Let κ be $(\omega+1)$ -Ramsey and $A \subseteq \kappa$. Let σ be a strategy for player I in $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ satisfying that whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with σ it holds that $A \in \mathcal{M}_0$ and $\mu_\alpha \in \mathcal{M}_{\alpha+1}$ for all $\alpha \leq \omega$. Then σ isn't winning as κ is $(\omega+1)$ -Ramsey, so we may fix a play $\sigma * \vec{\mu}_\alpha$ of $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ in which player II wins. Then by the choice of σ we get that μ_ω is a weakly amenable \mathcal{M}_ω -measure on κ , and by the rules of $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ it's also countably complete (it's even normal), which makes κ Ramsey by the above Theorem 4.18.

Since κ is Ramsey, $\mathcal{M}_\omega \models \text{'}\kappa \text{ is Ramsey'}$ as well. Letting $j : \mathcal{M}_\omega \rightarrow \mathcal{N}$ be the κ -powerset preserving embedding induced by μ_ω , we also get that $\mathcal{N} \models \text{'}\kappa \text{ is Ramsey'}$ by κ -powerset preservation. This then implies that κ is a stationary limit of Ramsey cardinals inside \mathcal{M}_ω , and thus also in V by elementarity. ■

As for the *consistency* strength of the strategic $(\omega+1)$ -Ramsey cardinals, we get the following result that they reach a measurable cardinal. The proof of the following is closely related to the proof due to Silver and Solovay that player II having a winning strategy in the *cut and choose game* is equiconsistent with a measurable cardinal — see e.g. p. 249 in Kanamori and Magidor (1978).

THEOREM 4.20 (N). *If κ is a strategic $(\omega+1)$ -Ramsey cardinal then, in $V^{\text{Col}(\omega, 2^\kappa)}$, there's a transitive class N and an elementary embedding $j : V \rightarrow N$ with $\text{crit } j = \kappa$. In particular, the existence of a strategic $(\omega+1)$ -Ramsey cardinal is equiconsistent with the existence of a measurable cardinal.*

PROOF. Set $\mathbb{P} := \text{Col}(\omega, 2^\kappa)$ and let σ be player II's winning strategy in $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$. Let $\dot{\mathcal{M}}$ be a \mathbb{P} -name of an ω -sequence $\langle \mathcal{M}_n \mid n < \omega \rangle$ of weak κ -models $\mathcal{M}_n \in V$ such that $\mathcal{M}_n < H_{\kappa^+}^V$ and $\mathcal{P}(\kappa)^V \subseteq \bigcup_{n < \omega} \mathcal{M}_n$, and let $\dot{\mu}$ be a \mathbb{P} -name for the ω -sequence of σ -responses to the \mathcal{M}_n 's in $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)^V$.

Assume that there's a \mathbb{P} -condition p which forces the generic ultrapower $\text{Ult}(V, \bigcup_n \dot{\mu}_n)$ to be illfounded, meaning that we can fix a \mathbb{P} -name \dot{f} for an ω -sequence $\langle f_n \mid n < \omega \rangle$ such that

$$p \Vdash \dot{X}_n := \{\alpha < \kappa \mid \dot{f}_{n+1}(\alpha) < \dot{f}_n(\alpha)\} \in \bigcup_{n < \omega} \dot{\mu}_n.$$

Now, in V , we fix some large regular $\theta \gg \kappa$ and a countable $\mathcal{N} < H_\theta$ such that $\dot{\mathcal{M}}, \dot{\mu}, \dot{f}, H_{\kappa^+}^V, \sigma, p \in \mathcal{N}$. We can find an \mathcal{N} -generic $g \subseteq \mathbb{P}^\mathcal{N}$ in V with $p \in g$ since \mathcal{N} is countable, so that $\mathcal{N}[g] \in V$. But the play $\dot{\mathcal{M}}_n^g * \dot{\mu}_n^g$ is a play of $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)^V$ which is according to σ , meaning that $\bigcup_{n < \omega} \dot{\mu}_n^g$ is normal and in particular countably complete (in V). Then $\bigcap_{n < \omega} \dot{X}_n^g \neq \emptyset$, but if $\alpha \in \bigcap_{n < \omega} \dot{X}_n^g$ then $\langle \dot{f}_n^g(\alpha) \mid n < \omega \rangle$ is a strictly decreasing ω -sequence of ordinals, \nexists . This means that $\text{Ult}(V, \bigcup_n \dot{\mu}_n)$ is indeed wellfounded.

This conclusion is well-known to imply that κ is a measurable in an inner model; see e.g. Lemma 4.2 in Kellner and Shelah (2011). ■

The above Theorem 4.20 then answers Question 9.2 in Holy and Schlicht (2018) in the negative, asking if λ -Ramseys are strategic λ -Ramseys for uncountable cardinals λ , as well as answering Question 9.7 from the same paper in the positive, asking whether strategic fully Ramseys are equiconsistent with a measurable.

5 The general case

Gitman's cardinals

In this subsection we define the strongly- and super Ramsey cardinals from Gitman (2011) and investigate further connections between these and the α -Ramsey cardinals. First, a definition.

DEFINITION 5.1 (Gitman). A cardinal κ is **strongly Ramsey** if every $A \subseteq \kappa$ is an element of a transitive κ -model \mathcal{M} with a weakly amenable \mathcal{M} -normal \mathcal{M} -measure μ on κ . If furthermore $\mathcal{M} < H_{\kappa^+}$ then we say that κ is **super Ramsey**. —

Note that since the model \mathcal{M} in question is a κ -model it is closed under countable sequences, so that the measure μ is automatically countably complete. The definition of the strongly Ramseys is thus exactly the same as the characterisation of Ramsey cardinals, with the added condition that the model is closed under $<\kappa$ -sequences. Gitman (2011) shows that every super Ramsey cardinal is a strongly Ramsey limit of strongly

Ramsey cardinals, and that κ is strongly Ramsey iff every $A \subseteq \kappa$ is an element of a transitive κ -model $\mathcal{M} \models \text{ZFC}$ with a weakly amenable \mathcal{M} -normal \mathcal{M} -measure μ on κ .

Now, a first connection between the α -Ramseys and the strongly- and super Ramseys is the result in Holy and Schlicht (2018) that fully Ramsey cardinals are super Ramsey limits of super Ramseys. The following result then shows that the strongly- and super Ramseys are sandwiched between the almost fully Ramseys and the fully Ramseys.

THEOREM 5.2 (N.-W.). *Every strongly Ramsey cardinal is a stationary limit of almost fully Ramseys.*

PROOF. Let κ be strongly Ramsey and let $\mathcal{M} \models \text{ZFC}$ be a transitive κ -model with $V_\kappa \in \mathcal{M}$ and μ a weakly amenable \mathcal{M} -normal \mathcal{M} -measure. Let $\gamma < \kappa$ have uncountable cofinality and $\sigma \in \mathcal{M}$ a strategy for player I in $\mathcal{G}_\gamma(\kappa)^\mathcal{M}$. Now, whenever player I plays $\mathcal{M}_\alpha \in \mathcal{M}$ let player II play $\mu \cap \mathcal{M}_\alpha$, which is an element of \mathcal{M} by weak amenability of μ . As $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$ the resulting play is inside \mathcal{M} , so \mathcal{M} sees that σ is not winning.

Now, letting $j_\mu : \mathcal{M} \rightarrow \mathcal{N}$ be the induced embedding, κ -powerset preservation of j_μ implies that μ is also a weakly amenable \mathcal{N} -normal \mathcal{N} -measure on κ . This means that we can copy the above argument to ensure that κ is also almost fully Ramsey in \mathcal{N} , entailing that it is a stationary limit of almost fully Ramseys in \mathcal{M} . But note now that λ is almost fully Ramsey iff it is almost fully Ramsey in a transitive ZFC-model containing $H_{(2^\lambda)^+}$ as an element by Theorem 5.5(e) in Holy and Schlicht (2018), so that κ being inaccessible, $V_\kappa \in \mathcal{M}$ and \mathcal{M} being transitive implies that κ really is a stationary limit of almost fully Ramseys. ■

Downwards absoluteness to K

Lastly, we consider the question of whether the α -Ramseys are downwards absolute to K , which turns out to at least be true in many cases. The below Theorem 5.4 then also answers Question 9.4 from Holy and Schlicht (2018) in the positive, asking whether α -Ramseys are downwards absolute to the Dodd-Jensen core model for $\alpha \in [\omega, \kappa]$ a cardinal. We first recall the definition of 0^\sharp .

DEFINITION 5.3. 0^\sharp is “the sharp for a strong cardinal”, meaning the minimal sound active mouse \mathcal{M} with $\mathcal{M} \restriction \text{crit}(\dot{F}^\mathcal{M}) \models \text{‘There exists a strong cardinal’}$, with $\dot{F}^\mathcal{M}$ being the top extender of \mathcal{M} . \dashv

THEOREM 5.4 (N.-W.). *Assume 0^\sharp does not exist. Let λ be a limit ordinal with uncountable cofinality and let κ be λ -Ramsey. Then $K \models \text{‘}\kappa \text{ is a } \lambda\text{-Ramsey cardinal’}$.*

PROOF. Note first that $\kappa^{+K} = \kappa^+$ by Schindler (1997), since κ in particular is weakly compact. Let $\sigma \in K$ be a strategy for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$, so that a play following σ will produce weak κ -models $\mathcal{M} < K \restriction \kappa^+$. We can then define a strategy $\tilde{\sigma}$ for player I in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ as follows. Firstly let $\tilde{\sigma}(\emptyset) := \text{Hull}^{H_{\kappa^+}}(K \restriction \kappa \cup \sigma(\emptyset))$. Assuming now that $\langle \tilde{\mathcal{M}}_\alpha, \tilde{\mu}_\alpha \mid \alpha < \gamma \rangle$ is a partial play of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde{\sigma}$, we have two cases. If $\tilde{\mu}_\alpha \in K$ for every $\alpha < \gamma$ then let $\langle \mathcal{M}_\alpha \mid \alpha < \gamma \rangle$ be the corresponding models played in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$

from which the $\tilde{\mathcal{M}}_\alpha$'s are derived and let

$$\tilde{\sigma}(\langle \tilde{\mathcal{M}}_\alpha, \tilde{\mu}_\alpha \mid \alpha < \gamma \rangle) := \text{Hull}^{H_{\kappa^+}}(K|_\kappa \cup \sigma(\langle \mathcal{M}_\alpha, \tilde{\mu}_\alpha \mid \alpha < \gamma \rangle)),$$

and otherwise let $\tilde{\sigma}$ play arbitrarily. As κ is λ -Ramsey (in V) there exists a play $\langle \tilde{\mathcal{M}}_\alpha, \tilde{\mu}_\alpha \mid \alpha \leq \lambda \rangle$ of $\mathcal{G}_\lambda^{\kappa^+}(\kappa)$ which is consistent with $\tilde{\sigma}$ in which player II won. Note that $\tilde{\mathcal{M}}_\lambda \cap K|_{\kappa^+} < K|_{\kappa^+}$ so let \mathcal{N} be the transitive collapse of $\tilde{\mathcal{M}}_\lambda \cap K|_{\kappa^+}$. But if $j : \mathcal{N} \rightarrow K|_{\kappa^+}$ is the uncollapse then $\text{crit } j$ is both an \mathcal{N} -cardinal and also $> \kappa$ because we ensured that $K|_\kappa \subseteq \mathcal{N}$. This means that $j = \text{id}$ because κ is the largest \mathcal{N} -cardinal by elementarity in $K|_{\kappa^+}$, so that $\tilde{\mathcal{M}}_\lambda \cap K|_{\kappa^+} = \mathcal{N}$ is a transitive elementary substructure of $K|_{\kappa^+}$, making it an initial segment of K .

Now, since $\mu := \tilde{\mu}_\lambda$ is a countably complete weakly amenable $K|_{o(\mathcal{N})}$ -measure¹³, the “beaver argument”¹⁴ shows that $\mu \in K$, so that we can then define a strategy τ for player II in $\mathcal{G}_\lambda^{\kappa^+}(\kappa)^K$ as simply playing $\mu \cap \mathcal{N} \in K$ whenever player I plays \mathcal{N} . Since $\mu = \tilde{\mu}_\lambda$ we also have that $\mu \cap \mathcal{M}_\alpha = \tilde{\mu}_\alpha \cap \mathcal{M}_\alpha$, so that σ will eventually play \mathcal{N} , making τ win against σ .¹⁵ ■

Note that the only thing we used $\text{cof } \lambda > \omega$ for in the above proof was to ensure that μ was countably complete. If now κ instead was either genuine- or normal α -Ramsey for any limit ordinal α then μ_α would also be countably complete and weakly amenable, so the same proof shows the following.

COROLLARY 5.5 (N.-W.). *Assume 0^\sharp does not exist and let α be any limit ordinal. Then every genuine- and every normal α -Ramsey cardinal is downwards absolute to K . In particular, if α is a limit of limit ordinals then every $<\alpha$ -Ramsey cardinal is downwards absolute to K as well.* \dashv

Indiscernible games

We now move to the strategic versions of the α -Ramsey hierarchy. The first thing we want to do is define α -very Ramsey cardinals, introduced in Sharpe and Welch (2011), and show the tight connection between these and the strategic α -Ramseys. We need a few more definitions. Recall the definition of a remarkable set of indiscernibles from Definition 4.15.

DEFINITION 5.6. A **good set of indiscernibles** for a structure \mathcal{M} is a set $I \subseteq \mathcal{M}$ of remarkable indiscernibles for \mathcal{M} such that $\mathcal{M}|_\iota < \mathcal{M}$ for any $\iota \in I$. \dashv

DEFINITION 5.7 (Sharpe-W.). Define the **indiscernible game** $G_\gamma^I(\kappa)$ in γ many rounds as follows

I	\mathcal{M}_0	\mathcal{M}_1	\mathcal{M}_2	\dots
II	I_0	I_1	I_2	\dots

¹³Here we use that $\mathcal{N} \triangleleft K$.

¹⁴See Lemmata 7.3.7–7.3.9 and 8.3.4 in Zeman (2002) for this argument.

¹⁵Note that τ is not necessarily a winning strategy — all we know is that it is winning against this particular strategy σ .

Here \mathcal{M}_α is an amenable structure of the form $(J_\kappa[A], \in, A)$ for some $A \subseteq \kappa$, $I_\alpha \in [\kappa]^\kappa$ is a good set of indiscernibles for \mathcal{M}_α and the I_α 's are \subseteq -decreasing. Player II wins iff they can continue playing through all the rounds. \dashv

DEFINITION 5.8 (Sharpe-W.). A cardinal κ is γ -**very Ramsey** if player II has a winning strategy in the game $G_\gamma^I(\kappa)$. \dashv

The next couple of results concerns the connection between the strategic α -Ramseys and the α -very Ramseys. We start with the following.

THEOREM 5.9 (N.). *Every $(\omega+1)$ -Ramsey is an ω -very Ramsey stationary limit of ω -very Ramseys.*

PROOF. Let κ be $(\omega+1)$ -Ramsey. We will describe a winning strategy for player II in the indiscernible game $G_\omega^I(\kappa)$. If player I plays $\mathcal{M}_0 = (J_\kappa[A_0], \in, A_0)$ in $G_\omega^I(\kappa)$ then let player I in $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ play

$$\mathcal{H}_0 := \text{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \kappa, A_0\}) < H_{\kappa^+}.$$

Let player I now follow a strategy in $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ which starts off with \mathcal{H}_0 and ensures that, whenever $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ is consistent with player I's strategy, then $\mu_\alpha \in \mathcal{M}_{\alpha+1}$ for all $\alpha \leq \omega$. Since player II is not losing in $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ there is a play $\vec{\mathcal{M}}_\alpha * \vec{\mu}_\alpha$ in which player I follows this strategy just described and where player II wins – write $\mathcal{H}_0^{(\alpha)} := \mathcal{M}_\alpha$ and $\mu_0^{(\alpha)} := \mu_\alpha$ for the models and measures in this play.

$$\begin{array}{ccccccc} \text{I} & \mathcal{H}_0^{(0)} & \dots & \mathcal{H}_0^{(\omega)} & \mathcal{H}_0^{(\omega+1)} & & \\ \text{II} & \mu_0^{(0)} & \dots & \mu_0^{(\omega)} & \mu_0^{(\omega+1)} & & \end{array}$$

By the choice of player I's strategy we get that $\mu_0^{(\omega)}$ is both weakly amenable, and it's also countably complete by the rules of $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ (it's even normal). Now Lemma 2.9 of Sharpe and Welch (2011) gives us a set of good indiscernibles $I_0 \in \mu_0^{(\omega)}$ for \mathcal{M}_0 , as $\mathcal{M}_0 \in \mathcal{H}_0^{(\omega)}$ and $\mu_0^{(\omega)}$ is a countably complete weakly amenable $\mathcal{H}_0^{(\omega)}$ -normal $\mathcal{H}_0^{(\omega)}$ -measure on κ . Let player II play I_0 in $G_\omega^I(\kappa)$. Let now $\mathcal{M}_1 = (J_\kappa[A_1], \in, A_1)$ be the next play by player I in $G_\omega^I(\kappa)$.

$$\begin{array}{ccc} \text{I} & \mathcal{M}_0 & \mathcal{M}_1 \\ \text{II} & I_0 & \end{array}$$

Since $\mu_0^{(\omega)} = \bigcup_n \mu_0^{(n)}$ we must have that $I_0 \in \mu_0^{(n_0)}$ for some $n_0 < \omega$. In the (n_0+1) 'st round of $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ we change player I's strategy and let player I play

$$\mathcal{H}_1 := \text{Hull}^{H_{\kappa^+}}(J_\kappa[A_0] \cup \{\mathcal{M}_0, \mathcal{M}_1, \kappa, A_0, A_1, \langle \mathcal{H}_0^{(k)}, \mu_0^{(k)} \mid k \leq n_0 \rangle\}) < H_{\kappa^+}$$

and otherwise continues following some strategy, as long as the measures played by player II keep being elements of the following models. Our play of the game $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ thus looks like the following so far.

$$\begin{array}{ccccccc}
\text{I} & \mathcal{H}_0^{(0)} & & \dots & & \mathcal{H}_0^{(n_0)} & \mathcal{H}_1 \\
\text{II} & & \mu_0^{(0)} & & \dots & & \mu_0^{(n_0)}
\end{array}$$

Now player II in $\mathcal{G}_{\omega+1}^{\kappa^+}(\kappa)$ is not losing at round n_0 , so there is a play extending the above in which player I follows their revised strategy and in which player II wins. As before we get a set $I'_1 \in \mu_1^{(n_1)}$ of good indiscernibles for \mathcal{M}_1 , where $n_1 < \omega$. Since $I_0 \in \mu_0^{(n_0)} \subseteq \mu_1^{(n_1)}$ we can let player II in $G_\omega^I(\kappa)$ play $I_1 := I_0 \cap I'_1 \in \mu_1^{(n_1)}$. Continuing like this, player II can keep playing throughout all ω rounds of $G_\omega^I(\kappa)$, making κ ω -very Ramsey.

As for showing that κ is a stationary limit of ω -very Ramseys, let $\mathcal{M} < H_{\kappa^+}$ be a weak κ -model with a weakly amenable countably complete \mathcal{M} -normal \mathcal{M} -measure μ on κ , which exists by Theorem 4.19 as κ is $(\omega+1)$ -Ramsey. Then by elementarity $\mathcal{M} \models \text{“}\kappa \text{ is } \omega\text{-very Ramsey”}$ and since κ being ω -very Ramsey is absolute between structures having the same subsets of κ it also holds in the μ -ultrapower, meaning that κ is a stationary limit of ω -very Ramseys by elementarity. ■

The above proof technique can be generalised to the following.

THEOREM 5.10 (N). *For limit ordinals α , every coherent $<\omega\alpha$ -Ramsey is $\omega\alpha$ -very Ramsey.*

PROOF. This is basically the same proof as the proof of Theorem 5.9. We do the “going-back” trick in ω -chunks, and at limit stages we continue our non-losing strategy in $\mathcal{G}_{\omega\alpha}^{\kappa^+}(\kappa)$ by using our winning strategy, which we have available as we are assuming coherent $<\omega\alpha$ -Ramsey. We need α to be a limit ordinal for this to work, as otherwise we would be in trouble in the last ω -chunk, as we cannot just extend the play to get a countably complete measure, which we need to use the proof of Theorem 5.9. ■

As for going from the α -very Ramseys to the strategic α -Ramseys we got the following.

THEOREM 5.11 (N). *For γ any ordinal, every coherent $<\gamma$ -very Ramsey¹⁶ is coherent $<\gamma$ -Ramsey.¹⁷*

PROOF. The reason why we work with $<\gamma$ -Ramseys here is to ensure that player II only has to satisfy a closed game condition (i.e. to continue playing throughout all the rounds). If $\gamma = \beta + 1$ then set $\zeta := \beta$ and otherwise let $\zeta := \gamma$. Let κ be ζ -very Ramsey and let τ be a winning strategy for player II in $G_\zeta^I(\kappa)$. Let $\mathcal{M}_\alpha < H_\theta$ be any move by player I in the α 'th round of $\mathcal{G}_\zeta(\kappa)$. Let $A_\alpha \subseteq \kappa$ encode all subsets of κ in \mathcal{M}_α

¹⁶Here the coherency again just means that the winning strategies σ_α for player II in $G_\alpha^I(\kappa)$ are \subseteq -increasing.

¹⁷Here a “coherent $<\gamma$ -very Ramsey cardinal” is defined from γ -very Ramseys in the same way as coherent $<\gamma$ -Ramsey cardinals is defined from γ -Ramseys. When γ is a limit ordinal then coherent $<\gamma$ -very Ramseys are precisely the same as γ -very Ramseys, so this is solely to “subtract one” when γ is a successor ordinal — i.e. a coherent $<(\gamma + 1)$ -very Ramsey cardinal is the same thing as a γ -very Ramsey cardinal.

and from now

$$\mathcal{N}_\alpha := (J_\kappa[A_\alpha], \in, A_\alpha),$$

which is a legal move for player I in $G_\zeta^I(\kappa)$, yielding a good set of indiscernibles $I_\alpha \in [\kappa]^\kappa$ for \mathcal{N}_α such that $I_\alpha \subseteq I_\beta$ for every $\beta < \alpha$. Now by section 2.3 in Sharpe and Welch (2011) we get a structure \mathcal{P}_α with $\mathcal{N}_\alpha \in \mathcal{P}_\alpha$ and a \mathcal{P}_α -measure $\tilde{\mu}_\alpha$ on κ , generated by I_α .¹⁸ Set $\mu_\alpha := \tilde{\mu}_\alpha \cap \mathcal{M}_\alpha$ and let player II play μ_α in $\mathcal{G}_\zeta(\kappa)$.

As the μ_α 's are generated by the I_α 's, the μ_α 's are \subseteq -increasing. We have thus created a strategy for player II in $\mathcal{G}_\zeta(\kappa)$ which does not lose at any round $\alpha < \gamma$, making κ coherent $<\gamma$ -Ramsey. ■

The following result is then a direct corollary of Theorems 5.10 and 5.11.

COROLLARY 5.12 (N.). *For limit ordinals α , κ is $\omega\alpha$ -very Ramsey iff it is coherent $<\omega\alpha$ -Ramsey. In particular, κ is λ -very Ramsey iff it is strategic λ -Ramsey for any λ with uncountable cofinality.* \dashv

We can now use this equivalence to transfer results from the α -very Ramseys over to the strategic versions. The *completely Ramsey cardinals* are the cardinals topping the hierarchy defined in Feng (1990). A completely Ramsey cardinal implies the consistency of a Ramsey cardinal, see e.g. Theorem 3.51 in Sharpe and Welch (2011). We are going to use the following characterisation of the completely Ramsey cardinals, which is Lemma 3.49 in Sharpe and Welch (2011).

THEOREM 5.13 (Sharpe-W.). *A cardinal is completely Ramsey if and only if it is ω -very Ramsey.* \dashv

This, together with Theorem 5.9, immediately yields the following strengthening of Theorem 4.19.

COROLLARY 5.14 (N.). *Every $(\omega+1)$ -Ramsey cardinal is a completely Ramsey stationary limit of completely Ramsey cardinals.* \dashv

The above Theorem 5.11 also yields the following consequence.

COROLLARY 5.15 (N.). *Every completely Ramsey cardinal is completely ineffable.*

PROOF. From Theorem 5.13 we have that being completely Ramsey is equivalent to being ω -very Ramsey, so the above Theorem 5.11 then yields that a completely Ramsey cardinal is coherent $<\omega$ -Ramsey, which we saw in Theorem 3.12 is equivalent to being completely ineffable. ■

Now, moving to the uncountable case, Corollary 5.12 yields that strategic ω_1 -Ramsey cardinals are ω_1 -very Ramsey, and Theorem 3.50 in Sharpe and Welch (2011) states that ω_1 -very Ramseys are measurable in the

¹⁸By generated here we mean that $X \in \tilde{\mu}_\alpha$ iff X contains a tail of indiscernibles from I_α .

core model K , assuming 0^\sharp doesn't exist, which then shows the following theorem. We also include the original direct proof of that theorem, due to Welch.

THEOREM 5.16 (W.). *Assuming 0^\sharp doesn't exist, every strategic ω_1 -Ramsey cardinal is measurable in K .*

PROOF. Let κ be strategic ω_1 -Ramsey, say τ is the winning strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Jump to $V[g]$, where $g \subseteq \text{Col}(\omega_1, \kappa^+)$ is V -generic. Since $\text{Col}(\omega_1, \kappa^+)$ is ω -closed, V and $V[g]$ have the same countable sequences of V , so τ is still a strategy for player II in $\mathcal{G}_{\omega_1}(\kappa)^{V[g]}$, as long as player I only plays elements of V .

Now let $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ be an increasing sequence of regular K -cardinals cofinal in κ^+ , let player I in $\mathcal{G}_{\omega_1}(\kappa)$ play $\mathcal{M}_\alpha := \text{Hull}^{H_\theta}(K \restriction \kappa_\alpha) < H_\theta$ and player II follow τ . This results in a countably complete weakly amenable K -measure μ_{ω_1} , which the “beaver argument”¹⁹ then shows is actually an element of K , making κ measurable in K . ■

A natural question is whether this behaviour persists when going to larger core models. It turns out that the answer is affirmative: every strategic ω_1 -Ramsey cardinal is also measurable in Steel's core model below a Woodin, a result due to Schindler which we include with his permission here. We will need the following special case of Corollary 3.1 from Schindler (2006).²⁰

THEOREM 5.17 (Schindler). *Assume that there exists no inner model with a Woodin cardinal, let μ be a measure on a cardinal κ , and let $\pi : V \rightarrow \text{Ult}(V, \mu) \cong N$ be the ultrapower embedding. Assume that N is closed under countable sequences. Write K^N for the core model constructed inside N . Then K^N is a normal iterate of K , i.e. there is a normal iteration tree \mathcal{T} on K of successor length such that $\mathcal{M}_{0^\infty}^\mathcal{T} = K^N$. Moreover, we have that $\pi_{0^\infty}^\mathcal{T} = \pi \restriction K$. \dashv*

THEOREM 5.18 (Schindler). *Assuming there exists no inner model with a Woodin cardinal, every strategic ω_1 -Ramsey cardinal is measurable in K .*

PROOF. Fix a large regular $\theta \gg 2^\kappa$. Let κ be strategic ω_1 -Ramsey and fix a winning strategy σ for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Let $g \subseteq \text{Col}(\omega_1, 2^\kappa)$ be V -generic and in $V[g]$ fix an elementary chain $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ of weak κ -models $M_\alpha < H_\theta^V$ such that $M_\alpha \in V$, ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ and $H_{\kappa^+}^V \subseteq M_{\omega_1} := \bigcup_{\alpha < \omega_1} M_\alpha$.

Note that V and $V[g]$ have the same countable sequences since $\text{Col}(\omega_1, 2^\kappa)$ is $<\omega_1$ -closed, so we can apply σ to the M_α 's, resulting in an M_{ω_1} -measure μ on κ . Let $j : M_{\omega_1} \rightarrow \text{Ult}(M_{\omega_1}, \mu)$ be the ultrapower embedding. Since we required that ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ we get that \mathcal{M}_{ω_1} is closed under ω -sequences in $V[g]$, making μ countably complete in $V[g]$. As we also ensured that $H_{\kappa^+}^V \subseteq \mathcal{M}_{\omega_1}$ we can lift j to an ultrapower embedding $\pi : V \rightarrow \text{Ult}(V, \mu) \cong N$ with N transitive.

Since V is closed under ω -sequences in $V[g]$ we get by standard arguments that N is as well, which means that Theorem 5.17 applies, meaning that $\pi \restriction K : K \rightarrow K^N$ is an iteration map with critical point κ ,

¹⁹See Lemmata 7.3.7–7.3.9 and 8.3.4 in Zeman (2002) for this argument.

²⁰That paper assumes the existence of a measurable as well, but by Jensen and Steel (2013) we can omit that here.

making κ measurable in K . ■

6 Questions and answers

In this section we give an update on previously posed open questions in the area, as well as posing further open questions. We provide answers for the following questions, which were posed in Holy and Schlicht (2018).

- (i) If γ is an uncountable cardinal and the challenger does not have a winning strategy in the game $\mathcal{G}_\gamma^\theta(\kappa)$, does it follow that the judge has one?
- (ii) If $\omega \leq \alpha \leq \kappa$, are α -Ramsey cardinals downwards absolute to the Dodd-Jensen core model?
- (iii) Does 2-iterability imply ω -Ramseyness, or conversely?
- (iv) Does κ having the strategic κ -filter property have the consistency strength of a measurable cardinal?

Here the “challenger” is player I and the “judge” is player II, so this is asking if every γ -Ramsey is strategic γ -Ramsey, when γ is an uncountable cardinal. Theorem 5.16 therefore gives a negative answer to (i) for all uncountable ordinals γ . Theorem 5.4 and Corollary 5.5 answer (ii) positively, for α -Ramseys with α having uncountable cofinality, and for $<\alpha$ -Ramseys when α is a limit of limit ordinals. Note that (ii) in the $\alpha = \omega$ case was answered positively in Holy and Schlicht (2018).

As for (iii), it’s mentioned in Holy and Schlicht (2018) that Gitman has showed that ω -Ramseys are not in general 2-iterable by showing that 2-iterables have strictly stronger consistency strength than the ω -Ramseys, which also follows from Theorem 4.9 and Theorem 4.8 in Gitman and Welch (2011). Corollary 3.5 shows that ω -Ramsey cardinals are Δ_0^2 -indescribable, and as 2-iterables are (at least) Π_3^1 -definable it holds that any 2-iterable ω -Ramsey cardinal is a limit of 2-iterables, so that in general 2-iterables can’t be ω -Ramsey either, answering (iii) in the negative. Lastly, Theorem 4.20 gives a positive answer to (iv).

QUESTION 6.1. It’s not too hard to see that, for a regular uncountable λ , κ is strategic λ -Ramsey iff there’s a $<\lambda$ -closed forcing \mathbb{P} such that, in $V^\mathbb{P}$, there’s a weakly amenable measure on κ with a wellfounded ultrapower. Can we get similar characterisations of strategic α -Ramseys for α countable? The proofs of Theorems 4.4 and 4.20 give plausible candidates.

QUESTION 6.2. Are genuine n -Ramsey cardinals limits of n -Ramsey cardinals? We conjecture this to be true, in analogy with the weakly ineffables being limits of weakly compacts. Since “weakly ineffable = Π_1^1 -indescribability + subtlety”, this might involve some notion of “ n -iterated subtlety”. The difference here is that n -Ramseys cannot be *equivalent* to Π_{2n+1}^1 -indescribables for consistency reasons, so there is some work to be done.

QUESTION 6.3. Fix some γ with countable cofinality and an uncountable $\kappa = \kappa^{<\kappa}$. For $\theta > \kappa$ say that κ is (γ, θ) -Ramsey if player I has no winning strategy in $\mathcal{G}_\gamma^\theta(\kappa)$, so that κ is γ -Ramsey iff it’s (γ, θ) -Ramsey for every $\theta > \kappa$. Do the (γ, θ) -Ramseys then eventually form a strict hierarchy? I.e. is there some $\theta > \kappa$ such

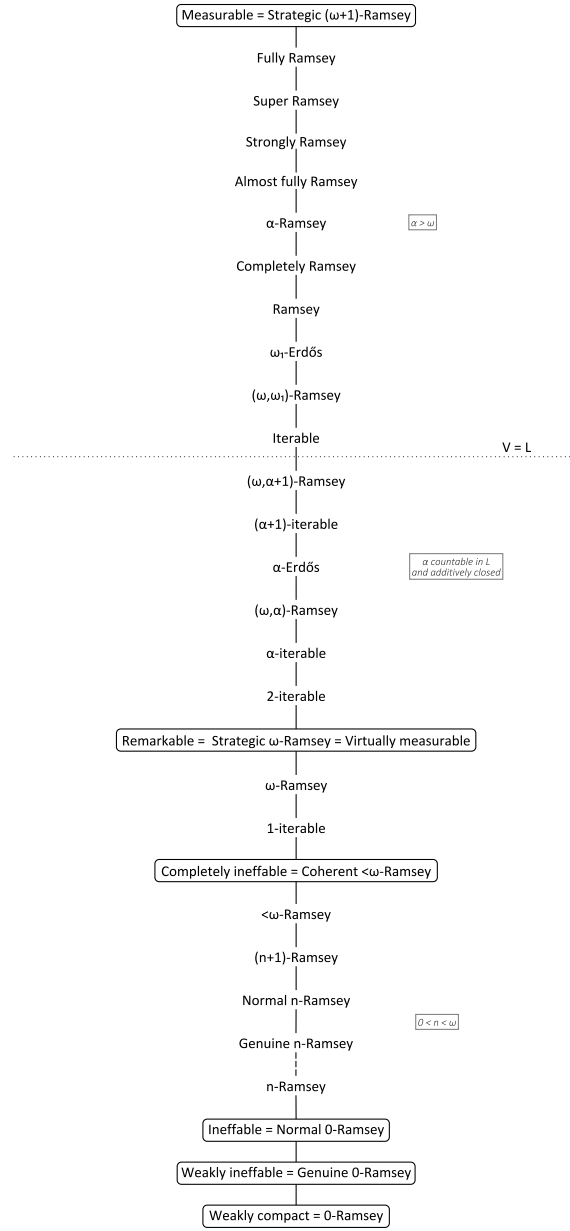
that $\text{ZFC} + \text{'there exists a } (\gamma, \theta_1)\text{-Ramsey cardinal'} \vdash \text{'there exists a } (\gamma, \theta_0)\text{-Ramsey cardinal'}$ holds for every $\theta_1 > \theta_0 \geq \theta$? Or, at the opposite end of the spectrum, do the (γ, θ) -Ramseys become eventually equivalent? I.e. is there a $\theta > \kappa$ such that κ is (γ, θ_0) -Ramsey iff it's (γ, θ_1) -Ramsey, for all $\theta_1, \theta_0 \geq \theta$?

Acknowledgements

The first author would like to thank Ralf Schindler, Philipp Schlicht and Peter Holy for their insightful comments, suggestions and corrections, and Yair Hayut for patiently explaining the above-mentioned Silver-Solovay result on the cut and choose game, which ultimately led to the proof of Theorem 4.20. We also thank the anonymous referee for their many helpful comments and corrections.

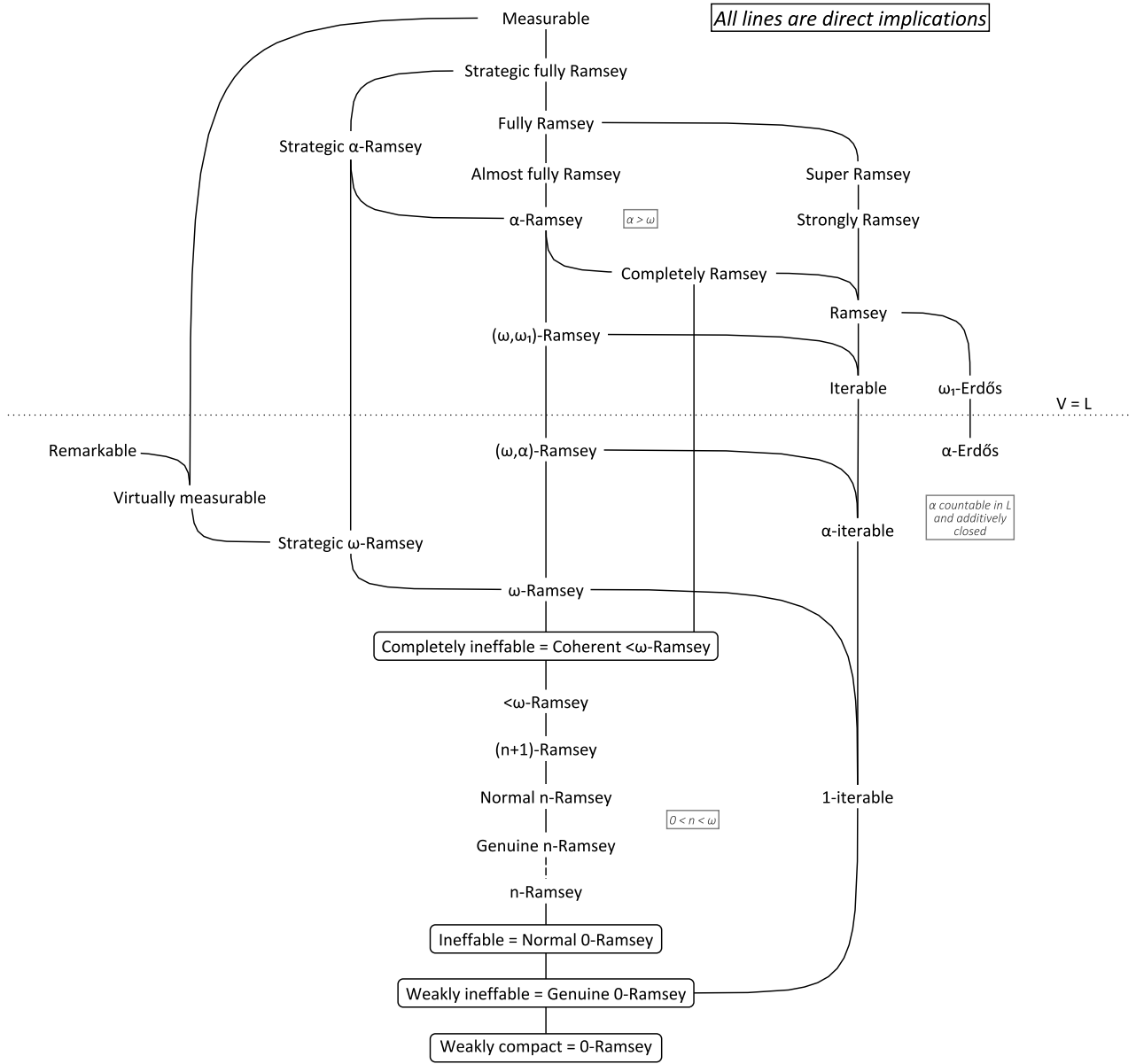
7 Diagrams

Consistency implications²¹



²¹Here dashed lines represent consistency implications which might be equiconsistencies.

Direct implications²²



²²Here dashed lines represent provable direct implications which might be equivalences.

References

- Abramson, F. G., Harrington, L. A., Kleinberg, E. M., and Zwicker, W. S. (1977). Flipping properties: a unifying thread in the theory of large cardinals. *Annals of Mathematical Logic*, 12:25–58.
- Dodd, A. J. (1982). The core model. *London Mathematical Society Lecture Note Series*, 61.
- Donder, D., Jensen, R. B., and Koppelberg, B. J. (1981). Some applications of the core model. In *Set theory and model theory*, pages 55–97. Springer.
- Feng, Q. (1990). A hierarchy of Ramsey cardinals. *Annals of Pure and Applied Logic*, 49(2):257–277.
- Gitman, V. (2011). Ramsey-like cardinals. *The Journal of Symbolic Logic*, 76(2):519–540.
- Gitman, V. and Schindler, R. (2015). Virtual large cardinals. *The Proceedings of the Logic Colloquium*.
- Gitman, V. and Welch, P. (2011). Ramsey-like cardinals II. *The Journal of Symbolic Logic*, 76(2):541–560.
- Holy, P. and Schlicht, P. (2018). A hierarchy of Ramsey-like cardinals. *Fundamenta Mathematicae*, 242:49–74.
- Jensen, R. and Steel, J. (2013). K without the measurable. *The Journal of Symbolic Logic*, 78(3):708–734.
- Kanamori, A. (2008). *The Higher Infinite: Large cardinals in set theory from their beginnings*. Springer Science & Business Media.
- Kanamori, A. and Magidor, M. (1978). The evolution of large cardinal axioms in set theory. In *Higher set theory*, pages 99–275. Springer.
- Kellner, J. and Shelah, S. (2011). More on the pressing down game. *Archive for Mathematical Logic*, 50(3-4):477–501.
- Mitchell, W. J. (1979). Ramsey cardinals and constructibility. *The Journal of Symbolic Logic*, 44(2):260–266.
- Schindler, R. (1997). Weak covering at large cardinals. *Mathematical Logic Quarterly*, 43:22–28.
- Schindler, R. (2006). Iterates of the core model. *The Journal of Symbolic Logic*, 71(1):241–251.
- Schindler, R.-D. (2000). Proper forcing and remarkable cardinals. *The Bulletin of Symbolic Logic*, 6(2):176–184.
- Sharpe, I. and Welch, P. D. (2011). Greatly Erdős cardinals with some generalizations to the Chang and Ramsey properties. *Annals of Pure and Applied Logic*, 162:863–902.
- Zeman, M. (2002). Inner models and large cardinals. In *Series in Logic and its Applications*, volume 5. de Gruyter, Berlin, New York.

UNIVERSITY OF BRISTOL
TYNDALL AVE
BRISTOL BS8 1TH, UK
E-mails: dan.nielsen@bristol.ac.uk & p.welch@bristol.ac.uk