## A THEORY OF PAIRS FOR NON-VALUATIONAL STRUCTURES

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ABSTRACT. Given a weakly o-minimal structure  $\mathcal{M}$  and its o-minimal completion  $\overline{\mathcal{M}}$ , we first associate to  $\overline{\mathcal{M}}$  a canonical language and then prove that  $Th(\mathcal{M})$  determines  $Th(\overline{\mathcal{M}})$ . We then investigate the theory of the pair  $(\overline{\mathcal{M}}, \mathcal{M})$ in the spirit of the theory of dense pairs of o-minimal structures, and prove, among other results, that it is near model complete, and every definable open subset of  $\overline{\mathcal{M}}^n$  is already definable in  $\overline{\mathcal{M}}$ .

We give an example of a weakly o-minimal structure which interprets  $\overline{M}$  and show that it is not elementarily equivalent to any reduct of an o-minimal trace.

### 1. INTRODUCTION

An expansion  $\mathcal{M}$  of an ordered group is *weakly o-minimal non-valuational* (below we use "non-valuational" for short) if it is weakly o-minimal (every definable subset of M is a finite union of convex sets) and does not admit any definable non-trivial convex sub-groups. Non-valuational structures were introduced in [6] and more systematically studied in [10] and [11]. In those works Wencel showed that to a non-valuational structure  $\mathcal{M}$ one can associate an o-minimal structure  $\overline{\mathcal{M}}$ , whose universe is  $\overline{M}$  – the definable Dedekind completion of  $\mathcal{M}$ – and with the additional property that the structure which  $\overline{\mathcal{M}}$  induces on (the natural embedding of) M (in  $\overline{\mathcal{M}}$ ) is precisely the structure  $\mathcal{M}$ . Wencel called the structure  $\overline{\mathcal{M}}$  the canonical o-minimal completion of  $\mathcal{M}$ . In [5] Keren shows that  $\overline{\mathcal{M}}$  has the same definable sets as the structure  $\mathcal{M}^*$ , whose atomic sets are all sets of the form  $\operatorname{cl}_{\overline{M}}(S) \subseteq \overline{M}^n$  for  $\mathcal{M}$ -definable  $S \subseteq M^n$ , (see Proposition 2.7 below). Both Wencel and Keren's constructions have the problem that the signatures of the resulting structures depend on the structure  $\mathcal{M}$ , rather than on its signature.

In the present paper we address this problem by considering, for  $A \subseteq M$ , structures of the form  $\mathcal{M}_A^*$  whose atomic sets are all sets of the form  $\operatorname{cl}_{\overline{M}}(S)$  for S an  $\mathcal{M}$ -definable set over A. The starting point of the present work, and the main result of Section 2 is:

**Theorem 1.** Let  $\mathcal{M}$  be a non-valuational structure. Then  $\mathcal{M}_{\emptyset}^*$  and  $\mathcal{M}^*$  have the same definable sets. Moreover, if  $\mathcal{M} \equiv \mathcal{N}$  then  $\mathcal{M}_{\emptyset}^* \equiv \mathcal{N}_{\emptyset}^*$ .

This result shows that to a non-valuational theory T we can associate an o-minimal theory  $T^*$  which can be viewed as an invariant of T. Consequently, any of the o-minimal properties of  $T^*$  can reflect on the weakly o-minimal T and vice versa. This plays a crucial role in the proof of Theorem 3 below.

Section 5 is dedicated to the study of the theory of the pair  $\mathcal{M}^P = (\mathcal{M}^*_{\emptyset}, \mathcal{M})$  for  $\mathcal{M}$  non-valuational, in the spirit of van den Dries' study of o-minimal dense pairs (see [9]). Our main result is the following:

**Theorem 2.** Let  $\mathcal{M}$  be non-valuational.

formulas of the form

(1) If M ≡ N then M<sup>P</sup> ≡ N<sup>P</sup>. We let T<sup>P</sup> = Th(M<sup>P</sup>) and assume Ñ = (N', N) ⊨ T<sup>P</sup>.
(2) If Y ⊆ (N')<sup>n</sup> is Ø-definable in Ñ then it can be written as a boolean combination of sets defined by

(1) 
$$\exists x_1 \cdots \exists x_k (\bigwedge_{i=1}^k x_i \in P \& \varphi(x_1, \dots, x_k, y),$$

and  $\varphi(x, y)$  is a formula of the o-minimal structure  $\mathcal{M}'$ . (3) If  $X \subseteq N^k$  is definable in  $\tilde{\mathcal{N}}$  over  $A \subseteq N$  then X is already definable in the weakly o-minimal  $\mathcal{N}$ .

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(4) If  $U \subseteq (N')^k$  is a definable open set in  $\tilde{\mathcal{N}}$  then U is already definable in the o-minimal structure  $\mathcal{N}'$ . In particular,  $\tilde{\mathcal{N}}$  has an o-minimal open core.

The above results show that pairs  $(\mathcal{M}', \mathcal{M})$  as above fit into the setting of recent works by Eleftherious, Gunaydin and Hieronymi (see for example [3]) on expansions of o-minimal structures by dense predicates.

Non-valuational structures arise naturally in the study of dense pairs of o-minimal structures. Namely, if  $\mathcal{M} \prec \mathcal{N}$  are o-minimal expansions of ordered groups and M is dense in N then the structure induced on M from  $\mathcal{N}$  is non-valuational (weak o-minimality follows from [1] and non-valuationality is easy, see e.g., [4]). Since every ordered group which is a reduct of a non-valuational structure, or even elementarily equivalent to one, is also such, a question arises whether every non-valuational structure arises in this manner.

First, some terminology. A non-valuational structure  $\mathcal{M}$  is called an *o-minimal trace* if there is a dense pair  $\mathcal{M}_0 \prec \mathcal{N}$  such that  $\langle M_0, < \rangle = \langle M, < \rangle$  (i.e., the structures  $\mathcal{M}_0$  and  $\mathcal{M}$  have the same underlying ordered set) and the induced structure on M in the pair  $(\mathcal{N}, \mathcal{M}_0)$  has the same definable sets as  $\mathcal{M}$  (see [4] for details). In [4] we showed that an ordered reduct of a non-valuational o-minimal trace need not be an o-minimal trace itself, and that the class of reducts of o-minimal traces is not closed under elementary equivalence. In the present paper we show that even after closing the class of o-minimal traces under reducts and elementary equivalence we still do not cover all non-valuational structures:

**Theorem 3.** Let  $\mathbb{Q}^{\sqrt{2}}$  be the expansion of  $(\mathbb{Q}, +)$  by the predicate  $y < \sqrt{2}x$ . Then  $\mathbb{Q}^{\sqrt{2}}$  is non-valuational and not elementarily equivalent to a reduct of an o-minimal trace.

Along the way we reveal a new dividing line between two types of non-valuational structures:

- *Tight* structures (of which  $\mathbb{Q}^{\sqrt{2}}$  is a typical example), in which  $\mathcal{M}^*$  is interpretable in  $\mathcal{M}$ . These are *small* (in the sense of [9]), and in that respect differ significantly from o-minimal traces.
- Non-tight structures, whose theory resembles to a much greater extent that of o-minimal traces.

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## 2. PRELIMINARIES

We fix a non-valuational structure  $\mathcal{M}$  and its definable completion  $\overline{M}$ . Recall that the elements of  $\overline{M}$  are all (unique) realizations of definable cuts in  $\mathcal{M}$ . These will be identified here with the definable open subsets of M that are bounded above and downward closed. The set  $\overline{M}$  is equipped with ordering by inclusion. The structure  $\langle M, < \rangle$  is naturally embedded into  $\overline{M}$  via the map  $a \mapsto (-\infty, a)$ , and from now on we will view M as a subset of  $\overline{M}$ . The topology on  $\overline{M}$  and  $\overline{M}^n$  are the order and the product topology, respectively. We let  $cl_{\overline{M}}(-), \partial_{\overline{M}}(-)$  denote the corresponding topological operations in  $\overline{M}^n$ . Unless otherwise stated, all definability below refers to the structure  $\mathcal{M}$ .

Recall that a partial function  $f: M^n \to \overline{M}$  is said to be *definable* if the set  $\{(x, y) \in M^{n+1} : y < f(x)\}$  is definable. Equivalently, the family of cuts  $\{y \in M : y < f(x)\}$ , for  $x \in M^n$ , is a definable family (and can be identified with a sort in  $\mathcal{M}$ ).

We start by collecting several useful facts concerning the relationship of  $\mathcal{M}$  and various structures on  $\overline{M}$ . We first recall the definition of a strong cell  $C \subseteq M^n$  from [10] <sup>1</sup> The definition will be inductive in n and for the induction step we will also associate inductively to each strong cell  $C \subseteq M^n$  its so-called *iterative convex hull*  $\overline{C}$ ,  $C \subseteq \overline{C} \subseteq \overline{M}^n$ . Having defined C and  $\overline{C}$  below, we say that an  $\mathcal{M}$ -definable function  $f : C \to \overline{M}$  is strongly continuous if it extends continuously to  $\overline{f} : \overline{C} \to \overline{M}$ , and in addition either  $f(C) \subseteq M$  or  $f(C) \subseteq \overline{M} \setminus M$ . We are now ready to state the definition:

**Definition 2.1.** A set  $C \subseteq M$  is a strong cell if it is either a point, in which case  $\overline{C} = C$ , or an open convex set, in which case  $\overline{C}$  is defined as the convex hull of C in  $\overline{M}$ .

Inductively, If  $C \subseteq M^n$  is a strong cell (with the associated  $\overline{C} \subseteq \overline{M}^n$ ) and  $f, g : C \to \overline{M}$  are strongly continuous with  $\overline{f}(x) < \overline{g}(x)$  for all  $x \in \overline{C}$  (note the strong assumption here!) then  $\Gamma_f(C)$  – the graph of f on C

<sup>&</sup>lt;sup>1</sup>We are using Wencel's definition, in a slightly different formulation than in [6].

- and  $(f,g)_C := \{(x,y) \in M^{n+1} : f(x) < y < g(y)\}$  are strong cells. In the first case the iterative convex hull is defined to be the graph of the extension  $\overline{f} : \overline{C} \to \overline{M}$ , and in the second case it is defined to be

$$\{(x, y) \in \overline{M}^{n+1} : x \in \overline{C} \& \overline{f}(x) < y < \overline{g}(x)\}.$$

- **Remark 2.2.** (1) It is easy to verify that for each strong cell  $C \subseteq M^n$  there exists a homeomorphic projection  $\pi_C : C \to D \subseteq M^k$  onto k of the coordinates,  $k \leq n$ , whose image is an open strong cell in  $M^k$ . In this case dim C := k. The coordinate functions of  $\pi_C^{-1}$  are strongly continuous on D.
  - (2) Notice that each strong cell C is a subset of M<sup>n</sup> that is definable in M, and furthermore the various functions f and g in the inductive definition of C are definable in M, even though they might take values in M \ M. However, in general C ⊆ M<sup>n</sup> is not definable in M in any obvious sense because it might not be contained in finitely many sorts in M.

We can now describe Wencel's canonical completion  $\overline{\mathcal{M}}$ , but we refine his definition so we have a better control of parameters.

**Definition 2.3.** Given  $A \subseteq M$ , we let  $\overline{\mathcal{M}}_A$  be the expansion of  $\overline{M}$  by all iterative convex hulls  $\overline{C} \subseteq \overline{M}^n$ , so that  $C \subseteq M^n$  is a strong cell defined over A.

It is easy to see that the order relation  $\langle$  is an atomic relation in  $\overline{\mathcal{M}}_A$ . Since  $\langle M, \langle , + \rangle$  is divisible, [6], and M is dense in  $\overline{M}$ , the group operation extends uniquely to  $\overline{M}$ , so it is strongly continuous, and its graph  $C_+$  is a strong cell whose iterative convex hull is the graph of a group operation on  $\overline{M}$  that we still denote by +.

We now collect some of the main results from [11]

**Fact 2.4.** Let  $\mathcal{M}$  be a weakly o-minimal non-valuational structure.

- (1) Every A-definable set has a decomposition into finitely many strong cells, each defined over A.
- (2) The structure  $\overline{\mathcal{M}}_M$  is o-minimal.
- (3) If  $X \subseteq \overline{M}^n$  is definable in  $\overline{\mathcal{M}}$  then  $X \cap M^n$  is definable in  $\mathcal{M}$ .

In [5], the language of  $\overline{\mathcal{M}}_A$  was replaced by another one, which we find more convenient to work with.

**Definition 2.5.** Given  $A \subseteq M$ , and an A-definable set  $X \subseteq M^n$  in  $\mathcal{M}$ , we associate to X a predicate symbol  $\hat{X}$ . We interpret  $\hat{X}$  in  $\overline{M}^n$  as the topological closure of X in  $\overline{M}^n$ , denoted by  $\operatorname{cl}_{\overline{M}}(X)$ , and let  $\mathcal{M}_A^*$  be the expansion of  $\overline{M}$  by all  $\hat{X}$ , for  $X \subseteq M^n$  definable over A.

It was proved in [5] that the structures  $\overline{\mathcal{M}}_M$  and and  $\mathcal{M}_M^*$  have the same definable sets. We re-prove here a more precise version. We first prove:

**Lemma 2.6.** If  $C \subseteq M^n$  is a strong cell then  $\operatorname{cl}_{\overline{M}}(C) = \operatorname{cl}_{\overline{M}}(\overline{C})$ .

*Proof.* Since  $C \subseteq \overline{C}$  it suffices to show that  $\overline{C} \subseteq \hat{C}$  for every strong cell C. We use induction on n.

If  $C \subseteq M$  the claim is obvious. Now, suppose that  $\overline{C} \subseteq \hat{C}$  for some strong cell C and let  $f_1, f_2 < f_3$  be strongly continuous such that the range of  $f_1$  is in M. We let  $C_1 = \Gamma(f_1)_C$  and  $C_2 = (f_2, f_3)_C$  be the associated strong cells, and will show that  $\overline{C}_1 \subseteq \hat{C}_1$  and  $\overline{C}_2 \subseteq \hat{C}_2$ .

Let  $(c,m) \in \overline{C} \times \overline{M}$ . If  $(c,m) \in \overline{C}_1$  then  $\overline{f}_1(c) = m$ . But then since  $c \in \overline{C}$  and  $\Gamma(f_1)_C$  is dense in  $\Gamma(\overline{f}_1)_{\overline{C}}$  (because C is dense  $\overline{C}$ ) then (c,m) is a limit point of  $f_1$  and therefore  $(c,m) \in \hat{C}_1$ . If  $(c,m) \in \overline{C}_2$  then  $\overline{f}_2(c) < m < \overline{f}_3(c)$ , and again since  $c \in \overline{C}$  and  $(f_2, f_3)_C$  is dense in  $(\overline{f}_2, \overline{f}_3)_{\overline{C}}$  then (c,m) is a limit point of  $(f_2, f_3)_C$  and therefore  $(c,m) \in \hat{C}_2$ .

We can now prove:

**Proposition 2.7.** For every  $A \subseteq M$ , the (o-minimal) structures  $\mathcal{M}_A^*$  and  $\overline{\mathcal{M}}_A$  have the same  $\emptyset$ -definable sets (so in particular the same definable sets).

*Proof.* We first show that every atomic set in  $\mathcal{M}_A^*$  is  $\emptyset$ -definable in  $\overline{\mathcal{M}}_A$ . So we take an A-definable  $X \subseteq M^k$ , and consider its closure  $\hat{X} \subseteq \overline{M}^k$ . By Fact 2.4, X can be written as the union  $\bigcup_{i=1}^k C_i$  of strong cells that are definable over A in  $\mathcal{M}$ . By Lemma 2.6, each  $\overline{C}_i$  is dense in  $\operatorname{cl}_{\overline{M}}(C_i)$ . It follows that  $\operatorname{cl}_{\overline{M}}(X) = \bigcup_{i=1}^k \operatorname{cl}_{\overline{M}}(\overline{C}_i)$ . Since each  $\overline{C}_i$  is  $\emptyset$ -definable in  $\overline{\mathcal{M}}_A$ , and the closure operation is itself definable, it follows that  $\operatorname{cl}_{\overline{M}}(X)$  is  $\emptyset$ -definable in  $\overline{\mathcal{M}}_A$ .

For the other inclusion, we need to see that for every strong cell  $C \subseteq M^n$  that is definable over A, the set  $\overline{C}$  is  $\emptyset$ -definable in  $\mathcal{M}^*_A$ . This is done by induction on n.

For 0-cells in  $\mathcal{M}$  this is clear. If  $C \subseteq M$  is a 1-cell then  $\overline{C}$  is an open interval (a, b) in  $\overline{M}$ . The interval [a, b] is  $\emptyset$ -definable in  $\mathcal{M}_A^*$ , hence so is  $\overline{C}$ . So we now assume that we have proved the result for all strong cells in  $M^n$  and we prove it for strong-cells in  $M^{n+1}$ . Let  $C \subseteq M^n$  be a strong cell defined over A. Let  $f : C \to M$  be a strongly continuous function definable in  $\mathcal{M}$  over A, and let Y be  $\Gamma_f$ , the graph of f. Then  $\overline{Y} := \{(x, \overline{f}(x)) : x \in \overline{C}\}$ . We have to show that  $\overline{Y}$  is  $\emptyset$ -definable in  $\mathcal{M}_A^*$ . As  $\overline{f}$  is continuous we get that  $\overline{Y} = \operatorname{cl}_{\overline{M}}(\Gamma_f) \cap (\overline{C} \times \overline{M})$ , which is  $\emptyset$ -definable in  $\mathcal{M}_A^*$  by the inductive hypothesis.

Now let  $f, g : C \to M$  be A-definable strongly continuous functions in  $\mathcal{M}$ , with f < g (unlike the above, we cannot assume here that they take values in M). We have to show that the iterative convex hull of  $Y := (f, g)_C$  is  $\emptyset$ -definable in  $\mathcal{M}^*_A$ . By definition,

$$\bar{Y} = \{(x, y) : x \in \bar{C}, \bar{f}(x) < y < \bar{g}(x)\}.$$

Since, by induction  $\overline{C}$  is  $\emptyset$ -definable in  $\mathcal{M}_A^*$ , it will suffice to show that  $\overline{f}$  (and similarly  $\overline{g}$ ) is  $\emptyset$ -definable in  $\mathcal{M}_A^*$ . If f is the constant function  $-\infty$ , then there is nothing to prove. So we assume this is not the case. By definition, the set

$$F := \{ (x, y) : x \in C, y < f(x) \}$$

is A-definable in  $\mathcal{M}$ . For every  $c \in \overline{C}$  let

$$s(c) := \sup\{y \in M : (c, y) \in \operatorname{cl}_{\bar{M}}(F)\}.$$

Since f is strongly continuous, s(c) is well defined, and by definition it coincides with f on C. Since C is dense in  $\overline{C}$  and  $\overline{f}$  is the unique continuous extension of f to  $\overline{C}$ , necessarily  $s = \overline{f}$ , and as s is  $\emptyset$ -definable in  $\mathcal{M}_A^*$ , we are done.

From now on we can use interchangeably the structures  $\mathcal{M}_A^*$  and  $\bar{\mathcal{M}}_A$ . Notice however, that the language of  $\bar{\mathcal{M}}_A$  depends on the specific structure  $\mathcal{M}$ , thus for different  $\mathcal{M}$  and  $\mathcal{N}$ , even if elementarily equivalent, the structures  $\bar{\mathcal{M}}_M$  and  $\bar{\mathcal{N}}_N$  are of different signature. One of the initial goals of this work was to obtain a uniform signature by showing that the definable sets in  $\bar{\mathcal{M}}_{\emptyset}$  ad  $\bar{\mathcal{M}}_M$  are the same. We need the following observations

**Proposition 2.8.** (1) Every  $\emptyset$ -definable set in  $\mathcal{M}$  can be written as a boolean combination of  $\emptyset$ -definable sets each of which is the closure of an open  $\emptyset$ -definable set. In particular, this is true if  $\mathcal{M}$  is o-minimal.

(2) The o-minimal structure  $\mathcal{M}_M^*$  eliminates quantifiers. Moreover, it is sufficient to take as atomic relations all  $\operatorname{cl}_{\overline{M}}(X)$  with  $X \subseteq M^n$  an open definable set.

*Proof.* (1) We first prove the result for an arbitrary definable open set  $X \subseteq M^n$ . Note that  $X = cl(X) \setminus \partial(X)$  (here  $\partial(X)$  is the boundary of X), and then that

$$\partial(X) = \operatorname{cl}(X) \cap \operatorname{cl}(M^n \setminus \operatorname{cl}(X)).$$

The set on the right is of the desired form, so we are done.

For an arbitrary definable  $X \subseteq M^n$ , we apply strong cell decomposition, so we may assume that X is a cell. Hence, X is either a point or the graph of a definable map f from an open cell  $C \subseteq M^{n-k}$  into  $M^k$  (the n-k coordinates need not be the first ones), and each of the coordinate functions of f are strongly continuous.

Thus it is sufficient to show that the graph of each strongly continuous  $f_i : C \to M$  is definable in the desired form. By the continuity of  $f_i$ , such a graph can be written as the complement in  $C \times M$  of the open set:

$$\{(x, y) \in C \times M : y > f_i(x)\} \cup \{(x, y) \in C \times M : y < f_i(x)\}.$$

Since each of the open sets can be defined in the required form, so is the graph of  $f_i$ , and hence so is X.

For (2), we first apply (1) to the o-minimal structure  $\mathcal{M}_M^*$  and reduce the problem to definable sets  $X \subseteq M^n$ , which are the closure of an open definable set  $U \subseteq \overline{M}^n$ . Since  $M^n$  is dense in  $\overline{M}^n$ ,  $\operatorname{cl}_{\overline{M}}(U) = \operatorname{cl}_{\overline{M}}(U \cap M^n)$ . By fact 2.4, the set  $U \cap \overline{M}^n$  is definable in  $\mathcal{M}$  (possibly over parameters). We now apply (1).

In the text the first part of the above proposition will be applied, mostly, when  $\mathcal{M}$  is, in fact, o-minimal.

**Lemma 2.9.** Let  $C \subseteq M^{k+n}$  be a strong cell,  $a \in \pi(C)$ , where  $\pi$  is the projection onto the first k-coordinate. Let  $C_a = \{x \in M^n : (a, x) \in C\}$ . Then

(1) 
$$C_a$$
 is a strong cell.  
(2)  $(\overline{C})_a = \overline{C_a}$ .

*Proof.* It is sufficient to prove the result for k = 1 (and then proceed by induction). This is straightforward from the definition of a strong cell.

**Theorem 2.10.** For every  $A \subseteq M$ , the structures  $\overline{\mathcal{M}}_M$  and  $\overline{\mathcal{M}}_A$  have the same definable sets.

*Proof.* Absorbing A to the language we, at this stage, assume that  $A = \emptyset$ . We first claim that for every  $n \in \mathbb{N}$ , we have

(2) 
$$\{Y \cap M^n : Y \subseteq \overline{M}^n \text{ definable in } \overline{\mathcal{M}}_{\emptyset}\} = \{Y \cap M^n : Y \subseteq \overline{M}^n \text{ definable in } \overline{\mathcal{M}}_M\}.$$

Since  $\overline{\mathcal{M}}_{\emptyset}$  is a reduct of  $\overline{\mathcal{M}}_M$  it is sufficient to prove the right-to-left inclusion. We first show: For every  $\emptyset$ -definable  $X \subseteq M^n$ , there exists a  $\emptyset$ -definable  $Y \subseteq \overline{M}^n$  in  $\overline{\mathcal{M}}_{\emptyset}$  such that  $Y \cap M^n = X$ . Indeed, X has a decomposition into  $\emptyset$ -definable strong cells (see Fact 2.4), and for each  $\emptyset$ -definable strong cell  $C_i$  we have  $\overline{C}_i \cap M^n = C_i$ , so  $Y = \bigcup_i \overline{C}_i$  is the desired set.

Now, let  $Z \subseteq M^n$  be definable in  $\overline{\mathcal{M}}_M$ . By Fact 2.4,  $Z \cap M^n$  is definable in  $\mathcal{M}$ , possibly over parameters. Hence, it is of the form  $X_a$ , for some  $X \subseteq M^{n+k}$  which is  $\emptyset$ -definable in  $\mathcal{M}$  and  $a \in M^k$ . By what we just shown, there is  $Y \subseteq \overline{M}^k$  which is  $\emptyset$ -definable in  $\overline{\mathcal{M}}_{\emptyset}$ , such that  $X = Y \cap M^{n+k}$ . Hence,

$$Z = (Y \cap M^{n+k})_a = Y_a \cap M^n,$$

and  $Y_a$  is definable in  $\overline{\mathcal{M}}_{\emptyset}$ . This ends the proof of (2).

We now make the following general observation:

**Lemma 2.11.** Let  $\langle N, \langle \rangle$  be a densely ordered set, with  $M \subseteq N$  a dense subset. Assume that  $\mathcal{N}_1$ ,  $\mathcal{N}_2$  are two *o*-minimal expansions of  $\langle N, \langle \rangle$  with the property that for every  $n \in \mathbb{N}$ , we have

(3) 
$$\{Y \cap M^n : Y \subset \overline{M}^n \text{ definable in } \mathcal{N}_1\} = \{Y \cap M^n : Y \subset \overline{M}^n \text{ definable in } \mathcal{N}_2\}.$$

Then  $\mathcal{N}_1$  and  $\mathcal{N}_2$  have the same definable sets.

*Proof.* It easily follows from the assumptions that we have

(4) 
$$\{Y \cap M^n : Y \subseteq \overline{M}^n \text{ open definable in } \mathcal{N}_1\} = \{Y \cap M^n : Y \subseteq \overline{M}^n \text{ open definable in } \mathcal{N}_2\}.$$

By Proposition 2.8 (1), it is enough to know that for every open  $U \subseteq N^n$ , the set cl(U) is definable in  $\mathcal{N}_1$  if and only if it is definable in  $\mathcal{N}_2$ . However, since M is dense in N, it is enough to consider sets of the form  $cl(U \cap M^n)$ . By (4), both collections of sets of the form  $U \cap M^n$ , where U is definable in either  $\mathcal{N}_1$  or in  $\mathcal{N}_2$ , are the same.

In order to prove Theorem 2.10, we apply Lemma 2.11 to the structures  $\overline{\mathcal{M}}_{\emptyset}$  and  $\overline{\mathcal{M}}_{M}$  using (2).

# 3. The structure $\mathcal{M}_A$ and elementary extensions

Again, we let  $\mathcal{M}$  be a fixed non-valuational structure. From now on we shall work with  $\mathcal{M}_A^*$  rather than  $\overline{\mathcal{M}}_A$ .

3.1. The canonical completion and elementary extensions. Let  $\mathcal{N}$  be an elementary extension of  $\mathcal{M}$ . Every definable cut C in  $\mathcal{M}$  has a natural realization  $C(\mathcal{N})$  in  $\mathcal{N}$  and so  $\overline{M}$  can be embedded into  $\overline{N}$ . Under this embedding, if  $n \in \overline{N}$  is the supremum of a cut in N which is definable over some  $A \subseteq M$  then n is already in  $\overline{M}$ . We have:



Where  $\iota$  is the natural embedding of (M, <) in  $(\overline{M}, <)$ . We now fix an arbitrary  $A \subseteq M$  and consider the structures  $\mathcal{M}_A^*$  and  $\mathcal{N}_A^*$ . Both structures are in the language  $\mathcal{L}_A^*$ , and we claim that  $\mathcal{M}_A^*$  is a substructure of  $\mathcal{N}_A^*$ . Indeed, first note that for a fixed  $x \in \overline{M}^n$ , and  $\epsilon > 0$  in M, the set  $B(x, \epsilon) \cap M^n = \{y \in M^n : |x - y| < \epsilon\}$  is definable in  $\mathcal{M}$  and moreover, it is uniformly definable as  $\epsilon$  varies in  $M_{>0}$  (x still fixed). It easily follows that for  $x \in \overline{M}^n$ , being in the closure of a definable  $X \subseteq M^n$  is a first order property. Namely, for  $x \in \overline{M}^n$ ,

$$x \in \operatorname{cl}_{\bar{M}}(X(M)) \Leftrightarrow x \in \operatorname{cl}_{\bar{N}}(X(N)).$$

Said differently,  $\hat{X}(N) \cap \overline{M}^n = \hat{X}(M)$ , so  $\mathcal{M}_A^*$  is a substructure of  $\mathcal{N}_A^*$ .

Our goal is to show that  $\mathcal{M}_A^*$  is in fact an elementary substructure of  $\mathcal{N}_A^*$ . We do that in several steps.

**Lemma 3.1.** Assume that  $A \subseteq M$  and that  $\mathcal{M}$  is  $|A|^+$ -saturated. If  $Y \subseteq \overline{M}^n$  is  $\emptyset$ -definable in  $\mathcal{M}^*_A$  then  $Y \cap M^n$  is A-definable in  $\mathcal{M}$ .

*Proof.* By fact 2.4,  $Y \cap M^n$  is definable in  $\mathcal{M}$ . By the saturation assumption it is enough to show that any automorphism of  $\mathcal{M}$  which fixes A point-wise leaves  $Y \cap M^n$  invariant. Let  $\alpha : M \to M$  be such an automorphism. We claim that  $\alpha$  has a (unique) extension to a bijection  $\bar{\alpha} : \bar{M} \to \bar{M}$  which is an automorphism of  $\mathcal{M}^*_A$ . Because  $\alpha$  is an automorphism of  $\mathcal{M}$  it sends definable cuts to definable cuts so extends naturally to  $\bar{\alpha} : \bar{M} \to \bar{M}$ . The map  $\bar{\alpha}$  is an order preserving bijection so in particular continuous on  $\bar{M}$ . To see that  $\bar{\alpha}$  is an automorphism of  $\mathcal{M}^*_A$ , let  $X \subseteq M^n$  be A-definable and consider its closure  $\hat{X}$ . Since  $\alpha(X) = X$ , continuity implies that  $\bar{\alpha}(\hat{X}) = \hat{X}$ , thus  $\bar{\alpha}$  is an automorphism of  $\mathcal{M}^*_A$ .

Since Y was  $\emptyset$ -definable in  $\mathcal{M}_A^*$  it is left invariant under  $\bar{\alpha}$ , and because  $\bar{\alpha}(M) = M$ , we have

$$\alpha(Y \cap M^n) = \bar{\alpha}(Y \cap M^n) = \bar{\alpha}(Y) \cap \bar{\alpha}(M^n) = Y \cap M^n.$$

**Lemma 3.2.** For  $A \subseteq M$  arbitrary, if  $\mathcal{M} \prec \mathcal{N}$  then  $\mathcal{M}_A^* \prec \mathcal{N}_A^*$ .

*Proof.* First note that we may assume that  $\mathcal{N}$  is sufficiently saturated. Indeed, we may consider  $\mathcal{N}' \succ \mathcal{N}$  which is saturated enough. The above would then imply that  $\mathcal{M}_A^* \prec (\mathcal{N}_A')^*$  and  $\mathcal{N}_A^* \prec (\mathcal{N}_A')^*$ , from which it follows that  $\mathcal{M}_A^* \prec \mathcal{N}_A^*$ .

By The Tarski-Vaught Criterion, it is enough to prove, for every nonempty  $Y \subseteq \overline{N}$  which is definable in  $\mathcal{N}_A^*$  over  $\overline{M}$ , that  $Y \cap \overline{M} \neq \emptyset$ .

Since  $\mathcal{N}_A^*$  is an o-minimal expansion of a group, Y contains some element  $b \in \operatorname{dcl}_{\mathcal{N}_A^*}(\overline{M})$ . So, there exists a finite tuple  $a = (a_1, \ldots, a_r)$  from  $\overline{M}$ , such that  $b \in \operatorname{dcl}_{\mathcal{N}_A^*}(a)$ . Each  $a_i$  realizes a cut in M, definable in  $\mathcal{M}$ over some finitely many parameters. Thus there is a finite  $F \subseteq M$  such that each  $a_i$  realizes a cut definable over F. If we now let  $A' = A \cup F \subseteq M \subseteq N$  then clearly every element in A' is  $\emptyset$ -definable in  $\mathcal{N}_{A'}^*$ , hence b is in  $\operatorname{dcl}_{\mathcal{N}_{A'}^*}(\emptyset)$  so the set  $(-\infty, b)$  is  $\emptyset$ -definable in  $\mathcal{N}_{A'}^*$ . Since  $\mathcal{N}$  is sufficiently saturated it follows from Lemma 3.1 that  $(-\infty, b) \cap N$  is A'-definable in  $\mathcal{N}$ .

Since  $\mathcal{M} \prec \mathcal{N}$  and  $A' \subseteq M$  it follows, as we already noted above, that  $b \in \overline{M}$ , so  $X \cap \overline{M} \neq \emptyset$ . Thus  $\mathcal{M}_A^* \prec \mathcal{N}_A^*$ .

**Note**: It only makes sense to compare  $\mathcal{M}_A^*$  and  $\mathcal{N}_A^*$  for  $A \subseteq M$ , since otherwise the two structures do not have a common language.

Finally, we can now prove:

**Theorem 3.3.** For  $A \subseteq M$  (with no saturation assumption), assume that  $X \subseteq \overline{M}^n$  is  $\emptyset$ -definable in the structure  $\mathcal{M}^*_A$ . Then  $X \cap M^n$  is A-definable in  $\mathcal{M}$ . In particular, if  $f : \overline{M}^n \to \overline{M}$  is  $\emptyset$ -definable in  $\mathcal{M}^*_A$  then  $f \upharpoonright M^n : M^n \to \overline{M}$  is A-definable in  $\mathcal{M}$ .

*Proof.* We consider an elementary extension  $\mathcal{N}$  of  $\mathcal{M}$  that is  $|A|^+$ -saturated. By Lemma 3.2, we have  $\mathcal{M}_A^* \prec \mathcal{N}_A^*$  and by Lemma 3.1, the set  $Y = X(\bar{N}) \cap N^n$  is definable in  $\mathcal{N}$  over A. Since  $\mathcal{M} \prec \mathcal{N}$  we can conclude that  $Y \cap M^n = X(\bar{N}) \cap M^n$  is also definable over A in  $\mathcal{M}$ . It is left to see that this last set equals  $X \cap M^n$ . Because  $\mathcal{M}_A^* \prec \mathcal{N}_A^*$  we have  $X(\bar{N}) \cap \bar{M}^n = X$ , and therefore

$$Y \cap M^n = X(\bar{N}) \cap M^n = X \cap M^n.$$

For the second clause, just note that the set  $\{x \in M^n : x < f(x)\}$  is the intersection of a  $\emptyset$ -definable subset of  $\overline{M}^n$  with  $M^n$ .

We now return to Proposition 2.8 and Theorem 2.10 and prove finer results:

**Proposition 3.4.** *For any*  $A \subseteq M$ *,* 

- (2) If  $X \subseteq \overline{M}^n$  is  $\emptyset$ -definable in  $\mathcal{M}^*_A$  then it is A-definable in  $\mathcal{M}^*_{\emptyset}$ .

*Proof.* (1) We may repeat the short argument in the proof of 2.8 with the additional data given by Theorem 3.3, that whenever  $X \subseteq \overline{M}^n$  is  $\emptyset$ -definable in  $\mathcal{M}^*_A$ , the set  $X \cap M^n$  is A-definable in  $\mathcal{M}$ . For (2), assume that Z is  $\emptyset$ -definable in  $\mathcal{M}^*_A$ . By (1), Z is a boolean combination of atomic sets (with no extra parameters), so it is sufficient to prove that each atomic such set Z is A-definable in  $\mathcal{M}^*_{\emptyset}$ . By the first paragraph of the proof of Proposition 2.7  $Z = \bigcup_{i=1}^k \operatorname{cl}_{\overline{M}}(\overline{C}_i)$  for some A-definable strong cells  $C_i \subseteq M^n$ . So  $C_i = (D_i)_a$  for some  $\emptyset$ -definable set  $D_i$  nd  $a \subseteq A$ . By strong cell decomposition, each  $D_i$  is itself a finite union of  $\emptyset$ -definable strong cells, so we may write each  $C_i$  as a union of the form  $\bigcup_i (D_{i,j})_a$ , where each  $D_{i,j}$  is a  $\emptyset$ -definable strong cell.

By Lemma 2.9 we know that  $(D_{i,j})_a = (\overline{D_{i,j}})_a$  for every j. The right-hand side of this equation is A-definable in  $\mathcal{M}^*_{\emptyset}$ , and hence so is its  $\overline{M}$ -closure. Therefore the closure of each  $C_i$  is a finite union of sets that are A-definable in  $\mathcal{M}^*_{\emptyset}$ . The conclusion follows.

Since any two elementarily equivalent structures have a common elementary extension, we can also conclude from Lemma 3.2:

**Corollary 3.5.** If  $\mathcal{M}$  is non-valuational and  $\mathcal{N} \equiv \mathcal{M}$  then  $\overline{\mathcal{M}}_{\emptyset} \equiv \overline{\mathcal{N}}_{\emptyset}$  (both are  $\overline{\mathcal{L}}_{\emptyset}$ -structures), and  $\mathcal{M}_{\emptyset}^* \equiv \mathcal{N}_{\emptyset}^*$  (as  $\mathcal{L}_{\emptyset}^*$ -structures)

Finally, we shall be using the following technical lemma:

# **Lemma 3.6.** For every $A \subseteq M$ , $dcl_{\mathcal{M}^*_{\alpha}}(A) \cap M = dcl_{\mathcal{M}}(A)$ .

*Proof.* Assume that  $a \in dcl_{\mathcal{M}^*_{\emptyset}}(A) \cap M$ . Then it follows that  $a \in dcl_{\mathcal{M}^*_A}(\emptyset)$  (since each element of A is  $\emptyset$ -definable in  $\mathcal{M}^*_A$ ). Hence, the interval  $(-\infty, a) \subseteq \overline{M}$  is  $\emptyset$ -definable in  $\mathcal{M}^*_A$ , so by Theorem 3.3, the intersection of  $(-\infty, a)$  with M is A-definable in  $\mathcal{M}$ . Because  $a \in M$ , we have  $a \in dcl_{\mathcal{M}}(A)$ .

For the converse, assume that  $a \in dcl_{\mathcal{M}}(A)$  (so in particular in M). Thus, the interval  $(-\infty, a)$  is definable in  $\mathcal{M}$  over A and its iterative convex hull, the interval  $\overline{(\infty, a)} \subseteq \overline{M}$ , is  $\emptyset$ -definable in  $\mathcal{M}_A$ . By Proposition 2.8(2), this interval is A-definable in  $\mathcal{M}_{\emptyset}^*$  so  $a \in dcl_{\mathcal{M}^*}(A) \cap M$ .

## 4. TIGHT WEAKLY O-MINIMAL STRUCTURES

As was pointed out before, the set M can be viewed as a union of sorts in  $\mathcal{M}$ , where each sort corresponds to a  $\emptyset$ -definable family of cuts in  $\mathcal{M}$ . In general, there might be infinitely many such sorts, but in some cases there are only finitely many such sorts.

### 4.1. Definition and basic properties.

**Definition 4.1.** A non-valuational structure  $\mathcal{M}$  is *tight* if there are finitely many  $\emptyset$ -definable families of cuts in  $\mathcal{M}$  such that every definable cut belongs to one of them.

Clearly, if  $\mathcal{M}$  is an o-minimal structure then it is (trivially) non-valuational and tight, since the family of definable cuts is just all intervals of the form  $(-\infty, a)$ , as a varies in  $\mathcal{M}$ .

It immediately follows that if  $\mathcal{M} \equiv \mathcal{N}$  then  $\mathcal{M}$  is tight if and only if  $\mathcal{N}$  is tight. Thus, we may use the term "tight" for T as well.

**Proposition 4.2.** The structure  $\mathcal{M}$  is tight if and only if there are finitely many  $\emptyset$ -definable functions  $f_i : M^{n_i} \to \overline{M}, i = 1, \ldots, k$ , such that  $\overline{M} \subseteq \bigcup_{i=1}^k Im(f_i)$ .

In particular,  $\mathcal{M}$  is tight then the structure  $\mathcal{M}^*$  is interpretable in  $\mathcal{M}$  without parameters.

*Proof.* The first clause is easy to verify. For the second clause, note first that the universe of M is a quotient of some  $M^n$  by a definable set, and furthermore the embedding of M in this quotient (i.e. the family of cuts  $\{C_x : x \in M\}$ , where  $C_x = \{y < x\}$ ) is definable in  $\mathcal{M}$ . It is easy to see that the ordering on  $\overline{M}$  is definable in  $\mathcal{M}$  and hence  $cl_{\overline{M}}(X)$  is definable in  $\mathcal{M}$  for every  $\mathcal{M}$ -definable  $X \subseteq M^n$ .

**Remark 4.3.** The above proof shows, in fact, that the pair  $(\mathcal{M}^*, \mathcal{M})$  is bi-interpretable with  $\mathcal{M}$ , i.e., not only is  $\mathcal{M}^*$  interpretable in  $\mathcal{M}$ , but so is the natural embedding of M in  $\overline{M}$ .

4.2. An example of a tight structure. We shall now see that there are examples of tight structures which are not o-minimal.

Let  $\mathbb{Q}_{vs} = \langle \mathbb{Q}, <, +, 1, \{\lambda_q\}_{q \in \mathbb{Q}} \rangle$  denote the group of rational numbers, viewed as an ordered vector space over itself, with a function symbol for every rational scalar. Let  $\mathbb{Q}^{\sqrt{2}}$  be the expansion of  $\mathbb{Q}_{vs}$  by the relation

$$P\sqrt{2} = \{(x,y) \in \mathbb{Q}^2 : y < \sqrt{2}x\}$$

We denote the langauge by  $\mathcal{L}_{\sqrt{2}}$ . (In [4, Section 3] a similar expansion of  $\mathbb{Q}_{vs}$  by the predicate  $P_{\pi}$  was investigated.)

The idea is to eventually identify  $P_{\sqrt{2}}$  with a map  $x \mapsto \sqrt{2}x$  from the structure  $\mathbb{Q}^{\sqrt{2}}$  into its canonical completion. Our goal is to show that  $\operatorname{Th}(\mathbb{Q}^{\sqrt{2}})$  is axiomatised by the following theory T:

- (1) The ordered  $\mathbb{Q}$ -vector space axioms.
- (2) An axiom expressing the fact that  $P_{\sqrt{2}}$  is "linear":

$$(\forall x_1, y_1, x_2, y_2) (((x_1, y_1) \in P_{\sqrt{2}} \land (x_2, y_2) \in P_{\sqrt{2}}) \to (x_1 + x_2, y_1 + y_2) \in P_{\sqrt{2}})$$

- (3) (Ensuring that we define the positive  $\sqrt{2}$ ):  $(\exists x, y)((x, y) \in P_{\sqrt{2}} \land x > 0 \land y > 0)$
- (4) For all  $r \in \mathbb{Q}$ , such that  $r < \sqrt{2}$ , we have  $\forall x (x > 0 \rightarrow (x, rx) \in P\sqrt{2})$ , and for all  $r \in \mathbb{Q}$  such that  $r > \sqrt{2}$ , we have  $\forall x (x > 0 \rightarrow (x, rx) \notin P\sqrt{2})$ .
- (5) For all  $x \neq 0$ , the set

$$\{y: (x,y) \in P\sqrt{2}\}$$

is closed downwards, and has no supremum. Furthermore,

Inf  $\{y_2 - y_1 : (x, y_1) \in P_{\sqrt{2}} \& (x, y_2) \notin P_{\sqrt{2}}\} = 0.$ 

(6) An axiom expressing the fact that the composition of  $x \mapsto \sqrt{2}x$  with itself yields the map  $x \mapsto 2x$ :

$$\forall (x, y > 0) \left( \left[ (\exists z > 0) P \sqrt{2}(x, z) \land P \sqrt{2}(z, y) \right] \iff y < 2x \right).$$

(7) The quantifier-free theory of  $\mathbb{Q}^{\sqrt{2}}$ .

# Clearly, $\mathbb{Q}^{\sqrt{2}}$ is a model of T.

For simplicity we write  $F = \mathbb{Q}(\sqrt{2})$ . Before we prove quantifier elimination we note that if  $\mathcal{M}$  is a model of T then we may consider the associated F-vector space  $V = F \otimes_{\mathbb{Q}} M$ . If we identify M with the  $\mathbb{Q}$ -subspace  $1 \otimes M$ , then each element of V can be written uniquely as  $x + \sqrt{2}y$  for  $x, y \in M$ . We can now endow V with an ordering by declaring  $x + \sqrt{2}y > 0$  when  $(y, -x) \in P\sqrt{2}$ . Indeed, the above axioms imply that this is a linear ordering of the vector space V, compatible with the ordering of F.

The definition of the ordering and Axiom (3) allows us to conclude:

**Claim 4.4.** (1) For  $x, y \in M$ , we have  $(x, y) \in P\sqrt{2} \Leftrightarrow$  if and only if  $y < \sqrt{2}x$  in V. (2) M is dense in V.

We can now endow V with an  $\mathcal{L}_{\sqrt{2}}$ -structure, by interpreting  $P_{\sqrt{2}}$  as we did over  $\mathbb{Q}$ . Clause (1) above then implies that  $\mathcal{M}$  is a substructure of V as an  $\mathcal{L}_{\sqrt{2}}$ -structures.

The following lemma is similar to [4, Proposition 3.3]:

### **Lemma 4.5.** The theory T is complete and has quantifier elimination.

*Proof.* Let  $Q_1, Q_2 \models T$  be  $\kappa$ -saturated models of the same cardinality. In order to prove quantifier elimination it suffices to prove (see for example [7, Corollary 3.1.6]):

If A is a substructure of  $Q_1$  and  $Q_2$  of cardinality smaller than  $\kappa$ , then for every  $a_1 \in Q_1$  there is  $a_2 \in Q_2$  such that  $a_1$  and  $a_2$  have the same quantifier-free type over A.

As above, consider the ordered F-vector spaces  $\mathcal{G}_i := F \otimes_{\mathbb{Q}} \mathcal{Q}_i$ . Since  $Q_i$  is dense in  $G_i$ , and  $\mathcal{G}_i$  is o-minimal, the saturation of  $\mathcal{Q}_i$  implies that  $\mathcal{G}_i$  is also  $\kappa$ -saturated. Let  $B_i$  be the F-span of A inside  $\mathcal{G}_i$ . Then  $B_1$  and  $B_2$  are isomorphic-over-A ordered vector spaces (both isomorphic to  $A + \sqrt{2}A$ , with the same ordering). Thus we may write  $B = B_1 = B_2$ 

Let  $p(x) := \operatorname{tp}_{\mathcal{G}_1}(a_1/B)$ . We may assume that  $a_1 \notin A$  and hence  $a_1 \notin B$  (note that  $B \cap Q_1 = A$ ). By the completeness of the theory of ordered F-vector spaces and saturation, we can find  $a_2 \in \mathcal{G}_2$  such that  $a_2 \models p(x)$ . In fact, because  $\mathcal{G}_2$  is  $\kappa$ -saturated and p is non-algebraic there is more than one such  $a_2$ , so since  $Q_2$  is dense in  $G_2$ , we can find such an  $a_2$  inside  $Q_2$ .

To see that T is complete we just notice that every model of T contains the structure  $\mathbb{Q}^{\sqrt{2}}$ , which is itself a model of T.

**Corollary 4.6.** The theory T is a tight weakly o-minimal non-valuational theory and  $T^*$  is the theory of ordered  $\mathbb{Q}^{\sqrt{2}}$ -ordered vector spaces (in the language  $\mathcal{L}_{\sqrt{2}}$ ).

*Proof.* The atomic subsets of Q, the universe of any  $Q \models T$ , are rays with or without endpoints. By quantifier elimination the definable subsets of Q are in the boolean algebra generated by those, proving the weak o-minimality. The same argument also shows that the only definable cuts are non-valuational, because so are the atomic cuts.

By the proof of lemma 4.5 and the preceding discussion, each model Q of T is a dense substructure of the o-minimal structure  $V = F \otimes_{\mathbb{Q}} Q$ . It is easy to verify that the intersection with Q of every ray  $(-\infty, a)$  in V is definable in Q, and hence every element of V realizes a definable cut in Q. Conversely, by quantifier elimination, the definable cuts in any model Q of T are of the form  $x_1 + r\sqrt{2}x_2$  for  $r \in \mathbb{Q}$  and  $x_1, x_2 \in Q$ , so they are realized in V. It follows that V is the canonical completion of Q, and its theory, in the language  $\mathcal{L}_{\sqrt{2}}$ , is that of an ordered F-vector space.

To see that T is tight we note that each definable cut in Q can also be written as  $x_1 + \sqrt{2}x_2$ , for  $x_1, x_2 \in Q$ , and that this is a definable family in T.

Note that the above construction worked because of the algebraicity of  $\sqrt{2}$ . If we consider  $\mathbb{Q}^t$ , the expansion of  $\mathbb{Q}_{vs}$  by  $x \mapsto tx$  where t realizes a cut defining a real transcendental number we would not obtain a tight structure. See the example  $\mathbb{Q}_{vs}^{\pi}$  in [4].

We now prove:

**Theorem 4.7.** The structure  $\mathbb{Q}^{\sqrt{2}}$  is not elementarily equivalent to a reduct of an o-minimal trace.

*Proof.* Assume towards a contradiction that there is a dense pair  $(\mathcal{R}, \mathcal{Q})$  of o-minimal expansions of groups such that  $\mathbb{Q}^{\sqrt{2}}$  is elementarily equivalent to a reduct of the trace which this pair induces on a structure  $\mathcal{Q}$ . By that we mean that there is some expansion  $\hat{\mathcal{Q}}$  of  $\langle \mathcal{Q}, <, + \rangle$  satisfying T, such that every definable set in  $\hat{\mathcal{Q}}$  is definable in the dense pair  $(\mathcal{R}, \mathcal{Q})$ . While some of these sets are already definable in the o-minimal structure  $\mathcal{Q}$  others may be the intersection with  $Q^n$  of subsets of  $\mathbb{R}^n$  that are definable in  $\mathcal{R}$  over parameters which are not in  $\mathcal{Q}$ . The order relation < and the group operation + are assumed to be definable in  $\mathcal{Q}$ .

Let us consider the predicate  $P_{\sqrt{2}}(\hat{Q})$ . It is a definable set in  $(\mathcal{R}, \mathcal{Q})$ , hence by [9, Theorem2], there is a definable  $Y_{\sqrt{2}} \subseteq R^2$  in the o-minimal structure  $\mathcal{R}$  such that  $Y_{\sqrt{2}} \cap \mathcal{Q} = \mathcal{P}_{\sqrt{2}}(\hat{Q})$ . Because  $\mathcal{Q}$  is dense in R (the universe of the o-minimal structure  $\mathcal{R}$ ), it easily follows that for every  $x \in Q$ , there is  $y(x) \in R$  such that

$$y(x) = \sup\{y \in \mathcal{Q} : (x, y) \in Y_{\sqrt{2}}\}.$$

By taking the closure of the graph of y(x) we obtain an  $\mathcal{R}$ -definable function, which we will denote by  $\lambda_{\sqrt{2}}$ :  $R \to R$ , which gives y(x) for every  $x \in \mathcal{Q}$ . It is not hard to see that  $\lambda_{\sqrt{2}}$  is a definable automorphism of  $\langle R, + \rangle$  satisfying  $\lambda_{\sqrt{2}} \circ \lambda_{\sqrt{2}}(x) = 2x$ .

We now consider two cases. If the function  $\lambda_{\sqrt{2}}$  is  $\emptyset$ -definable in  $\mathcal{R}$  then it comes from a definable function in the o-minimal structure  $\mathcal{Q}$ , and in particular, for every  $x \in \mathcal{Q}$ , the set  $\{y \in \mathcal{Q} : (x, y) \in P\sqrt{2}\}$  has a supremum in  $\mathcal{Q}$ . This contradicts the axioms of T.

On the other hand, if  $\lambda_{\sqrt{2}}$  is not  $\emptyset$ -definable then by [8], one can define in the o-minimal structure  $\mathcal{R}$  a multiplication function  $\cdot$  on  $\mathbb{R}^2$ , making  $\langle \mathbb{R}, <, +, \cdot \rangle$  a real closed field, call it K. A-priori the multiplication function might not be  $\emptyset$ -definable but in that case there is a  $\emptyset$ -definable family of such multiplications all of which expand  $\langle \mathbb{R}, + \rangle$  to a real closed field. By definable choice we may find one such multiplication function that is  $\emptyset$ -definable.

Since  $\lambda_{\sqrt{2}}$  is an  $\mathcal{R}$ -definable automorphism of the additive group of K it must be of the form  $x \mapsto c \cdot x$  for some scalar  $c \in K$ . Because  $\lambda_{\sqrt{2}} \circ \lambda_{\sqrt{2}}(x) = 2x$ , and because  $\lambda_{\sqrt{2}}$  takes positive values on x > 0, the scalar c is necessarily  $\sqrt{2}$  (in the sense of K). In particular,  $\lambda_{\sqrt{2}}$  is  $\emptyset$ -definable in  $\mathcal{R}$ , yielding a contradiction as before.  $\Box$ 

5. The theory of  $(\mathcal{M}^*, \mathcal{M})$ 

From now on, given a complete non-valuational theory T we will denote by  $T^*$  the theory of the associated o-minimal completion, in the language  $\mathcal{L}^*_{\emptyset}$  (by Corollary 3.5, the theory T indeed determines  $T^*$ ). We write  $\overline{\mathcal{M}}$  and  $\mathcal{M}^*$ , for the structure  $\overline{\mathcal{M}}_{\emptyset}$ , and  $\mathcal{M}^*_{\emptyset}$ , respectively.

While  $\mathcal{M}$  and  $\mathcal{M}^*_{\emptyset}$  initially have different signatures it will be convenient to treat them in the same langauge. We thus modify the language of  $\mathcal{M}$ .

**Lemma 5.1.** Let  $\mathcal{M}$  be a weakly o-minimal non-valuational structure. Let  $\mathcal{M}_0$  be the reduct of  $\mathcal{M}$  generated by all  $\emptyset$ -definable closed sets. Then every  $\emptyset$ -definable set in  $\mathcal{M}$  is  $\emptyset$ -definable in  $\mathcal{M}_0$ . In particular,  $\mathcal{M}$  and  $\mathcal{M}_0$  have the same  $\emptyset$ -definable sets.

*Proof.* This follows from the proof of Proposition 2.8.

So from now on we will assume that  $\mathcal{M}$  is given in the signature consisting of a function symbol for +, the ordering <, and a predicate for each  $\emptyset$ -definable closed set in  $\mathcal{M}^n$ . We let  $\mathcal{L}$  be the associated language, so we may use the same language for  $\mathcal{M}^*$ . By Proposition 4.5, the structure  $\mathcal{M}^*$  eliminates quantifiers.

We let  $\mathcal{L}^P = \mathcal{L} \cup \{P\}$ , where P is a unary predicate. We consider the  $\mathcal{L}^P$ -structure

$$\mathcal{M}^P = \langle \mathcal{M}^*, \mathcal{M} \rangle,$$

where the interpretation of P is M. As we will see, the theory of  $\mathcal{M}^P$  depends only on T. We propose the following axiomatization for this theory:

Let  $T^{d}$  be the  $\mathcal{L}^{P}$ -language axiomatized as follows (we write  $(\mathcal{M}', \mathcal{M})$  for models of  $T^{d}$ ),

(1)  $\mathcal{M} \models T, \mathcal{M}' \models T^*$ .

(2) M dense in M'.

(3) Every definable cut in  $\mathcal{M}$  has a supremum in M'.

(4) (when T is tight) Every element of  $\mathcal{M}'$  realizes a definable cut in  $\mathcal{M}$ .

Our goal is to prove:

**Theorem 5.2.** The theory  $T^d$  is complete.

5.1. The tight case. Assume that T is tight. As we saw in Proposition 4.2, the structure  $\mathcal{M}^*$  is interpretable in  $\mathcal{M}$  without parameters. Using axiom (4) above we immediately conclude:

**Lemma 5.3.** Assume that T is tight.

(1) If  $(\mathcal{M}', \mathcal{M}) \models T^d$  then necessarily  $\mathcal{M}' = \mathcal{M}^*$ .

(2) For all  $\mathcal{M}, \mathcal{N} \models T$ , we have  $(\mathcal{N}^*, \mathcal{N}) \equiv (\mathcal{M}^*, \mathcal{M})$ .

## 5.2. The general case.

**Theorem 5.4.** If  $\mathcal{M}^d = (\mathcal{M}', \mathcal{M})$  and  $\mathcal{N}^d = (\mathcal{N}', \mathcal{N})$  are models of  $T^d$ , then  $\mathcal{M}^d \equiv \mathcal{N}^d$ .

*Proof.* We may assume that T is non-tight. We may assume that  $\mathcal{M}^d$  and  $\mathcal{N}^d$  are  $\kappa$ -saturated for sufficiently large  $\kappa$ .

Notice that every  $\mathcal{M}$ -definable cut is realized in  $\mathcal{M}'$  exactly once, hence there is a natural embedding of  $\mathcal{M}^*$  into  $\mathcal{M}'$ , and the same holds for  $\mathcal{N}'$  and  $\mathcal{N}$ . However, by saturation, unless  $\mathcal{M}$  is tight it is not the case that  $\mathcal{M}'$  equals  $\mathcal{M}^*$ , since it realizes cuts which are not definable as well. Our goal is to show that there are  $(B, A) \prec (\mathcal{M}', \mathcal{M})$  and  $(D, C) \prec (\mathcal{N}', \mathcal{N})$  which are isomorphic.

Notice first that both M and  $M' \setminus M$  are dense in M', for i = 1, 2. Indeed, this follows from the fact that T is non-valuational, so if  $c \in \overline{M} \setminus M$  is any element then  $c + M \subseteq \overline{M}$  is dense in  $\overline{M}$ , so also in M'.

Since  $\mathcal{M}^* \models T^*$  and  $\mathcal{M}^*$  eliminates quantifiers, the pair  $(\mathcal{M}', \mathcal{M}^*)$  is an elementary dense pair of o-minimal structures, so we shall apply to it the theory of dense pairs as in [9].

We first need:

Lemma 5.5. Let  $(\mathcal{M}', \mathcal{M}) \models T^d$ . Let  $M_0 \prec \mathcal{M}$ . Then  $\operatorname{dcl}_{\mathcal{M}_0^*}(M_0) = \overline{\mathcal{M}}_0$ . Moreover,  $\operatorname{dcl}_{\mathcal{M}'}(M_0) = \overline{M}_0$ .

*Proof.* It will suffice to prove the first part of the lemma as the second part follows from the fact that  $\mathcal{M}_0^* \prec \mathcal{M}'$ . First we show the right-to-left inclusion. For that we need:

**Claim 5.6.** Assume that  $f : M_0^n \to M_0$  is a  $\emptyset$ -definable function in  $\mathcal{M}_0$ . Then there are in  $\mathcal{M}_0$  finitely many  $\emptyset$ -definable strong cells of the form  $C_1, \ldots, C_k \subseteq M_0^n$ , with  $M_0^n \subseteq \bigcup_i \overline{C}_i$ , and in  $\mathcal{M}_0^*$  there are finitely many  $\emptyset$ -definable functions  $\overline{f}_i : \overline{C}_i \to \overline{M}_0$ , such that for all  $x \in C_i$ ,  $\overline{f}_i(x) = f(x)$ .

*Proof.* We decompose  $M_0^n$  into  $\emptyset$ -definable strong cells,  $C_1, \ldots, C_k$ , on each of which f is strongly continuous. For each i, the graph of  $\bar{f} \upharpoonright \bar{C}_i$  is the iterative convex hull of  $\Gamma(f \upharpoonright C_i)$ , so it is  $\emptyset$ -definable in  $\mathcal{M}_0^*$ .

Assume now that  $b \in M_0$ , then by definition of the completion, the cut  $Y = \{x \in M_0 : x < b\}$  is definable in  $\mathcal{M}_0$ , over a tuple of parameters a. We may assume that  $Y = Y_a$  for a  $\emptyset$ -definable family of sets  $\{Y_t : t \in T\}$ and  $\emptyset$ -definable set  $T \subseteq M_0^m$ , and that we have  $b = \sup Y_a$ . It follows that there is in  $\mathcal{M}_0$  a  $\emptyset$ -definable function  $f : T \to \overline{M}_0$ , such that f(a) = b.

By the above claim, we have  $T = \bigcup C_i$  a union of  $\emptyset$ -definable strong cells in  $\mathcal{M}$ , and there are  $f_i : \overline{C}_i \to \overline{M}_0$ all  $\emptyset$ -definable in  $\mathcal{M}_0^*$ , such that

(5) 
$$\bigwedge_{i=1}^{k} \forall x \in C_i \ \bar{f}(x) = f(x).$$

In particular, there is  $i \in \{1, \ldots, k\}$  such that  $a \in C_i$  and  $b = f_i(a)$  is in  $\operatorname{dcl}_{\mathcal{M}_0^*}(\mathcal{M}_0)$ . Thus,  $\mathcal{M}_0 \subseteq \operatorname{dcl}_{\mathcal{M}_0^*}(\mathcal{M}_0)$ .

For the converse, we assume that g(a) = b for some  $\emptyset$ -definable function g in  $\mathcal{M}_0^*$  and  $a \in \mathcal{M}_0^m$ . We want to show that  $b \in \overline{\mathcal{M}}_0$ , namely that b is the supremum of a definable cut in the structure  $\mathcal{M}_0$ .

The function g is  $\emptyset$ -definable in the o-minimal structure  $\mathcal{M}_0^*$ , so by Theorem 3.3, the set

$$Y = \{(x, y) \in M_0^{n+1} : y < g(x)\}$$

is  $\emptyset$ -definable in  $\mathcal{M}_0$  and we have

(6)

$$\forall x \in M_0^n \ g(x) = \sup(Y_x)$$

It follows that  $b = g(a) = \sup Y_a$ , with  $Y \subseteq M_0^{n+1}$  a  $\emptyset$ -definable set in  $\mathcal{M}_0$ . Hence,  $b \in \overline{\mathcal{M}}_0$ . We will also need:

**Claim 5.7.** For  $A \subseteq M$  and  $a \in M$ , the  $\mathcal{M}$ -type of a over A is determined by the cut of a in dcl<sub> $\mathcal{M}'$ </sub>(A).

*Proof.* Assume that a and b in M realize the same cut over  $dcl_{\mathcal{M}'}(A)$ . To see that a and b realize the same  $\mathcal{M}$ -type over A, it is sufficient, by the weak o-minimality of  $\mathcal{M}$ , to show, for every cut  $C \subseteq M$  definable in  $\mathcal{M}$  over A, that  $a \in C$  iff  $b \in C$ . Using our assumptions, it is enough to prove that the supremum of C exists in M' and belongs to  $dcl_{\mathcal{M}'}(A)$ .

If C has a supremum s in M then  $s \in dcl_{\mathcal{M}}(A) \cap M$ , and therefore (Lemma 3.6)  $s \in dcl_{\mathcal{M}^*}(A)$ . Since  $\mathcal{M}^*$  is an elementary substructure of  $\mathcal{M}'$  we have  $s \in dcl_{\mathcal{M}'}(A)$ .

If C has no supremum in M then, by definition, its supremum is realized in  $\overline{M}$ . As C is definable in  $\mathcal{M}$  over A, its closure in  $\overline{M}$  is  $\emptyset$ -definable in  $\mathcal{M}^*_A$ , so by 4.5(2) it is definable in  $\mathcal{M}^*_{\emptyset}$  over A. But then  $\sup C \in \operatorname{dcl}_{\mathcal{M}^*}(A) = \operatorname{dcl}_{\mathcal{M}'}(A)$ . This finishes the proof.

The rest of the proof follows closely the arguments from [9]. In order to proceed we borrow the following terminology:

**Definition 5.8.** For  $B \subseteq M'$  and  $A = B \cap M$ , we say that (B, A) is *free* if  $\dim_{\mathcal{M}'}(B'/A) = \dim_{\mathcal{M}'}(B'/M)$  for every finite  $B' \subseteq B$ . Namely, every subset of B which is  $\mathcal{M}'$ -independent over A remains independent over M. We make the same definitions for subsets of N' and N.

We consider all  $(B, A) \subseteq (M', M)$  (and similarly (D, C) in (N', N)) which satisfy:

(i)  $B \cap M = A$ .

(ii)  $\operatorname{dcl}_{\mathcal{M}'}(B) = B.$ 

(iii) (B, A) is free.

We now begin the construction of the intended isomorphism. By saturation, there is  $\mathcal{M}_0 \prec \mathcal{M}$ , of cardinality smaller than  $\kappa$  that is isomorphic to some  $\mathcal{N}_0 \prec \mathcal{N}$ .

If we let  $A_0 := M_0 B_0 := \overline{A}_0$  and  $C_0 =:= \overline{N}_0$ ,  $D_0 := \overline{C}_0$ . Then (i) holds. By Lemma 5.5 dim<sub> $\mathcal{M}'$ </sub> ( $B_0/A_0$ ) = 0, so ( $B_0, A_0$ ) is (trivially) free. Also, by this lemma,  $B_0$  is definably closed in  $\mathcal{M}'$ , so ( $B_0, A_0$ ) satisfy (i),(ii),(iii). Similarly, ( $D_0, C_0$ ) satisfies (i),(ii),(iii).

Our goal is to use back-and-forth and Tarski-Vaught in order to build isomorphic elementary substructures of  $(\mathcal{M}', \mathcal{M})$  and  $(\mathcal{N}', \mathcal{N})$ . Towards that goal we need to prove the following result:

**Lemma 5.9.** Assume that  $(B, A) \subseteq (\mathcal{M}', \mathcal{M})$  and  $(D, C) \subseteq (\mathcal{N}', \mathcal{N})$  satisfy (i),(ii),(iii), and isomorphic (namely, there is an  $\mathcal{L}$ -isomorphism  $\alpha : B \to D$  sending A onto C), with  $|A| < \kappa$ . Then, for every  $b \in M'$ , there are  $B' \subseteq M', A' \subseteq M$  with  $b \in B'$ , and there are  $D' \subseteq N', C' \subseteq N$ , such that (B', A'), (D', C') satisfy (i),(ii),(iii), and there is an isomorphism  $\alpha' : (B', A') \to (D', C')$  extending  $\alpha$ .

(We also have the analogous result for (D, C) and  $d \in N'$ .)

*Proof.* We divide the argument into several cases:

## Case I. $b \in M$ .

First, we find  $d \in N$  such that  $\alpha(\operatorname{tp}_{\mathcal{M}'}(b/B)) = \operatorname{tp}_{\mathcal{N}'}(d/D)$  (so by Lemma 5.7, also  $\alpha(\operatorname{tp}_{\mathcal{M}}(b/A)) = \operatorname{tp}_{\mathcal{N}}(d/C)$ ). Indeed, this is possible because N is dense in N' and N' is  $\kappa$ -saturated. The function  $\alpha$  then extends naturally to an isomorphism  $\alpha'$  of the o-minimal structures  $B' := \operatorname{dcl}_{\mathcal{M}'}(Bb)$  and  $D' := \operatorname{dcl}_{\mathcal{N}'}(Dd)$ . We let  $A' = B' \cap M$  and  $C' = D' \cap N$ . In order to see that  $\alpha'$  is an isomorphism of (B', A') and (D', C') it is left to verify is that for every  $a \in B'$ ,

(7) 
$$a \in M \Leftrightarrow \alpha'(a) \in N$$

So, we take  $a \in \operatorname{dcl}_{\mathcal{M}'}(Bb)$  and prove (7).

Assume first that  $a \in \operatorname{dcl}_{\mathcal{M}'}(Ab)$ . By Lemma 5.5,  $a \in \overline{M}$ , so we have  $a \in \operatorname{dcl}_{\mathcal{M}^*}(Bb)$ . Hence, there exists a  $\emptyset$ -definable function F of (n + 1)-variables in  $\mathcal{M}^*$ , and  $e \in (\overline{M})^n$ , with F(b, e) = a. The function F is definable in  $\mathcal{M}^*$ , and, by 3.3, its restriction to  $M^{n+1}$  is  $\emptyset$ -definable in  $\mathcal{M}$  (as a function into  $\overline{M}$ ). Thus, we can definably in  $\mathcal{M}$  partition its domain into  $\emptyset$ -definable strong cells on each of which F takes either values in M or in  $\overline{M} \setminus M$ . This partition is part of the weakly o-minimal theory T, and thus holds in both  $\mathcal{M}$  and  $\mathcal{N}$ . Since  $\alpha(\operatorname{tp}_{\mathcal{M}}(b/A)) = tp_{\mathcal{N}}(d/C)$  it follows that  $a = F(b, e) \in M$  if and only if  $\alpha'(a) = F(d, \alpha(e)) \in N$ .

Assume now that  $a \in \operatorname{dcl}_{\mathcal{M}'}(Bb) \setminus \operatorname{dcl}_{\mathcal{M}'}(Ab)$  (so  $\alpha'(a) \in \operatorname{dcl}_{\mathcal{N}'}(Dd) \setminus \operatorname{dcl}_{\mathcal{N}'}(Cd)$ ). We claim that  $a \notin M$  and  $\alpha'(a) \notin N$ .

Indeed, assume towards a contradiction that  $a \in M$ , and let  $Y \subseteq B$  be a minimal finite set which is  $dcl_{\mathcal{M}'}$ independent over Ab such that  $a \in dcl_{\mathcal{M}'}(YAb)$ . Because  $a \notin dcl_{\mathcal{M}'}(Ab)$  the set Y is nonempty so fix  $y_0 \in Y$ . We have  $a \in dcl_{\mathcal{M}'}(Y'y_0Ab)$ , with  $Y' = Y \setminus \{y_0\}$ , so by exchange (and minimality of Y'),  $y_0 \in dcl_{\mathcal{M}'}(Y'Aba)$ . Because  $a, b \in M$  and  $A \subseteq M$ , it follows that Y is not independent over M, even though it is independent over A. This contradicts the fact that (B, A) was free, so  $a \notin M$ . The same argument shows that  $\alpha'(a) \notin N$ .

Thus, we showed that  $\alpha' : (B', A') \to (D', C')$  is an isomorphism. It is clear, that the pairs satisfy (i) and (ii), so we are left to see that they are free. So, we take  $Y \subseteq B'$  independent over A' and claim that it remains independent over M. Indeed, because  $b \in A'$  (since  $b \in M$ ), it must be the case that  $Y \subseteq B$ , and the result follows immediately from the freeness of (B, A) (because  $A \subseteq A'$ ). This ends Case I.

# Case II. $b \in \operatorname{dcl}_{\mathcal{M}'}(BM)$ .

In this case, there is  $\bar{m} = (m_1, \ldots, m_k) \in M^k$  such that  $b \in \operatorname{dcl}_{\mathcal{M}'}(B\bar{m})$ . We first apply Case I to each  $m_i$ , and thus may assume that  $\bar{m} \subseteq B$ , and in particular may assume that b is already in B.

## Case III. $b \notin \operatorname{dcl}_{\mathcal{M}'}(BM)$

Notice first that in this case  $\mathcal{M}$  (and hence also  $\mathcal{N}$ ) is not tight (since in the tight case  $M' = \overline{M} = \operatorname{dcl}_{\mathcal{M}'}(M)$ ). We let  $B' = \operatorname{dcl}_{\mathcal{M}'}(Bb)$  and  $A' = B' \cap M$ . Our goal is to show that (B', A') satisfies (i),(ii),(iii), so we need to show that it is free.

We first claim that A' = A. Indeed, if  $a \in \operatorname{dcl}_{\mathcal{M}'}(Bb) \cap M$  then either  $a \in \operatorname{dcl}_{\mathcal{M}'}(B)$ , so  $a \in A$ , or if not then by exchange,  $b \in \operatorname{dcl}_{\mathcal{M}'}(Ba)$ , contradicting the assumption on b.

Assume now that  $Y \subseteq B'$  is independent over A' = A. If  $Y \subseteq B$  then Y is independent over M, and otherwise, we may assume that it is of the form Y'b with  $Y' \subseteq B$ . By freeness of (B, A) we have Y' independent over M and by assumption on b we may conclude that Y'b independent over M. Thus, (B', A') is indeed free. Next, we claim that we may find in  $\mathcal{N}'$  an element d such that  $\alpha(\operatorname{tp}_{\mathcal{M}'}(b/B)) = \operatorname{tp}_{\mathcal{N}'}(d/D)$  and in addition  $d \notin \operatorname{dcl}_{\mathcal{N}'}(DN)$ . It is here that we use the fact that  $\mathcal{N}$  is non-tight. We prove:

**Lemma 5.10.** Let  $D \subseteq N'$  be of cardinality smaller than  $\kappa$ . Then for every  $\mathcal{M}'$ -type p(x) over B, there is a realization of  $\alpha(p)$  which is not in  $dcl_{\mathcal{N}'}(DN)$ .

*Proof.* By the saturation of (N', N) it is sufficient to prove that  $X \nsubseteq \operatorname{dcl}_{\mathcal{N}'}(DN)$  for every infinite set  $X \subseteq N'$  that is definable in  $\mathcal{N}'$  over D. For that it is clearly sufficient to show that  $X \nsubseteq \operatorname{dcl}_{\mathcal{N}'}(D\overline{N})$ . By applying the theory of dense pairs to the pair of o-minimal structures  $(\mathcal{N}', \overline{\mathcal{N}})$ , we may conclude from [9, Lemma 4.1], that no interval in  $\mathcal{N}'$  is in the image of  $\overline{N}^n$  under an  $\mathcal{N}'$ -definable map. This is easily seen to imply the result we want.

This ends the proof of Lemma 5.9.

Going back to our proof of completeness of  $T^d$ , we find  $d \in N'$  with  $\alpha(\operatorname{tp}_{\mathcal{M}'}(b/B)) = \operatorname{tp}_{\mathcal{N}'}(d/D)$  and with  $d \notin \operatorname{dcl}_{\mathcal{N}'} DN$ . We let  $D' = \operatorname{dcl}_{\mathcal{N}'}(Dd)$  and  $C' = D' \cap N'$  (which equals C), so as before (D', C') is free. It is left to see that the natural extension of  $\alpha$  to  $\alpha' : B' \to D'$  preserves  $M \cap B'$ . However,  $B' \cap M = A'$  so by applying what we already know to both  $\alpha$  and  $\alpha^{-1}$  we conclude that  $x \in M' \Leftrightarrow \alpha'(x) \in N'$ . This ends the proof of Theorem 5.4.

Notice that the proof above showed that any isomorphism of weakly o-minimal structures  $M_1 \prec M$  and  $M_2 \prec N$  can be extended to an isomorphism of elementary substructures  $(B, A) \prec (M', M)$  and  $(D, C) \prec (N', N)$ . Lemma 5.9 also implies:

**Lemma 5.11.** Assume that  $(\mathcal{M}', \mathcal{M}), (\mathcal{N}', \mathcal{N}) \models T^d$  and  $(B, A) \subseteq (\mathcal{M}', \mathcal{M}), (D, C) \subseteq (\mathcal{N}', \mathcal{N})$  satisfy (i), (ii), (ii). If  $\alpha : B \to D$  is an  $\mathcal{L}$ -isomorphism sending A to C and  $\alpha(b) = d$  for some  $b \in B^n$  then

$$\alpha(\operatorname{tp}_{(\mathcal{M}',\mathcal{M})}(b/\emptyset)) = \operatorname{tp}_{(\mathcal{N}',\mathcal{N})}(d/\emptyset).$$

We can now prove analogues of several theorems from [9]. The proofs are very similar to the original ones.

**Theorem 5.12.** Let  $\mathcal{M}^d = (\mathcal{M}', \mathcal{M})$  be a model of  $T^d$ .

(1) In  $\mathcal{M}^d$ , every  $\emptyset$ -definable subset of  $(M')^n$  is a boolean combination of sets defined by formulas of the form

(8) 
$$\exists x_1 \cdots \exists x_k (\bigwedge_{i=1}^k x_i \in P \& \varphi(x_1, \dots, x_k, y),$$

where |y| = n and  $\varphi(x, y)$  is an  $\mathcal{L}$  formula.

- (2) Let  $B \subseteq M'$  be such that  $(B, B \cap M)$  is free. Then every subset of  $M^k$  that is definable in  $\mathcal{M}^d$  over  $B \subseteq M'$  is of the form  $Y \cap M^k$  for some  $Y \subseteq (M')^k$  that is definable in  $\mathcal{M}'$  over B.
- (3) Every subset of  $M^k$  that is definable in  $\mathcal{M}^d$  over  $A_0 \subseteq M$  is definable in the structure  $\mathcal{M}$  over  $A_0$ .
- (4) Every subset of  $M^n$  that is definable in  $(\mathcal{M}^*, \mathcal{M})$  (here  $\mathcal{M}^*$  is the completion of  $\mathcal{M}$ ) is definable in the structure  $\mathcal{M}$ .

*Proof.* Without loss of generality,  $(\mathcal{M}', \mathcal{M})$  is sufficiently saturated.

(1) By standard model theoretic considerations it is enough to prove the following: For any  $b, d \in (M')^k$ , assume that b satisfies a formula of the form (8) if and only if d does. Then b and d have the same type in  $\mathcal{M}^*$  over  $\emptyset$ .

Let  $r = \dim_{\mathcal{M}'}(b/M)$ . We can find  $a \subseteq M$  finite such that  $\dim_{\mathcal{M}'}(b/a) = r$ . It follows that if we let  $B = \operatorname{dcl}_{\mathcal{M}'}(ab)$  and  $A = \operatorname{dcl}_{\mathcal{M}'}(a)$  then (B, A) is free and  $A = B \cap M$ .

We consider the  $\mathcal{L}$ -type of (b, a) over  $\emptyset$ . Because b and d realize the same formulas of the form (8), and because of saturation we can find  $c \in M$  such that  $\operatorname{tp}_{\mathcal{M}'}(b, a/\emptyset) = \operatorname{tp}_{\mathcal{M}'}(d, c/\emptyset)$ . The pair (D, C), with  $D = \operatorname{dcl}_{\mathcal{M}'}(cd)$ and  $C = \operatorname{dcl}_{\mathcal{M}'}(c)$  is free with  $C = D \cap M$ . Just like in the proof of Lemma 5.9, the natural  $\mathcal{L}$ -isomorphism of Band D (sending (b, a) to (d, c)) sends A to C.

By Lemma 5.11, the  $\mathcal{L}^{P}$ -types of b and d in  $(\mathcal{M}', \mathcal{M})$  are the same. Thus we proved (1).

(2) By standard model theoretic arguments it is sufficient to prove: If  $b_1, b_2 \in M^k$  satisfy the same  $\mathcal{M}'$ -type over B then they satisfy the same  $L^P$ -type over B. For that, let  $A = B \cap M$ . It is sufficient to show that there are  $(B_1, A_1), (B_2, A_2) \prec (\mathcal{M}', \mathcal{M})$ , with  $(B, A) \subseteq (B_i, A_i)$  and  $b_i \in B_i$  for i = 1, 2, and there is an  $\mathcal{L}$ -isomorphism between  $(B_1, A_1)$  and  $(B_2, A_2)$ , which fixes B point-wise, and sending  $b_1$  to  $b_2$ .

We are now in the setting of Case I of the proof of Lemma 5.9, with our  $b_1, b_2$  replacing b, d there. Thus, we may first find two free pairs  $(B'_1, A'_1)$  and  $(B'_2, A'_2)$  with  $B \subseteq B'_i$  and  $b_i \in B'_i$ , i = 1, 2, and an isomorphism  $\alpha : (B'_1, A'_1) \to (B'_2, A'_2)$  extending the identity map, with  $\alpha(b_1) = b_2$ . We now proceed exactly as in the proof of Theorem 5.4 and obtain the desired  $(B_1, A_1), (B_2, A_2) \prec (\mathcal{M}', \mathcal{M})$ . Thus,  $b_1$  and  $b_2$  realize the same  $\mathcal{L}^P$  type over B and we may conclude (1).

For (3), let  $X \subseteq M^k$  be definable in  $(\mathcal{M}', \mathcal{M})$  over  $A_0 \subseteq M$ . Notice that the mere definability of X in  $\mathcal{M}$  follows immediately from (2) but we want to show that X is definable over the same  $A_0$ . For that, it is sufficient to prove that any  $a_1, a_2 \in M$  which realize the same  $\mathcal{M}$ -type over  $A_0$  realize the same  $\mathcal{L}^P$ -type over  $A_0$ .

To do that, we first find a small model  $\mathcal{M}_1 \prec \mathcal{M}$  containing  $A_0 a_1, a_2$ , and an automorphism  $\alpha$  of  $\mathcal{M}_1$  over  $A_0$ , sending  $a_1$  to  $a_2$ . As we commented previously, we may the extend  $\alpha$  to an isomorphism of two structures  $(B, A), (D, C) \prec (\mathcal{M}', \mathcal{M})$ . This is clearly sufficient.

To see (4), we note that every element of  $\mathcal{M}^*$  is in  $dcl_{\mathcal{M}^*}(N)$  and hence every definable subset of  $\overline{M}^k$  in  $\mathcal{M}^*$  can be defined over M. We now apply (3).

Note that (3) above fails if we omit the requirement that  $M_0 \subseteq M$ , since in the non-tight case, in general,  $\mathcal{M}'$  will realize cuts which are not definable in  $\mathcal{M}$  and thus their intersection with M is not definable in  $\mathcal{M}$ .

We also point out:

**Lemma 5.13.** If  $\mathcal{M}^d = (\mathcal{M}', \mathcal{M}) \models T^d$  then it is definably complete.

*Proof.* If  $X \subseteq M'$  is definable in  $\mathcal{M}^d$  and bounded below then the intersection of its convex hull with M is definable in  $\mathcal{M}^d$ , and thus has the form  $Y \cap M$  for some  $Y \subseteq M'$  which is definable in  $\mathcal{M}'$ . Without loss of generality, Y is also convex and thus InfY = InfX. This suffices, by o-minimality of  $\mathcal{M}'$ .

We can now conclude, using Boxall and Hieronymi, [2]:

**Theorem 5.14.** Let  $\mathcal{M}^d = (\mathcal{M}', \mathcal{M}) \models T^d$ . If  $U \subseteq (M')^n$  is open and definable in  $\mathcal{M}^d$  then it is definable in  $\mathcal{M}'$ . More precisely, if an open U is defined in  $\mathcal{M}^d$  over  $B \subseteq M'$  such that  $(B, B \cap M)$  is free, then U is definable in  $\mathcal{M}'$  over B. In particular,  $\mathcal{M}^P$  has an o-minimal open core.

*Proof.* This is an immediate corollary of [2, Corollary 3.2] and what we proved so far. We extract from their argument a direct proof, which is underlined by the following simple corollary of cell decomposition.

**Fact 5.15.** If  $Y \subseteq (M')^n$  is definable in  $\mathcal{M}'$  and dim Y < n then  $Y \cap M^n$  has empty interior in  $M^n$ .

We now first claim that  $cl_{M'}(U)$  is definable in  $\mathcal{M}'$  over B. Indeed, by Theorem 5.12 (2), there is  $Y \subseteq (M')^n$  definable in  $\mathcal{M}'$  over B such that  $Y \cap M^n = U \cap M^n$ . By the above observation,  $\dim Y = n$ .

Since  $M^n$  is dense in  $(M')^n$ , the set  $Int(Y) \cap M^n$  is dense in the open set Int(Y). We claim that it is also dense in U. Indeed, we know that  $Y \cap M^n = U \cap M^n$  is open in  $M^n$  and dense in U, and by o-minimality  $\dim_{\mathcal{M}'}(Y \setminus Int(Y)) < n$ . It thus follows from Fact 5.15, that  $Int(Y) \cap M^n$  is dense in U. So,

$$\operatorname{cl}_{M'}(U) = \operatorname{cl}_{M'}(Int(Y) \cap M^n) = \operatorname{cl}_{M'}(Int(Y)).$$

Because Y was definable in  $\mathcal{M}'$  over B,  $cl_{\mathcal{M}'}(U)$  is definable in  $\mathcal{M}'$  over B.

We thus showed that the closure of every  $\mathcal{M}^d$ -definable open set over  $B \subseteq M'$  is definable in  $\mathcal{M}'$  over B. It follows that every  $\mathcal{M}^d$ -definable continuous function  $f : (M')^n \to M$  is definable in  $\mathcal{M}'$ , over the same parameters. Indeed, the closure of the open set  $\{(x, y) \in (M')^{n+1} : y < f(x)\}$  is exactly  $\{(x, y) \in (M')^{n+1} : y < f(x)\}$ , from which the definability of f follows.

Finally, we show that every closed  $F \subseteq (M')^n$  set which is  $\mathcal{M}^d$ -definable over  $B \subseteq M'$  is definable in  $\mathcal{M}'$  over B. For every  $x \in M^n$  we let  $f(x) = d(x, F) = Inf\{d(x, y) : y \in F\}$ . By Lemma 5.13, this is a well defined function in  $\mathcal{M}^d$  (over B), and since F is closed, the function f is continuous and F is its zero set. Because f is definable in  $\mathcal{M}'$  over B, so is the set F.

Since every definable set in  $\mathcal{M}'$  can be defined over some  $B \subseteq M'$  with  $(B, B \cap M)$  free, the theorem follows.

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