### DESTRUCTIBILITY OF THE TREE PROPERTY AT $\aleph_{\omega+1}$

YAIR HAYUT AND MENACHEM MAGIDOR

ABSTRACT. We construct a model in which the tree property holds in  $\aleph_{\omega+1}$ and it is destructible under  $\operatorname{Col}(\omega, \omega_1)$ . On the other hand we discuss some cases in which the tree property is indestructible under small or closed forcings.

### 1. INTRODUCTION

A partial order  $\langle T, \leq_T \rangle$  is called a *tree*, if it has a minimal element and for every  $t \in T$ , the set  $\{s \in T \mid s \leq_T t\}$  is well ordered by  $\leq_T$ . The order type of the chain of elements that lie below t in the tree order is called the *level* of t and denoted by  $\text{Lev}_T(t)$ . For a cardinal  $\kappa$ , T is called a  $\kappa$ -tree if  $\sup_{t \in T} (\text{Lev}_T(t) + 1) = \kappa$  and the cardinality of each level of T is strictly below  $\kappa$ .

By a theorem of Kőnig, every  $\omega$ -tree has a cofinal branch (namely, a cofinal chain). On the other hand, a theorem of Aronszajn states that there is an  $\omega_1$ -tree that has no cofinal branches. Such a tree is called *Aronszajn tree*. For any larger successor cardinal,  $\kappa > \omega_1$ , it is independent of ZFC whether there is  $\kappa$ -tree with no cofinal branches. This question is related to other combinatorial topics and in order to get the consistency of the non-existence of  $\kappa$ -Aronszajn tree, one must assume the consistency of some large cardinals. If every  $\kappa$  tree has a cofinal branch, we say that  $\kappa$  has the tree property.

By a theorem of Silver, if uncountable cardinal  $\kappa$  has the tree property then  $\kappa$ is weakly compact in L. On the other end, Mitchell proved that if  $\kappa$  is weakly compact and  $\mu < \kappa$  is regular then there is a generic extension in which  $\kappa = \mu^{++}$ and the tree property holds at  $\kappa$ , thus showing that the tree property at the double successor of a regular cardinal is equiconsistent with the existence of a weakly compact cardinal. Where  $\kappa$  is the successor of a singular cardinal, the situation is more complicated. In [4], Magidor and Shelah showed that it is consistent, relative to some large cardinals, that the tree property holds at  $\aleph_{\omega+1}$ . The large cardinals assumption was later reduced by Sinapova and Neeman to the existence of an  $\omega$ sequence of supercompact cardinals (see, e.g. [5] for the Prikry-free version). In both constructions,  $\aleph_1$  plays a special role. It reflects, in some sense, the properties of  $\aleph_{\omega+1}$ .

In section 3 we will show that it is consistent to have a model in which the tree property holds at  $\aleph_{\omega+1}$ , but after collapsing  $\aleph_1$ , it fails. This extends a work by Cummings, Foreman and the second author [2, Theorem 14]. In this paper they show that it is possible that a weak square is added by a small forcing. Our arguments are very similar to the arguments there. In [6], Rinot shows that it is consistent that there is no special Aronszajn tree on  $\aleph_{\omega_1+1}$  and a  $\sigma$ -closed  $\aleph_2$ -Knaster forcing of cardinality  $\aleph_3$  introduces one. We note that we do not know how to apply a similar argument for this case.

In section 4 we discuss three cases in which the tree property at a successor of a singular cardinal is somewhat indestructible. In 4.1 we will show that it is consistent that the tree property holds at  $\aleph_{\omega^2+1}$  and it is indestructible under any forcing of cardinality  $\langle\aleph_{\omega^2}$ . In 4.2 we will show that the tree property at  $\aleph_{\omega+1}$  can be made indestructible under small  $\sigma$ -closed forcings.

### 2. Preliminaries

The following notation, due to Magidor and Shelah [4], plays an important role in the investigation of the tree property at successors of singular cardinals. For more information about narrow systems and their connections to squares we refer to [3].

**Definition 1.** Let  $\lambda$  be a regular cardinal. A *system* is a triplet  $S = \langle I, \kappa, \mathcal{R} \rangle$  such that:

- (1)  $I \subseteq \lambda$  unbounded.  $\kappa < \lambda$ .
- (2)  $\mathcal{R}$  is a collection of partial order relations on  $I \times \kappa$ .
- (3) Each  $R \in \mathcal{R}$  is a tree like partial order. R respects the lexicographic order on  $I \times \kappa$ . Namely,  $\langle \alpha, \zeta \rangle R \langle \beta, \xi \rangle$  implies  $\alpha \leq \beta$  and if  $\alpha = \beta$  then  $\zeta = \xi$ . Moreover, if  $\langle \beta, \xi \rangle, \langle \gamma, \rho \rangle R \langle \alpha, \zeta \rangle$  and  $\beta \leq \gamma$  then  $\langle \beta, \xi \rangle R \langle \gamma, \rho \rangle$ .
- (4) For every  $\alpha < \beta$  in *I* there are  $\zeta, \xi < \kappa$  and  $R \in \mathcal{R}$  such that  $\langle \alpha, \zeta \rangle R \langle \beta, \xi \rangle$ .

A branch through S is a set of elements in  $I \times \kappa$  which is a chain relative to some  $R \in \mathcal{R}$ . We say that a branch b meets the  $\alpha$ -th level of S if  $b \cap \{\alpha\} \times \kappa \neq \emptyset$ . A branch is *cofinal* if it meets cofinally many levels.

A system S is *narrow* if  $\max(\kappa^+, |\mathcal{R}|^+) < \lambda$ .

**Definition 2.** Let  $\lambda$  be a regular cardinal. We say that the *narrow system property* holds at  $\lambda$  if every narrow system of height  $\lambda$  has a cofinal branch.

Unlike the tree property, the narrow system property is indestructible by any small forcing. Let  $\mathbb{P}$  be a forcing notion with  $|\mathbb{P}|^+ < \lambda$  and let  $\dot{S}$  be a name for a narrow system. Let  $\dot{\mathcal{R}}$  be the collection of names of relations in S and let I be the set of all ordinals that can be levels of the  $\mathbb{P}$ . Let us define the narrow system  $\hat{S}$  in the natural way: the relations of  $\hat{S}$  are indexed by  $\mathbb{P} \times \dot{\mathcal{R}}$ , and let  $\langle \alpha, \beta \rangle (p, R) \langle \gamma, \delta \rangle$  iff  $p \Vdash \langle \alpha, \beta \rangle R \langle \gamma, \delta \rangle$  for  $R \in \dot{\mathcal{R}}$ . A branch in the system  $\hat{S}$  corresponds to a condition  $p \in \mathbb{P}$  and a set of element in S which are forced to be a branch in the generic extension by p.

## 3. Destructible tree property

**Theorem 3.** Let  $\kappa = \kappa_0 < \kappa_1 < \cdots$  be an  $\omega$ -sequence of supercompact cardinals. Then there is a forcing extension in which the tree property holds at  $\aleph_{\omega+1}$  and the forcing  $\operatorname{Col}(\omega, \omega_1)$  adds a special  $\aleph_{\omega+1}$ -Aronszajn tree.

We will prove something slightly stronger. We will define a forcing poset that forces that in the generic extension there is a partial weak square on  $\aleph_{\omega+1}$  whose domain contains all ordinals with cofinality above  $\omega_1$ , while the tree property holds at  $\aleph_{\omega+1}$ . If we further extend the universe and collapse  $\omega_1$  to be countable, then we can complete all the missing places in this square sequence by just adding  $\omega$ sequences. By a theorem of Shelah and Ben-David [1, Theorem 3], without violating the continuum hypothesis at  $\aleph_{\omega}$ , we cannot hope to have this kind of partial square with only one club at each ordinal, while having the tree property.

Let  $\mu = \sup \kappa_n$  and let  $\lambda = \mu^+$ .

We begin with some definitions:

**Definition 4.** A partial square on a set  $S \subseteq \lambda$  with width  $\langle \eta \rangle$  is a sequence  $C = \langle C_{\alpha} | \alpha < \lambda \rangle$  such that:

- (1) For every  $\alpha < \lambda$ ,  $\mathcal{C}_{\alpha}$  is a set of cardinality  $< \eta$ . If  $\alpha \in S$  then  $\mathcal{C}_{\alpha} \neq \emptyset$ .
- (2) Every  $D \in \mathcal{C}_{\alpha}$  is a closed and unbounded subset of  $\alpha$  and  $\operatorname{otp} D < \alpha$ .
- (3) If  $\beta \in \operatorname{acc} D$ ,  $D \in \mathcal{C}_{\alpha}$  then  $D \cap \beta \in \mathcal{C}_{\beta}$ .

When  $\lambda = \mu^+$ , we may assume that  $\operatorname{otp} D \leq \mu$  for every  $D \in \mathcal{C}_{\alpha}$ .

Since successor ordinals are never accumulation points of a club, the values of the square sequence at successor points are irrelevant. We will assume that  $C_{\alpha+1} = \{\alpha\}$  for every  $\alpha$ , for consistency.

We want to force a partial square for the set  $S^{\lambda}_{>\kappa}$  with width  $< \mu$ .

**Definition 5.** Let S be the following forcing notion. A condition  $s \in S$  is a sequence  $s = \langle c_i \mid i \leq \gamma \rangle$  for some ordinal  $\gamma < \mu^+$  such that all three requirements for the partial square sequence hold for every  $\alpha \leq \gamma$ . Namely,

- (1)  $\forall \alpha \leq \gamma, c_{\alpha} \text{ is a set of less than } \mu \text{ sets. If } cf \alpha \geq \kappa, \text{ then } c_{\alpha} \neq \emptyset.$
- (2) For every D ∈ c<sub>α</sub>, otp D ≤ μ and D is a closed and unbounded subset of α.
  (3) If β ∈ acc D, D ∈ c<sub>α</sub> then D ∩ β ∈ c<sub>β</sub>.

We order S by end extension.

We will think of the conditions  $s \in S$  as functions, so for  $s = \langle c_i | i \leq \gamma \rangle$  we will write dom  $s = \gamma + 1$  and  $s(i) = c_i$  for  $i \in \text{dom } s$ .

**Lemma 6.**  $\mathbb{S}$  is  $\kappa$ -directed closed.

Given a partial square C, we will define a threading forcing,  $\mathbb{T}_{\eta}$ . This forcing will add a club at  $\lambda$  with order type  $\eta$  such that all its initial segments are from C.

**Definition 7.** Let  $\mathbb{T}_{\eta} = \{D \mid \exists \alpha, D \in \mathcal{C}_{\alpha}, 1 < \operatorname{otp} D < \eta\}$ , ordered by end extension.

The following lemma is standard:

**Lemma 8.** Let  $\mathbb{S}, \mathbb{T}_{\eta}$  be as above. Then:

- (1)  $\mathbb{S}$  is  $\lambda$ -distributive.
- (2) Let C be the generic partial square added by S, and let  $\eta$  be a regular cardinal.  $\mathbb{S} * \mathbb{T}_{\eta}$  is equivalent to an  $\eta$ -directed closed forcing. Moreover, for every  $\rho < \mu, \mathbb{S} * \mathbb{T}_{\eta}^{\rho}$  (where we use full support power in  $V^{\mathbb{S}}$ ) contains an  $\eta$ -directed closed dense subset.

*Proof.* Let us show that S is  $\lambda$ -distributive. We will show that it is  $\eta$ -strategically closed for every regular  $\eta < \lambda$ . We will do this by showing the second part of the lemma – that  $S * \mathbb{T}_{\eta}$  contains a  $\eta$ -closed dense set.

Let us observe first that the set of conditions  $\langle s, \check{t} \rangle \in \mathbb{S} * \mathbb{T}_{\eta}$ , dom $(s) = \gamma + 1$ ,  $t \in s(\gamma)$  is dense. For every condition  $\langle s, \dot{t} \rangle$ ,

 $s \Vdash ``t is a member of some set in the square sequence",$ 

and therefore  $\dot{t}$  is forced to be a member of the ground model.

Thus, there is an extension of s, s', which decides the value of  $\dot{t}$  to be equal to an element in V, that we will denote by t. The closed set t might have no extension in  $s'(\max \operatorname{dom} s')$  but we can extend s' to s'' where dom  $s'' = \operatorname{dom} s' + \omega + 1$ , and thas an extension in the top element of s''. Let call this extension t'. Thus we have a condition  $\langle s'', t' \rangle \leq \langle s, t \rangle$  and  $\langle s'', t' \rangle$  has the desired form.

The set

 $D = \{ \langle s, \check{t} \rangle \in \mathbb{S} * \mathbb{T}_{\eta} \mid \max t = \max \operatorname{dom} s \}$ 

is  $\eta$ -directed closed. Let  $\rho < \eta$  and let  $\{\langle s_i, \check{t}_i \rangle \mid i < \rho\} \subset D$  be a directed set. Let us assume that sup dom  $s_i$  is a limit ordinal (otherwise, the sequence is fixed on a tail). The condition  $\langle s_\star, t_\star \rangle$ , where  $t_\star = \bigcup t_i$  and  $s_\star = (\bigcup s_i)^{\frown} \langle \{t_\star\} \rangle$  is a condition in D, stronger than  $s_i$  for all i.

The claim that  $\mathbb{S} * \mathbb{T}^{\rho}_{\eta}$  contains a  $\eta$ -closed dense subset (for all  $\rho < \mu$ ), is proved by the same method. For this case, we consider

$$D = \{ \langle s, \langle t_{\alpha} \mid \alpha < \rho \rangle \rangle \mid \forall \alpha < \rho, \max t_{\alpha} = \max \operatorname{dom} s \}.$$

By the same argument, using the fact that the bound on the cardinality of the set  $s(\max \operatorname{dom} s)$ , for  $s \in \mathbb{S}$ , is greater than  $\rho$ , we conclude that D is dense and  $\eta$ -directed closed in  $\mathbb{S} * \mathbb{T}_{\eta}^{\rho}$ .

Let us move now toward the proof of 3. Let  $\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$  be supercompact cardinals. By using Laver's preparation, we may assume that they are Laver-indestructible, i.e. that for every  $n < \omega$  and every  $\kappa_n$ -directed closed forcing  $\mathbb{P}$ ,  $\Vdash_{\mathbb{P}} \check{\kappa}_n$  is supercompact. Let  $\mathbb{M} = \prod_{i < \omega} \operatorname{Col}(\kappa_i, < \kappa_{i+1})$  a full support product of Levy collapses.

## **Lemma 9.** After forcing with $\mathbb{S} \times \mathbb{M}$ , the narrow system property holds at $\lambda$ .

*Proof.* Let  $H_S \subseteq \mathbb{S}$ ,  $H_M \subseteq \mathbb{M}$  be mutually generic filters. Let  $G = H_S \times H_M$ . Let us denote by  $H_i \subseteq \operatorname{Col}(\kappa_{i-1}, <\kappa_i)$  be the *i*-th coordinate of the generic filter  $H_M$  (i > 0). Let  $H^i$  be the generic filters for all the parts of  $\mathbb{M}$  except the *i*-th coordinate, namely  $H^i = \langle H_m \mid m \neq i \rangle$ .

Let  $S \in V[G]$  be a narrow system on  $I \times \eta$ , with relations  $\mathcal{R}$ . Let us assume, towards a contradiction, that S has no cofinal branch in V[G]. Since the set I will play no role later in the proof, we will restrict ourselves to the notation-wise simpler case in which  $I = \lambda$ . Let  $n \geq 2$  be large enough such that  $\kappa_{n-2} \geq |\eta \times \mathcal{R}|^+$  in  $V^{\mathbb{S} \times \mathbb{M}}$ .

Let  $W_n = V[H_S][H^n]$ . Let us force over  $W_n$  with  $\mathbb{T}_{\kappa_n}^{\kappa_n-2}$ . Let  $K = \langle K_i \mid i < \kappa_{n-2} \rangle$  be the sequence of pairwise mutually generic filters. We stress that the product,  $\mathbb{T}_{\kappa_n}^{\kappa_n-2}$ , is taken over V[G] and not over  $W_n$ .

Fix  $\xi < \kappa_{n-2}$ .  $W_n[K_{\xi}] \models \kappa_n$  is supercompact since:

(1)  $\mathbb{S} * \mathbb{T}_{\kappa_n}^{\kappa_{n-2}}$  contains a dense  $\kappa_n$ -directed closed subset,

(2)  $\prod_{n < i < \omega} \operatorname{Col}(\kappa_i, < \kappa_{i+1})$  is  $\kappa_n$ -directed closed.

(3)  $\prod_{i < n-1}^{-} \operatorname{Col}(\kappa_i, < \kappa_{i+1})$  has cardinality  $\kappa_{n-1}$  which is  $< \kappa_n$ .

We are using the indestructibility in the two first items and Lévy-Solovay Theorem in the last one.

Let  $j: W_n[K_{\xi}] \to M$  be a  $\lambda$ -supercompact embedding with crit  $j = \kappa_n$ . Since  $\operatorname{Col}(\kappa_{n-1}, < \kappa_n)$  is  $\kappa_n$ -c.c., after forcing with

$$\operatorname{Col}(\kappa_{n-1}, < j(\kappa_n)) = \operatorname{Col}(\kappa_{n-1}, < \kappa_n) \times \operatorname{Col}(\kappa_{n-1}, [\kappa_n, j(\kappa_n)))$$

we may extend the elementary embedding j to a  $\lambda$ -supercompact elementary embedding  $\tilde{j}: W_n[H_n][K_{\xi}] \to M[\tilde{j}(H_n)]$ . Since  $W_n[H_n] = V[G], S \in W_n[H_n]$ , so  $\tilde{j}(S)$  is defined.

Let  $L = \langle L_i | i < \kappa_{n-2} \rangle$  be a generic filter for  $\operatorname{Col}(\kappa_{n-1}, [\kappa_n, j(\kappa_n)))^{\kappa_{n-2}}$ . Note that the forcing that adds L is  $\kappa_{n-1}$ -closed over V, the ground model.

Let  $\delta = \sup \tilde{j}'' \lambda < \tilde{j}(\lambda)$ . Let  $\leq_i \in \mathcal{R}$  and let

$$b_{i,\epsilon} = \{ \langle \alpha, \beta \rangle \mid \langle j(\alpha), \beta \rangle \leq_i \langle \delta, \epsilon \rangle \text{ in } j(\mathcal{S}) \}.$$

Since  $|\mathcal{R}|, \eta < \kappa_{n-2} < \operatorname{crit} \tilde{j}$ , for some  $i, \epsilon, b_{i,\epsilon}$  is a cofinal branch and moreover  $\bigcup_{i,\epsilon} \{\alpha \mid \exists \beta, \langle \alpha, \beta \rangle \in b_{i,\epsilon}\} = \lambda$ .

We say that forcing with  $\operatorname{Col}(\kappa_{n-1}, [\kappa_n, j(\kappa_n))) \times \mathbb{T}_{\kappa_n}$  adds a system of branches for S. By removing a bounded part we may assume that all the branches in this system of branches are new and cofinal.

In particular the forcing  $\operatorname{Col}(\kappa_{n-1}, [\kappa_n, < j(\kappa_n)))^{\kappa_{n-2}} \times \mathbb{T}_{\kappa_n}^{\kappa_{n-2}}$  introduces  $\kappa_{n-2}$ many distinct realizations for the system of branches  $\{\dot{b}_j \mid j \in J\}$ . Note that in order to claim that there is no pair of system of branches which are equal we only used the pairwise mutual genericity.

We conclude that in V[G][H][K][L] there are  $\kappa_{n-2}$  different systems of branches,  $\{b_j^{\alpha} \mid \alpha < \kappa_{n-2}, j \in J\}$ . In this model  $\kappa_{n-2} \ge |\eta \times \mathcal{R}|^+$  is regular and cf  $\lambda \ge \kappa_{n-1}$ . Since for every  $\alpha < \beta < \kappa_{n-2}$ , and every relation  $\le_i \in \mathcal{R}, b_i^{\alpha}, b_i^{\beta}$  split at some point below  $\lambda$ , and since there are only  $\kappa_{n-2}$  realizations and only  $|\mathcal{R}|$  relations in  $\mathcal{R}$ , there is  $\rho_{\star} < \lambda$  such that for every  $\xi \geq \rho_{\star}$ , and for every  $\alpha, \beta, b_i^{\alpha}(\xi) \neq b_i^{\beta}(\xi)$  (where it is possible that only one of them is defined). By the Pigeonhole Principle there are  $\alpha, \beta < \kappa_{n-2}$  such that  $\langle \rho_{\star}, \xi \rangle \in b_i^{\alpha}, b_i^{\beta}$  for the same  $\xi, i$ , because there are only  $|\mathcal{R}| \times \eta$  many possibilities for this pair. This is a contradiction to the choice of  $\rho_{\star}$ . We conclude that it is impossible that there was not cofinal branch in  $\mathcal{S}$  in the ground model, as wanted.

Let  $W = V^{\mathbb{S} \times \mathbb{M}}$ . Note that  $\kappa$  is supercompact in W, by the Laver indestructibility of  $\kappa$ .

**Theorem 10.** There is  $\rho < \kappa$  such that forcing with  $\operatorname{Col}(\omega, \rho^{+\omega}) \times \operatorname{Col}(\rho^{+\omega+1}, < \kappa)$ over W forces the tree property at  $\aleph_{\omega+1}$ . Further collapsing the new  $\aleph_1$  introduces a weak square at  $\aleph_{\omega+1}$ .

Proof. Assume otherwise. Let  $\mathbb{L}_{\rho} = \operatorname{Col}(\omega, \rho^{+\omega}) \times \operatorname{Col}(\rho^{+\omega+1}, < \kappa)$ . For every  $\rho < \kappa$ , let  $\dot{T}_{\rho}$ , be a  $\mathbb{L}_{\rho}$ -name for an Aronszajn tree at  $\lambda$ . Since  $\kappa$  is supercompact, there is  $j: W \to M$  such that  ${}^{\lambda}M \subseteq M$ . By our assumption, M models that  $\Vdash_{j(\mathbb{L})_{\kappa}} "j(\dot{T})_{\kappa}$  is an Aronszajn tree". Let  $\delta = \sup j"\lambda < j(\lambda)$ , and let  $t = \langle \delta, 0 \rangle$ .

Work in M. For every  $\alpha < \lambda$ , pick a condition  $p_{\alpha} = \langle c_{\alpha}, q_{\alpha} \rangle$  such that

$$\exists \zeta < j(\kappa^{+\omega}), \, p_{\alpha} \Vdash_{j(\mathbb{L}_{\kappa})} \langle j(\alpha), \zeta \rangle \leq_{j(\dot{T})_{\kappa}} \check{t}$$

Let us denote this  $\zeta$  by  $\zeta_{\alpha}$ . We may pick the conditions  $p_{\alpha}$  in a way that  $q_{\alpha}$  is a decreasing sequence. Since  $\lambda$  is regular and  $|\operatorname{Col}(\omega, \kappa^{+\omega})| = \kappa^{+\omega} < \lambda$ , there is a cofinal set  $I \subseteq \lambda$ ,  $n < \omega$  and  $c_{\star} \in \operatorname{Col}(\omega, \kappa^{+\omega})$  such that for every  $\alpha \in I$ ,  $c_{\alpha} = c_{\star}$ and  $\zeta_{\alpha} < j(\kappa^{+n})$ .

By elementarity, for every  $\alpha, \beta \in I$ , there are  $\gamma, \gamma' < \kappa^{+n}$ ,  $\rho < \kappa$  and  $p \in \mathbb{L}_{\rho}$  such that  $p \Vdash_{\mathbb{L}_{\rho}} \langle \alpha, \gamma \rangle \leq_{\dot{T}_{\rho}} \langle \beta, \gamma' \rangle$ .

This defines a narrow system in W: The domain of the system is  $I \times \kappa^{+n}$ . The indices set is  $\bigcup_{\rho < \kappa} \mathbb{L}_{\rho} \times \{\rho\}$ .  $\langle \alpha, \xi \rangle \leq_{p,\rho} \langle \beta, \zeta \rangle$  iff  $p \Vdash_{\mathbb{L}_{\rho}} \langle \alpha, \xi \rangle \leq_{\dot{T}_{\rho}} \langle \beta, \zeta \rangle$ .

By the narrow system property there is a cofinal branch in W. Namely there are  $\rho < \kappa$ ,  $p \in \mathbb{L}_{\rho}$  and  $\gamma < \kappa^{+n}$  such that for every  $\alpha, \beta \in I$ ,  $p \Vdash_{\mathbb{L}_{\rho}} \langle \alpha, \gamma \rangle \leq \langle \beta, \gamma \rangle$ .

This proves that the tree property holds at  $\aleph_{\omega+1}$  in the generic extension.

For the last claim, note that after collapsing  $\aleph_1$ , for every  $\gamma < \aleph_{\omega+1}$  either cf  $\gamma = \omega$  or  $\mathcal{C}_{\gamma} \neq \emptyset$ . Thus, one can complete the partial square to a full  $\Box_{\aleph_{\omega}, <\aleph_{\omega}}$  by adding cofinal  $\omega$ -sequences.

# 4. Indestructible tree property

In this section we will build three models in which the tree property at a successor of singular cardinal is indestructible under certain class of forcing notions. We start by building a model in which the tree property holds at  $\aleph_{\omega^2+1}$  and it is indestructible under any forcing  $\mathbb{P}$  of cardinality less than  $\aleph_{\omega^2}$ . Similarly, we will construct a model for the tree property at  $\aleph_{\omega+1}$  in which the tree property still holds after any  $\sigma$ -closed forcing of cardinality  $< \aleph_{\omega}$ .

We remark that we do not know whether it is possible to force the tree property at  $\aleph_{\omega+1}$  to be indestructible under any  $\aleph_{\omega+1}$ -closed forcing notions.

4.1. Indestructible Tree Property for  $\aleph_{\omega^2+1}$ . In this subsection, we will show that in Sinapova's model for the tree property at  $\aleph_{\omega^2+1}$  [7] (but without the failure of SCH, as in [8]), the tree property is indestructible under small forcings. We start with some simple observations:

**Lemma 11.** Let  $\lambda$  be a cardinal such that the tree property holds at  $\lambda^+$  and it is indestructible by any forcing of the form  $\operatorname{Col}(\omega, \rho)$  for  $\rho < \lambda$ . Then the tree property

at  $\lambda^+$  is indestructible by any forcing of size  $< \lambda$ . Moreover, it is enough to assume that for every  $\rho < \lambda$  there is  $\rho \leq \rho' < \lambda$  such that  $\operatorname{Col}(\omega, \rho')$  forces the tree property at  $\lambda^+$ .

*Proof.* Let  $\mathbb{P}$  be a forcing notion of cardinality  $< \lambda$ . Let  $\mu = |\mathbb{P}|$ . Col $(\omega, \rho)$  adds a generic filter for  $\mathbb{P}$ . Let  $G \subseteq \mathbb{P}$  be a generic filter. The quotient forcing  $\operatorname{Col}(\omega, \rho)/G$ has cardinality at most  $\rho$  and therefore it does not add a cofinal branch to any  $\lambda^+$ -Aronszajn tree. Since the tree property holds after forcing with  $\operatorname{Col}(\omega, \rho)$  and the forcing  $\operatorname{Col}(\omega, \rho)/G$  does not add a branch to Aronszajn tree – the tree property holds in V[G] as well. 

**Theorem 12.** Let  $\kappa = \kappa_0 < \kappa_1 < \cdots$  be a sequence of  $\omega$  supercompact cardinals. Let  $\mu = \sup \kappa_n$  and  $\lambda = \mu^+$ . There is a generic extension in which  $\kappa = \aleph_{\omega^2}$ .  $\lambda = \aleph_{\omega^2+1}$  and for every  $\rho < \mu$ , the tree property holds after forcing with  $\operatorname{Col}(\omega, \rho)$ .

In order to prove this theorem, we will work with Sinapova's model for the tree property at  $\aleph_{\omega^2+1}$  from [7]. We will not need to violate SCH at this point, so the proof is somewhat simpler at some points.

The main idea behind the indestructibility is that one can define a projection  $f: \mathbb{P} \times \operatorname{Col}(\omega, \rho) \to \mathbb{P}_n$  that shifts the Prikry sequence by n steps to the left, where  $\mathbb{P}_n$  is a "shifted" version of the forcing  $\mathbb{P}$  which forces the tree property as well. This way, we can analyze the sets that were added by a forcing of the form  $\operatorname{Col}(\omega, \rho)$ simply by shifting the first element of the Prikry sequence to be above  $\rho$ .

We start with a well known fact:

**Lemma 13.** Let  $\mathbb{M} = \prod_{n < \omega} \operatorname{Col}(\kappa_n, < \kappa_{n+1})$  - a full support product of Levy collapses. In  $V^{\mathbb{M}}$  the narrow branch property holds at  $\lambda^+$ .

The proof is similar to the proof of Lemma 9 and appears in [5].

Work in  $V^{\mathbb{M}}$ . The cardinal  $\kappa = \kappa_0$  is still supercompact, by the Laver indestructibility. Let  $\mathcal{U}$  be a normal measure on  $P_{\kappa}\lambda$  in  $V^{\mathbb{M}}$ . Let  $\mathcal{U}_n$  be the projection of  $\mathcal{U}$ to  $P_{\kappa}\kappa_n$  for  $n < \omega$ .

Let  $j_n: W \to N_n \cong \text{Ult}(W, \mathcal{U}_n)$  be the elementary embedding derived from  $\mathcal{U}_n$ . Let us construct an  $N_n$ -generic filter  $H_n$  for the forcing  $\operatorname{Col}(\kappa^{+\omega+2}, < j(\kappa))^{N_n}$ . This is possible by the standard arguments: the forcing notion  $\operatorname{Col}(\kappa^{+\omega+2}, < j(\kappa))^{N_n}$  is  $\kappa^{+n+1}$ -closed in W and has only  $\kappa^{+n+1}$ -dense subsets in  $N_n$  (as counted by  $V^{\mathbb{M}}$ ).

Let us define the main forcing notion  $\mathbb{P}$ :

A condition  $p \in \mathbb{P}$  has the following form

$$p = \langle d_0, a_0, c_0, \dots, a_{n-1}, c_{n-1}, A_n, C_n, \dots \rangle$$

where:

- (1)  $a_i \in P_{\kappa} \kappa^{+i}$  and  $A_i \in \mathcal{U}_i$ . Let  $\rho_i = a_i \cap \kappa$  if i < n and  $\rho_i = \kappa$  otherwise.
- (2)  $d_0 \in \operatorname{Col}(\omega, \rho_0^{+\omega})$  if  $\rho_0 < \kappa$  and otherwise  $d_0 \in \operatorname{Col}(\omega, \kappa)$ . (3)  $c_i \in \operatorname{Col}(\rho_i^{+\omega+2}, < \rho_{i+1})$
- (4)  $C_i: A_i \to W$  such that  $C_i(a) \in \operatorname{Col}((a \cap \kappa)^{+\omega+2}, <\kappa)$  for every  $a \in A_i$  and  $[C_i]_{\mathcal{U}_i} \in H_i.$

n is called the length of p and we denote len(p) = n. A condition p is stronger than  $q \ (p \le q)$  if:

- (1)  $\operatorname{len}(p) \ge \operatorname{len}(q)$

- (1)  $\operatorname{Rel}(p) \geq \operatorname{Rel}(q)$ (2)  $d_0^p \leq d_0^q$ . (3)  $a_i^p = a_i^q$  and  $c_i^p \leq c_i^q$  for every  $i < \operatorname{len}(q)$ . (4)  $a_i^p \in A_i^q$  and  $c_i^p \leq C_i^q(a_i)$  for  $\operatorname{len}(q) \leq i < \operatorname{len}(p)$ (5)  $A_i^p \subseteq A_i^q$  for  $i \geq \operatorname{len}(p)$ . (6)  $C_i^p(a) \leq C_i^q(a)$  for every  $a \in A_i^p$ .

For the proof of Theorem 12, we will also need to consider the following shifted version of  $\mathbb{P}$ . For every  $s < \omega$ , we define the forcing  $\mathbb{P}_s$ .

A condition  $p \in \mathbb{P}_s$  has the following form

$$p = \langle d_0, a_0, c_0, \dots, a_{n-1}, c_{n-1}, A_n, C_n, \dots \rangle$$

where:

- (1)  $a_i \in P_{\kappa} \kappa^{+i+s}$  and  $A_i \in \mathcal{U}_{i+s}$ . Let  $\rho_i = a_i \cap \kappa$  if i < n and  $\rho_i = \kappa$  otherwise.

- (2)  $d_0 \in \operatorname{Col}(\omega, \rho_0^{+\omega})$  if  $\rho_0 < \kappa$  and otherwise  $d_0 \in \operatorname{Col}(\omega, \kappa)$ . (3)  $c_i \in \operatorname{Col}(\rho_i^{+\omega+2}, < \rho_{i+1})$ (4)  $C_i \colon P_\kappa \kappa^{+i+s} \to W$  such that  $C_i(a) \in \operatorname{Col}((a \cap \kappa)^{+\omega+2}, < \kappa)$  for every  $a \in A_i$ and  $[C_i]_{\mathcal{U}_{i+s}} \in H_{i+s}$ .

We order the conditions in the same way as we did for  $\mathbb{P}$ . Note that  $\mathbb{P}_0 = \mathbb{P}$ .

**Theorem 14** (Sinapova). For every  $s < \omega$ ,  $\mathbb{P}_s$  forces that  $\lambda = \aleph_{\omega^2+1}$  and the tree property holds in  $\lambda$ .

*Proof.* We will give a sketch of the proof. We will show that the claim holds for s = 0. The argument for general s is the same, by notation-wise more complicated.

Let  $p \in \mathbb{P}$  be a condition and let  $\dot{T}$  be a name for a  $\lambda$ -Aronszajn tree. Let n be the length of p. Let  $j: V \to M$  be a  $\lambda$ -supercompact embedding, with critical point  $\kappa$  which is compatible with  $\mathcal{U}_n$  (namely  $\mathcal{U}_n$  is the  $P_{\kappa}\kappa^{+n}$  measure which is derived from j).

In M, let us look at the forcing  $j(\mathbb{P})$  below a condition  $q \leq j(p)$  of length n+1such that  $a_n^q = j'' \kappa^{+n}$ . In other words, q is an extension of j(p) that forces that the n + 1-th element of the diagonal Prikry sequence is  $j'' \kappa^{+n}$ . The forcing  $j(\mathbb{P})/q$ preserves  $\lambda$  as a regular cardinal and realizes j(T) to be a  $j(\lambda)$ -Aronszajn tree.

Let us denote  $\delta = \sup j \, \lambda \langle j(\lambda) \rangle$  and let us look at the name of a partial branch  $\{\langle j(\alpha), \zeta_{\alpha}\rangle \mid M^{j(\mathbb{P})} \models \langle j(\alpha), \zeta_{\alpha}\rangle \leq_{j(\dot{T})} \langle \delta, 0\rangle\}.$ 

Using the Prikry property, we may find a direct extension of  $q, q^*$ , such that for every  $\alpha < \lambda$  the value of  $k < \omega$  such that  $\zeta_{\alpha} < j(\kappa^{+k})$  is determined by  $q^{\star}$  up to forcing with the first n lower parts of  $j(\mathbb{P})$   $(n < \omega)$ . Since there are less than  $\lambda$ many possible values for the first n coordinates of the conditions below  $q^{\star}$ , there is a cofinal subset of  $\lambda$ , I, a natural number  $n_{\star} < \omega$  large enough and a fixed lower part  $a_{\star}$  of length  $n_{\star} \geq n+1$  such that

$$I = \{ \alpha < \lambda \mid \exists r \le q^{\star}, \operatorname{stem}(r) = a_{\star}, r \Vdash \exists \zeta < j(\kappa^{+n_{\star}}), \langle j(\alpha), \zeta \rangle \le \langle \delta, 0 \rangle \}.$$

In particular, for every  $\alpha, \beta \in I$ , M thinks that there is an extension of  $j(p), q^{\star\star}$ of length n+1 and ordinals  $\zeta, \zeta' < j(\kappa^{+n_\star})$  such that  $q^{\star\star} \Vdash \langle j(\alpha), \zeta \rangle \leq_{j(\dot{T})} \langle j(\beta), \zeta' \rangle$ . Reflecting this to V we conclude that for every  $\alpha, \beta \in I$  there is a condition  $q' \leq p$ with stem of length n+1 and  $\zeta, \zeta' < \kappa^{+n_{\star}}$  such that  $q' \Vdash \langle \alpha, \zeta \rangle \leq_{\dot{T}} \langle \beta, \zeta' \rangle$ .

This defines a narrow system on  $I \times \kappa^{+n_{\star}}$ , indexed by the stems of length n+1which are stems of some condition which is stronger than p. By the narrow system property, there is a cofinal branch. So there is  $I' \subseteq I$ , a stem  $s_{\star}$  and an ordinal  $\zeta_{\star} < \kappa^{+n_{\star}}$  such that for every  $\alpha < \beta$  in I' there is a condition q with stem  $s_{\star}$  forcing  $\langle \alpha, \zeta_{\star} \rangle \leq_{\dot{T}} \langle \beta, \zeta_{\star} \rangle.$ 

Next we will build inductively a sequence of conditions  $\langle p_{\alpha} \mid \alpha \in I' \setminus \rho \rangle$  (for some  $\rho < \lambda$ ), such that for every  $\alpha < \beta$ ,

$$p_{\alpha} \wedge p_{\beta} \Vdash \langle \alpha, \zeta_{\star} \rangle \leq_{\dot{T}} \langle \beta, \zeta_{\star} \rangle$$

The construction is done by induction on  $m < \omega$ , where at each step we define  $p_{\alpha} \upharpoonright m$  in a way that for all  $\alpha, \beta$  (except a bounded segment) there is a condition q with  $q \upharpoonright m = p_{\alpha} \upharpoonright m \land p_{\beta} \upharpoonright m$  such that

$$q \Vdash \langle \alpha, \zeta_\star \rangle \leq_{\dot{T}} \langle \beta, \zeta_\star \rangle.$$

Extending  $p_{\alpha} \upharpoonright m$  to  $p_{\alpha} \upharpoonright (m+1)$  is done by defining a narrow system corresponding to the possible extension and using the branch in order to define the relevant value for all  $\alpha \in I'$  above the first level that the branch meets.

Eventually, we obtain a sequence of conditions  $\{p_{\alpha} \mid \alpha \in I' \setminus \rho\}$ , for some  $\rho < \lambda$ ,  $p_{\alpha} \leq p$ . Using the chain condition of the forcing  $\mathbb{P}$  we conclude that there is an extension of p that forces that for unbounded many ordinals  $\alpha < \lambda$ ,  $p_{\alpha}$  will be in the generic filter. But then  $\{\langle \alpha, \zeta_* \rangle \mid p_{\alpha} \in G\}$  is a cofinal branch in  $\dot{T}$  (where G is the generic filter for  $\mathbb{P}$ ).

In order to show the indestructibility, we need to show that there is a simple connection between the different shifts of the forcing:

**Lemma 15.** Let  $p \in \mathbb{P}$ ,  $\operatorname{len}(p) = n + 1$ ,  $n \geq 1$  and let  $f \in \operatorname{Col}(\omega, \rho_n^{+\omega})$ . There is a condition  $q \in \mathbb{P}_n$ , of length one such that  $\rho_0^q = \rho_n^p$ , such that  $\mathbb{P}_n/q \cong (\mathbb{P}/p) \times (\operatorname{Col}(\omega, \rho_n^{+\omega})/f)$ .

*Proof.* Let  $\eta = (\rho_n^p)^{+\omega}$ .

The forcing  $\mathbb{P}/p$  is the product  $\mathbb{C}/p^{\leq n} \times \mathbb{P}^{\geq n}/p^{\geq n}$  where

$$\mathbb{C} = \operatorname{Col}(\omega, (\rho_0^p)^{+\omega}) \times \prod_{i < n} \operatorname{Col}((\rho_i^p)^{+\omega+2}, < \rho_{i+1}^p)$$

and  $\mathbb{P}^{\geq n}$  is the set of the *n*-upper part of the conditions of  $\mathbb{P}$ . More precisely, a condition  $s \in \mathbb{P}^{\geq n}$  is an  $\omega$ -sequence of the form

$$s = \langle a_n^s, c_n^s, \dots a_{l-1}^s, c_{l-1}^s, A_l^s, C_l^s, \dots \rangle,$$

where  $l \ge n$  and  $a_i^s, c_i^s, A_i^s, C_i^s$  are as in the definition of  $\mathbb{P}$  (in particular,  $a_i^s \in P_{\kappa}\kappa^{+i}$ ).

The conditions  $p^{\geq n} \in \mathbb{P}^{\geq n}, p^{< n} \in \mathbb{C}$  are defined as follows:

$$p^{
$$p^{\geq n} = \langle a_n^p, c_n^p, A_{n+1}^p, C_{n+1}^p, \dots, A_l^p, C_l^p, \dots \rangle.$$$$

Clearly,  $|\mathbb{C}| \leq \eta$  and thus  $(\mathbb{C}/p^{< n}) \times (\operatorname{Col}(\omega, \eta)/f) \cong \operatorname{Col}(\omega, \eta)$ . Let us fix an isomorphism  $\pi_0$ :  $\operatorname{Col}(\omega, \eta) \to (\mathbb{C}/p^{< n}) \times (\operatorname{Col}(\omega, \eta)/f)$ . Note that  $\pi_0(\emptyset) = (p^{< n}, f)$ .

Let  $q \in \mathbb{P}_n$  be the condition  $\langle \emptyset \rangle^{\frown} p^{\geq n}$ .

By the definition of  $\mathbb{P}_n$  and  $\mathbb{P}^{\geq n}$ ,

$$\mathbb{P}_n/q \cong \operatorname{Col}(\omega, \eta) \times \left(\mathbb{P}^{\geq n}/p^{\geq n}\right).$$

Combining this with the isomorphism  $\pi_0$ , we obtain the isomorphism:

$$\mathbb{P}_n/q \cong (\mathbb{P}/p) \times (\operatorname{Col}(\omega, \eta)/f).$$

**Theorem 16.**  $\mathbb{P}$  forces the tree property at  $\aleph_{\omega^2+1}$  to be indestructible by any forcing of size  $\langle \aleph_{\omega^2}$ .

*Proof.* Is it enough to show that it is the case for  $\operatorname{Col}(\omega, \aleph_{\omega \cdot n})$ . Recall that  $\aleph_{\omega \cdot n} = \rho_n^{+\omega}$  so we are in the situation of Lemma 15. This means that after forcing with  $\operatorname{Col}(\omega, \aleph_{\omega \cdot n})$  the tree property holds, as the iteration is isomorphic to the forcing notion  $\mathbb{P}_n$  below some condition.

4.2. Indestructible Tree property for  $\aleph_{\omega+1}$  under small  $\sigma$ -closed forcings. Let us construct a model very similar to subsection 4.1, in which we have the tree property at  $\aleph_{\omega+1}$  and it will be indestructible under any  $\sigma$ -closed forcing of cardinality  $\langle \aleph_{\omega}$ . The additional restriction on the forcing notions (namely that the forcing is  $\sigma$ -closed), implies that those forcing notions cannot collapse  $\omega_1$ .

**Theorem 17.** It is consistent, relative to the existence of  $\omega$  many supercompact cardinals, that the tree property holds at  $\aleph_{\omega+1}$  and it is indestructible under any  $\sigma$ -closed forcing of cardinality  $< \aleph_{\omega}$ .

*Proof.* We will start with a model of the narrow system property at  $\kappa^{+\omega+1}$  for  $\kappa$ a supercompact cardinal. This can be obtained, for example, by forcing with the product of the Levy collapses between the supercompact cardinals as in Lemma 13. Let  $\mathcal{U}_0$  be a normal ultrafilter on  $\kappa$  generated from a  $\kappa^{+\omega+1}$ -supercompact elementary embedding,  $j: V \to M$ .

Let us show that for every  $n < \omega$ , there is a large set  $A_n \in \mathcal{U}_0$  such that for every  $\rho \in A_n$ , forcing with  $\mathbb{L}_{\rho} = \operatorname{Col}(\omega, \rho^{+\omega}) \times \operatorname{Col}(\rho^{+\omega+1}, \kappa^{+n})$  forces the tree property at  $\kappa^{+\omega+1}$ .

Assume that this is not the case and let  $\dot{T}_{\rho}$  be a counter example for every bad choice of  $\rho$ , for a fixed  $n < \omega$ . Since the set of bad choices is in  $\mathcal{U}_0$ ,  $\kappa$  is a bad choice of ordinal in M. Let us force with  $j(\mathbb{L})_{\kappa}$ , and let M[H] be the generic extension. Let  $T = j(\dot{T})^{H}_{\kappa}$  be an Aronszajn tree at  $j(\kappa^{+\omega+1})$ . Let  $\delta = \sup j \, {}^{*}\kappa^{+\omega+1}$  and for every  $\alpha < \kappa^{+\omega+1}$  let  $\beta_{\alpha} < j(\kappa^{+\omega})$  be the element in the level  $j(\alpha)$  below  $\langle \delta, 0 \rangle$ .

Using the same arguments as in the proof of Theorem 10, there is a cofinal set  $I \subseteq \kappa^{+\omega+1}$ , a decreasing sequence of conditions  $q_{\alpha} \in \operatorname{Col}(\kappa^{+\omega+1}, j(\kappa)^{+n})$ , a condition  $p \in \operatorname{Col}(\omega, \kappa^{+\omega})$  and a natural number  $N < \omega$  such that for every  $\alpha \in I$ there is  $\beta < j(\kappa^{+N})$  such that  $(p, q_{\alpha}) \Vdash \langle j(\alpha), \beta \rangle \leq_T \langle \delta, 0 \rangle$ . Reflecting this back to V, we conclude that for every  $\alpha, \alpha' \in I$ :

 $\exists \beta, \beta' < \kappa^{+N}, \ \rho < \kappa, \ p \in \mathbb{L}_{\rho} \text{ such that } p \Vdash_{\mathbb{L}_{\rho}} \langle \alpha, \beta \rangle \leq_{T_{\rho}} \langle \alpha', \beta' \rangle.$ 

This gives us a narrow system, similar to the one in the proof of Theorem 10. A branch through this system provides us an ordinal  $\rho$  which was a bad choice, a condition  $r \in \mathbb{L}_{\rho}$ , a cofinal set  $J \subseteq I$  and for all  $\alpha \in J$  an ordinal  $\beta_{\alpha} < \kappa^{+N}$  such that for all  $\alpha, \alpha' \in J$ ,

$$r \Vdash_{\mathbb{L}_{\alpha}} \langle \alpha, \beta_{\alpha} \rangle, \langle \alpha', \beta_{\alpha'} \rangle$$
 are compatible.

This is a contradiction to the fact that this  $\dot{T}_{\rho}$  was a name for an  $\lambda$ -Aronszajn tree.

Let  $A = \bigcap_{n < \omega} A_n$  and let  $\rho \in A$ . Forcing with  $\operatorname{Col}(\omega, \rho^{+\omega}) \times \operatorname{Col}(\rho^{+\omega+1}, \kappa)$ forces the tree property. For every small  $\sigma$ -closed forcing notion  $\mathbb{Q}$  there is n such that  $\operatorname{Col}(\rho^{+\omega+1},\kappa) * \mathbb{Q}$  is a regular subforcing of  $\operatorname{Col}(\rho^{+\omega+1},\kappa^{+n})$  and since the tree property holds after this forcing and since the quotient is small and thus cannot add branches to Aronszajn trees - we are done.  $\square$ 

## 5. Open questions

In Section 4.1 we proved that the tree property at  $\aleph_{\omega^2+1}$  can be made indestructible under any small forcing poset.

Question 1. Is it consistent that the tree property at  $\aleph_{\omega+1}$  is indestructible under any forcing of cardinality  $\langle \aleph_{\omega} \rangle$ 

On the other hand, one can ask whether it is possible to extend the results of Theorem 3.

Question 2. Is it consistent that the tree property holds at  $\aleph_{\omega+1}$  but there is a small forcing (of cardinality  $\langle \aleph_{\omega} \rangle$ ), that does not collapse cardinals and adds an  $\aleph_{\omega+1}$ -Aronszajn tree?

Note that in all the currently known models for the tree property at  $\aleph_{\omega+1}$ , adding a single Cohen real does not add an Aronszajn tree at  $\aleph_{\omega+1}$ . So we ask the following stronger version of Question 2:

**Question 3.** Is it consistent that the tree property holds at  $\aleph_{\omega+1}$  but adding a Cohen real adds an  $\aleph_{\omega+1}$ -Aronszajn tree?

This question is particularly interesting when we assume that  $\aleph_{\omega}$  is strong limit since then adding a Cohen real cannot add a weak square for  $\aleph_{\omega}$ , assuming that there is no weak square in the ground model.

## 6. Acknowledgments

We would like to thank the anonymous referee for improving the readability and accuracy of this paper.

#### References

- Shai Ben-David and Saharon Shelah, Souslin trees and successors of singular cardinals, Ann. Pure Appl. Logic 30 (1986), no. 3, 207–217. MR 836425 (87h:03078)
- James Cummings, Matthew Foreman, and Menachem Magidor, Squares, scales and stationary reflection, J. Math. Log. 1 (2001), no. 1, 35–98. MR 1838355 (2003a:03068)
- [3] Chris Lambie-Hanson, Squares and narrow systems, J. Symb. Log. 82 (2017), no. 3, 834–859.
   MR 3694332
- Menachem Magidor and Saharon Shelah, The tree property at successors of singular cardinals, Arch. Math. Logic 35 (1996), no. 5-6, 385–404. MR 1420265
- [5] Itay Neeman, The tree property up to  $\aleph_{\omega+1},$  J. Symb. Log. **79** (2014), no. 2, 429–459. MR 3224975
- [6] Assaf Rinot, A cofinality-preserving small forcing may introduce a special Aronszajn tree, Arch. Math. Logic 48 (2009), no. 8, 817–823. MR 2563820 (2011i:03042)
- [7] Dima Sinapova, The tree property and the failure of the singular cardinal hypothesis at ℵ<sub>ω<sup>2</sup></sub>,
   J. Symbolic Logic 77 (2012), no. 3, 934–946. MR 2987144
- [8] \_\_\_\_\_, The tree property at ℵ<sub>ω+1</sub>, J. Symbolic Logic 77 (2012), no. 1, 279–290. MR 2951641 E-mail address, Yair Hayut: yair.hayut@mail.huji.ac.il

EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, THE HEBREW UNIVERSITY OF JERUSALEM, GIVAT RAM. JERUSALEM, 9190401, ISRAEL

#### E-mail address, Menachem Magidor: mensara@savion.huji.ac.il

EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, THE HEBREW UNIVERSITY OF JERUSALEM, GIVAT RAM. JERUSALEM, 9190401, ISRAEL