ON THE EXISTENCE OF LARGE ANTICHAINS FOR DEFINABLE QUASI-ORDERS

BENJAMIN D. MILLER AND ZOLTÁN VIDNYÁNSZKY

ABSTRACT. We generalize Harrington-Marker-Shelah's Dilworthstyle characterization of the existence of non-empty perfect antichains to co-analytic quasi-orders, establish the analogous theorem at the next definable cardinal, and consider generalizations beyond the first level of the projective hierarchy.

Introduction

A quasi-order is a reflexive transitive binary relation. Associated with every such relation R on a set X are the equivalence relation $x \equiv_R y \iff (x R y \text{ and } y R x)$ and the incomparability relation $x \perp_R y \iff (\neg x R y \text{ and } \neg y R x)$. We say that a set $Y \subseteq X$ is an R-antichain if $R \upharpoonright Y$ is the diagonal on Y, and an R-chain if $\perp_R \upharpoonright Y$ is empty. A subset of a topological space is perfect if it is closed and has no isolated points, Borel if it is in the σ -algebra generated by the open sets, analytic if it is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$, and co-analytic if its complement is analytic. In §1, we generalize [HMS88, Theorem 5.1] from Borel to co-analytic quasi-orders:

Theorem 1. Suppose that X is a Hausdorff space and R is a co-analytic quasi-order on X. Then exactly one of the following holds:

- (1) The space X is a union of countably-many Borel R-chains.
- (2) There is a non-empty perfect R-antichain.

Our proof uses only Baire category arguments and the \mathbb{G}_0 dichotomy [KST99, Theorem 6.3], which itself has a classical proof [Mil12]. An interesting new wrinkle is that, while such arguments typically utilize just one application of the \mathbb{G}_0 dichotomy, ours requires infinitely many.

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A homomorphism from a binary relation R on X to a binary relation S on Y is a function $\phi\colon X\to Y$ such that $(\phi\times\phi)(R)\subseteq S$, and a reduction of R to S is a homomorphism from R to S that is also a homomorphism from R to R to R borel equivalence relation R on an analytic Hausdorff space R is smooth if there is a Borel-measurable reduction of R to equality on R0, and an analytic set R1 is R2. We establish the analog of [HMS88, Theorem 5.1] at the next Borel cardinal:

Theorem 2. Suppose that X is an analytic Hausdorff space and R is a Borel quasi-order on X. Then exactly one of the following holds:

- (1) There is a smooth Borel superequivalence relation of \equiv_R whose equivalence classes are R-chains.
- (2) There is an \equiv_R -non-smooth perfect set whose quotient by \equiv_R is an (R/\equiv_R) -antichain.

Our proof uses only Baire category arguments and the \mathbb{E}_0 dichotomy [HKL90, Theorem 1.1], which itself has a classical proof [Mil12], and reveals that the theorem holds for the rather degenerate reason that its two alternatives are equivalent to those of the \mathbb{E}_0 dichotomy (for \equiv_R).

A subset of a topological space X is κ -Borel if it is in the κ -complete algebra generated by the open sets, κ -Souslin if it is a continuous image of a closed subset of $\kappa^{\mathbb{N}}$, co- κ -Souslin if its complement is κ -Souslin, bi- κ -Souslin if it is both κ -Souslin and co- κ -Souslin, and \aleph_0 -universally Baire if its pre-image under every continuous function $\phi \colon 2^{\mathbb{N}} \to X$ has the Baire property. Let \mathbb{E}_0 denote the equivalence relation on $2^{\mathbb{N}}$ given by $c \mathbb{E}_0 d \iff \exists n \in \mathbb{N} \forall m \geq n \ c(m) = d(m)$. An embedding is an injective reduction. In §3, we note that our arguments also yield:

Theorem 3. Suppose that κ is an aleph, X is a Hausdorff space, and R is an \aleph_0 -universally-Baire co- κ -Souslin quasi-order on X. Then at least one of the following holds:

- (1) The space X is a union of κ -many R-chains.
- (2) There is a non-empty perfect R-antichain.

Theorem 4. Suppose that κ is an aleph, X is a Hausdorff space, and R is an \aleph_0 -universally-Baire bi- κ -Souslin quasi-order on X. Then at least one of the following holds:

- (1) There is a homomorphism from \equiv_R to equality on 2^{κ} such that the pre-image of every singleton is an R-chain.
- (2) There is a continuous embedding $\pi: 2^{\mathbb{N}} \to X$ of \mathbb{E}_0 into \equiv_R such that $\pi(2^{\mathbb{N}})/\equiv_R$ is an (R/\equiv_R) -antichain.

In the special case that X is analytic, one can use the (non-classical) arguments of [Kan97] to establish the strengthenings in which the objects in condition (1) of these results are κ^+ -Borel/ κ^+ -Borel measurable.

A subset of an analytic Hausdorff space is Σ_1^1 if it is analytic, Π_n^1 if its complement is Σ_n^1 , Σ_{n+1}^1 if it is a continuous image of a Π_n^1 set, and Δ_n^1 if it is both Π_n^1 and Σ_n^1 . Let δ_n^1 denote the supremum of the lengths of all Δ_n^1 pre-wellorderings of $\mathbb{N}^{\mathbb{N}}$. We say that a Δ_{2n+1}^1 equivalence relation E on an analytic Hausdorff space X is smooth if there exists $\kappa < \delta_{2n+1}^1$ for which there is a Δ_{2n+1}^1 -measurable reduction of E to equality on 2^{κ} , and an analytic set $A \subseteq X$ is E-smooth if $E \upharpoonright A$ is smooth. Taking the known structure theory of the projective sets as a black box, we note that our arguments also provide classical proofs of the relevant special cases of the Kanovei-style strengthenings of Theorems 3 and 4 necessary to obtain:

Theorem 5 (AD). Suppose that $n \in \mathbb{N}$, X is an analytic Hausdorff space, and R is a Π^1_{2n+1} quasi-order on X. Then exactly one of the following holds:

- (1) The space X is a union of $(< \delta^1_{2n+1})$ -many Δ^1_{2n+1} R-chains.
- (2) There is a non-empty perfect R-antichain.

Theorem 6 (AD). Suppose that $n \in \mathbb{N}$, X is an analytic Hausdorff space, and R is a Δ^1_{2n+1} quasi-order on X. Then exactly one of the following holds:

- (1) There is a smooth Δ^1_{2n+1} superequivalence relation of \equiv_R whose equivalence classes are R-chains.
- (2) There is an \equiv_R -non-smooth perfect set whose quotient by \equiv_R is an (R/\equiv_R) -antichain.

In a future paper with Müller, we will establish the version of the \mathbb{G}_0 dichotomy necessary to obtain such results from PD.

We work in ZF + DC throughout.

1. Perfect antichains

For each discrete set D and sequence $s \in D^{<\mathbb{N}}$, we use \mathcal{N}_s to denote the basic open set consisting of all extensions of s in $D^{\mathbb{N}}$. We use the notation $\forall^*x \in X \ P(x)$ to indicate that $\{x \in X \ | \ \neg P(x)\}$ is meager, and $\exists^*x \in X \ P(x)$ to indicate that $\{x \in X \ | \ P(x)\}$ is non-meager. Fix sequences $\mathfrak{s}_n \in 2^n$ such that $\forall s \in 2^{<\mathbb{N}} \exists n \in \mathbb{N} \ s \sqsubseteq \mathfrak{s}_n$, and define $\mathbb{G}_0 = \bigcup_{n \in \mathbb{N}} \{(\mathfrak{s}_n \frown (i) \frown c, \mathfrak{s}_n \frown (1-i) \frown c) \ | \ c \in 2^{\mathbb{N}} \ \text{and} \ i < 2\}$. While our proof of the characterization of the existence of a non-empty perfect antichain requires infinitely-many applications of the \mathbb{G}_0 dichotomy, we need only one to establish the following:

Theorem 7. Suppose that X is a Hausdorff space, R is a co-analytic quasi-order on X, and X is not a union of countably-many Borel R-chains. Then there are compact sets $K_i \subseteq X$ that are not unions of countably-many Borel R-chains such that $\prod_{i < 2} K_i \subseteq \bot_R$.

Proof. As \perp_R is analytic and X is not a union of countably-many Borel R-chains, the \mathbb{G}_0 dichotomy yields a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from \mathbb{G}_0 to \perp_R . As the set $R_0 = (\phi \times \phi)^{-1}(R)$ is co-analytic, it has the Baire property (see, for example, [Kec95, 21.6]), thus so too does \perp_{R_0} .

Lemma 8. The relation \perp_{R_0} is non-meager.

Proof. Suppose, towards a contradiction, that \perp_{R_0} is meager, and fix non-empty open sets $U_i \subseteq 2^{\mathbb{N}}$ for which R_0 is comeager in $\prod_{i<2} U_i$ (see, for example, [Kec95, Proposition 8.26]). The Kuratowski-Ulam theorem (see, for example, [Kec95, Theorem 8.41]) ensures that the sets $C_0 = \{c \in 2^{\mathbb{N}} \mid \exists^* d \in U_0 \ c \ R_0 \ d\}$ and $C_1 = \{d \in 2^{\mathbb{N}} \mid \forall^* c \in U_0 \ c \ R_0 \ d\}$ have comeager union, and [Kec95, Theorem 16.1] and the Kuratowski-Ulam theorem imply that they have the Baire property. The Kuratowski-Ulam theorem also ensures that C_0 is non-meager, since otherwise $\forall^* c, d \in U_0 \ (\neg c \ R_0 \ d \ \text{and} \ d \ R_0 \ c)$, and C_1 is non-meager. As the \mathbb{E}_0 -saturation of every non-meager set with the Baire property is comeager (see, for example, [Kec95, Theorem 8.47]), there are comeagerly-many $c \in 2^{\mathbb{N}}$ for which the sets $C_i \cap [c]_{\mathbb{E}_0}$ non-trivially partition $[c]_{\mathbb{E}_0}$. As a straightforward induction reveals that \mathbb{E}_0 is the equivalence relation generated by \mathbb{G}_0 , it follows that $(\prod_{i<2} C_i) \cap \mathbb{G}_0 \neq \emptyset$, contradicting the fact that $\prod_{i<2} C_i \subseteq R_0$.

Lemma 9. There are continuous homomorphisms $\phi_i : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from \mathbb{G}_0 to itself for which $\prod_{i \leq 2} \phi_i(2^{\mathbb{N}}) \subseteq \perp_{R_0}$.

Proof. By Lemma 8, there are non-empty open sets $U_i \subseteq 2^{\mathbb{N}}$ and dense open sets $V_n \subseteq \prod_{i < 2} U_i$ such that $\bigcap_{n \in \mathbb{N}} V_n \subseteq \bot_{R_0}$. Recursively construct $u_{i,n} \in 2^{<\mathbb{N}}$ and $k_{i,n} \in \mathbb{N}$ such that $\prod_{i < 2} \mathcal{N}_{\phi_{i,n}(t_i)} \subseteq V_n$ for all $t_0, t_1 \in 2^n$ and $\phi_{i,n}(s_n) = s_{k_{i,n}}$ for all i < 2, where $\phi_{i,n} : 2^n \to 2^{<\mathbb{N}}$ is given by $\phi_{i,n}(t) = u_{i,0} \cap \bigoplus_{m < n} (t(m)) \cap u_{i,m+1}$. Then the functions $\phi_i : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ given by $\phi_i(c) = \bigcup_{n \in \mathbb{N}} \phi_{i,n}(c \upharpoonright n)$ are as desired.

It only remains to observe that if the functions ϕ_i are as in Lemma 9, then the sets $K_i = (\phi \circ \phi_i)(2^{\mathbb{N}})$ are as desired.

We now establish our characterization of the existence of a nonempty perfect antichain: Proof of Theorem 1. Conditions (1) and (2) are clearly mutually exclusive. To see $\neg(1) \Longrightarrow (2)$, note that if condition (1) fails, then X is the projection onto either coordinate of the complement of \equiv_R , thus analytic. Fix a continuous surjection $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$, and recursively appeal to Theorem 7 to obtain functions $\psi_n \colon 2^n \to \mathbb{N}^n$ and sequences $(F_s)_{s \in 2^n}$ of closed subsets of X with the following properties:

- (1) $\forall s \in 2^n F_s$ is not a union of countably-many Borel R-antichains.
- (2) $\forall s \in 2^n \ F_s \subseteq \phi(\mathcal{N}_{\psi_n(s)}).$
- (3) $\forall s \in 2^n \ F_{s \cap (0)} \cup F_{s \cap (1)} \subseteq F_s$.
- (4) $\forall s \in 2^n \ F_{s \cap (0)} \times F_{s \cap (1)} \subseteq \bot_R$.
- (5) $\forall i < 2 \forall s \in 2^n \ \psi_n(s) \sqsubseteq \psi_{n+1}(s \smallfrown (i)).$

Define $\psi \colon 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by $\psi(c) = \bigcup_{n \in \mathbb{N}} \psi_n(c \upharpoonright n)$, as well as $\pi = \phi \circ \psi$, noting that $\pi(c) \in \bigcap_{n \in \mathbb{N}} F_{c \upharpoonright n}$ for all $c \in 2^{\mathbb{N}}$. To see that $\pi(2^{\mathbb{N}})$ is a perfect R-antichain, observe that if $c, d \in 2^{\mathbb{N}}$ are distinct, then there is a maximal natural number $n \in \mathbb{N}$ for which $c \upharpoonright n = d \upharpoonright n$, so the fact that $\pi(c) \in F_{s \cap (c(n))}$ and $\pi(d) \in F_{s \cap (d(n))}$ ensures that $\pi(c) \perp_R \pi(d)$. \boxtimes

2. Non-smooth antichains

We now establish our characterization of the existence of a non-smooth perfect set whose quotient is an antichain:

Proof of Theorem 2. To see that conditions (1) and (2) are mutually exclusive, note that if E is a Borel superequivalence relation of \equiv_R whose classes are R-chains, and $A \subseteq X$ is a set whose quotient by \equiv_R is an (R/\equiv_R) -antichain, then $\equiv_R \upharpoonright A = E \upharpoonright A$. When A is analytic, it follows that if E is smooth, then so too is $\equiv_R \upharpoonright A$.

To see $\neg(1) \Longrightarrow (2)$, note that if (1) fails, then \equiv_R is non-smooth, so the \mathbb{E}_0 dichotomy yields a continuous embedding $\phi \colon 2^{\mathbb{N}} \to X$ of \mathbb{E}_0 into \equiv_R . As the set $R_0 = (\phi \times \phi)^{-1}(R)$ is Borel, it has the Baire property, thus so too does \perp_{R_0} .

Lemma 10. The relation \perp_{R_0} is comeager.

Proof. If there exist $n \in \mathbb{N}$ and $s, t \in 2^n$ for which R_0 is comeager in $\mathcal{N}_s \times \mathcal{N}_t$, then the fact that $\mathbb{E}_0 \subseteq R_0$ ensures that R_0 is comeager in $\mathcal{N}_{s'} \times \mathcal{N}_{t'}$ for all $s', t' \in 2^n$, and therefore comeager, thus so too is \equiv_{R_0} , contradicting the fact that the latter set is \mathbb{E}_0 .

Lemma 11. There is a continuous embedding $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of \mathbb{E}_0 into itself that is also a homomorphism from $\sim \mathbb{E}_0$ to \perp_{R_0} .

Proof. By Lemma 10, there are dense open sets $U_n \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that $\bigcap_{n \in \mathbb{N}} U_n \subseteq \bot_{R_0}$. We can clearly assume that these sets are decreasing and disjoint from the diagonal. Recursively construct $u_{i,n} \in 2^{<\mathbb{N}}$ such

that $|u_{0,n}| = |u_{1,n}|$ and $\prod_{i < 2} \mathcal{N}_{\psi_{n+1}(t_i \smallfrown (i))} \subseteq U_n$ for all $t_0, t_1 \in 2^n$, where $\psi_{n+1} \colon 2^{n+1} \to 2^{<\mathbb{N}}$ is given by $\psi_{n+1}(t) = \bigoplus_{m \le n} (t(m)) \smallfrown u_{t(m),m}$. Then the map $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ given by $\psi(c) = \bigcup_{n \in \mathbb{N}} \psi_n(c \upharpoonright n)$ is as desired. \boxtimes

It only remains to observe that if the function ψ is as in Lemma 11, then the set $(\phi \circ \psi)(2^{\mathbb{N}})$ is as desired.

3. Generalizations

Simplifications of the classical proofs of the \mathbb{G}_0 and \mathbb{E}_0 dichotomies can be used to obtain a continuous homomorphism from \mathbb{G}_0 to every κ -Souslin graph on a Hausdorff space with no κ -coloring, and a continuous embedding of \mathbb{E}_0 into every bi- κ -Souslin equivalence relation on a Hausdorff space which is not reducible to equality on 2^{κ} [Mil12]. By using these facts in lieu of the usual dichotomies in our proofs of Theorems 1 and 2, we obtain proofs of Theorems 3 and 4.

If AD holds and $n \in \mathbb{N}$, then a subset of an analytic Hausdorff space is Δ^1_{2n+1} if and only if it is δ^1_{2n+1} -Borel [Mar70, Mos71], and there is a cardinal κ^1_{2n+1} for which $\delta^1_{2n+1} = (\kappa^1_{2n+1})^+$ [Kec74] and a subset of an analytic Hausdorff space is Σ^1_{2n+1} if and only if it is κ^1_{2n+1} -Souslin (see, for example, [Jac08, Theorem 2.21]). It follows that continuous images of δ^1_{2n+1} -Borel sets are κ^1_{2n+1} -Souslin, a fact which alone ensures that the classical proofs of the \mathbb{G}_0 and \mathbb{E}_0 dichotomies yield the special cases of the Kanovei-style generalizations thereof at κ^1_{2n+1} . By using these in lieu of the usual dichotomies in our proofs of Theorems 1 and 2, we obtain proofs of the Kanovei-style strengthenings of Theorems 3 and 4 at κ^1_{2n+1} . As AD also ensures that every subset of a topological space is \aleph_0 -universally Baire (see, for example, [Kec95, Theorem 38.17]), Theorems 5 and 6 easily follow.

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BENJAMIN D. MILLER, KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC, UNIVERSITÄT WIEN, WÄHRINGER STRASSE 25, 1090 WIEN, AUSTRIA

E-mail address: benjamin.miller@univie.ac.at URL: http://www.logic.univie.ac.at/benjamin.miller

ZOLTÁN VIDNYÁNSZKY, KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC, UNIVERSITÄT WIEN, WÄHRINGER STRASSE 25, 1090 WIEN, AUSTRIA

E-mail address: zoltan.vidnyanszky@univie.ac.at URL: http://www.logic.univie.ac.at/~vidnyanszz77/