# FORCING AXIOMS, APPROACHABILITY, AND STATIONARY SET REFLECTION

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ABSTRACT. We prove a variety of theorems about stationary set reflection and concepts related to internal approachability. We prove that an implication of Fuchino-Usuba relating stationary reflection to a version of Strong Chang's Conjecture cannot be reversed; strengthen and simplify some results of Krueger about forcing axioms and approachability; and prove that some other related results of Krueger are sharp. We also adapt some ideas of Woodin to simplify and unify many arguments in the literature involving preservation of forcing axioms.

# 1. Introduction

Foreman-Todorcevic [13] introduced several natural variants of the class IA of internally approachable sets of size  $\omega_1$ . These are the classes of internally club sets (IC), the internally stationary sets (IS), and the internally unbounded sets (IU). The inclusions

$$IA \subseteq IC \subseteq IS \subseteq IU$$

follow from ZFC, and if the Continuum Hypothesis holds, then IA =\* IC =\* IS =\* IU.<sup>1</sup> The chain of inclusions in (1) is closely related to Shelah's *Approachability Ideal I*[ $\omega_2$ ], as follows:

**Lemma 1** (folklore; see remarks after Observation 22 for a proof). Assume  $2^{\omega_1} = \omega_2$ . The assertion that the approachability property fails at  $\omega_2$ —i.e. that  $\omega_2 \notin I[\omega_2]$ —is equivalent to the assertion that  $IU \setminus IA$  is stationary in  $\wp_{\omega_2}(H_{\omega_2})$ . In other words, failure of approachability property at  $\omega_2$  is equivalent to asserting that **at least one** of the three inclusions in (1) is strict in  $P(\wp_{\omega_2}(H_{\omega_2}))/NS$ .

In light of Lemma 1, a separation of adjacent classes in the chain of inclusions from (1) can be viewed as a very strong failure of approachability.

Foreman and Todorcevic independently proved that the Proper Forcing Axiom (PFA) implies failure of the approachability property at  $\omega_2$ ; in particular, PFA implies that at least one of the inclusions in (1) must be strict. Krueger [21] improved this, by showing that PFA in fact separates IA from IC in a global fashion. Given subclasses  $\Gamma$  and  $\Gamma'$  of  $\{W: |W| = \omega_1 \subset W\}$  such that  $\Gamma \subseteq \Gamma'$ , let us say that the inclusion  $\Gamma \subseteq \Gamma'$  is **globally strict** iff  $(\Gamma' \cap \wp_{\omega_2}(H_\theta)) \setminus (\Gamma \cap \wp_{\omega_2}(H_\theta))$  is stationary for every regular  $\theta \ge \omega_2$ . Answering a question of Foreman-Todorcevic, Krueger proved ([21], [20], and [22]) that *each* of the three inclusions in (1) can be globally strict, under various strong forcing axioms. As mentioned above, PFA globally separates IA from IC, but stronger forcing axioms were used for the following separations:

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<sup>&</sup>lt;sup>1</sup>Meaning that for any regular  $\theta \geq \omega_2$ , for all but nonstationarily many  $W \in [H_{\theta}]^{\omega_1}$ , W is in one of those four classes if and only if it is in all of them.

<sup>&</sup>lt;sup>2</sup>See the discussion preceding Theorem 5.3 in König-Yoshinobu [19].

**Theorem 2** (Krueger [22], Theorem 5.2).  $PFA^+$  implies that the inclusion  $IC \subseteq IU$ —i.e. between the second and fourth class in the chain (1)—is globally strict. In particular,  $PFA^+$  implies there is a disjoint stationary sequence on  $\omega_2$ .

**Theorem 3** (Krueger; corollary of Theorem 0.2 of [20] and Theorem 6.3 of [22]). Martin's Maximum (MM) implies that the inclusion  $IS \subseteq IU$  is globally strict. In particular, there is a disjoint club sequence on  $\omega_2$ .

**Theorem 4** (Krueger [20], Theorem 0.3).  $PFA^{+2}$  implies that the inclusion  $IC \subseteq IS$  is globally strict.

We prove that the assumptions of his Theorems 2 and 3 are sharp, but the assumption of Theorem 4 is not:

**Theorem 5.** The assumption of PFA<sup>+</sup> in Theorem 2 cannot be replaced by PFA.

**Theorem 6.** The assumption of MM in Theorem 3 cannot be replaced by  $PFA^{+\omega_1}$ .

**Theorem 7.** The conclusion of Theorem 4 also follows from PFA<sup>+</sup> and from Martin's Maximum, but not from PFA.

Note that the PFA<sup>+</sup> portion of Theorem 7—i.e. that PFA<sup>+</sup> implies that the inclusion IC  $\subseteq$  IS is globally strict—strengthens both Theorems 2 and 4. Figure 1 summarizes the theorems above.

FIGURE 1. When the inclusions from (1) are globally strict.

Foreman and Todorcevic also considered stationary reflection principles for the classes in (1). Given a (possibly finite) cardinal  $\mu \leq \omega_1$  and a subclass  $\Gamma$  of  $\{W : |W| = \omega_1 \subset W\}$ , let  $\mathrm{RP}^{\mu}_{\Gamma}$  assert that for all regular  $\theta \geq \omega_2$  and every  $\mu$ -sized collection  $\mathcal{S}$  of stationary subsets of  $[H_{\theta}]^{\omega}$ , there is a  $W \in \Gamma$  such that  $S \cap [W]^{\omega}$  is stationary in  $[W]^{\omega}$  for every  $S \in \mathcal{S}$ . We usually write  $\mathrm{RP}_{\Gamma}$  for  $\mathrm{RP}^1_{\Gamma}$ . Now (1) clearly implies

Krueger, answering another question of Foreman-Todorcevic [13], proved that the implication  $RP_{IC} \implies RP_{IS}$  cannot be reversed; in fact:

**Theorem 8** (Krueger [20]).  $RP_{IS}^{\omega_1}$  does not imply  $RP_{IC}$ .

Fuchino-Usuba [15] proved another result related to (2). They introduced a game-theoretic principle denoted  $G^{\downarrow\downarrow}$ , proved it is equivalent to a version of Strong Chang's Conjecture (see Definition 25 below), and also equivalent to a reflection principle that they did not name, but which we call RP<sub>internal</sub> (see Section 2). They proved:

 $<sup>^3</sup>$ Separation of IC from IS under Martin's Maximum was claimed in Theorem 4.4 part (3) of Viale [32], but the argument given there implicitly used the stronger assumption  $MM^{+2}$ . See Section 5.3 for a discussion.

<sup>&</sup>lt;sup>4</sup>The theorem stated in Theorem 5.1 of [20] just says that  $RP_{IS}^{\omega_1}$  does not imply  $RP_{IA}$ , but the proof there clearly shows that even  $RP_{IC}$  fails in his model.

Theorem 9 (Fuchino-Usuba [15]).

$$RP_{IC} \implies RP_{internal} \iff G^{\downarrow\downarrow} \implies RP_{IS}$$

We show below that the left implication of Theorem 9 cannot be reversed. This was already implicit in Krueger's model from Theorem 8, but our Theorem 11 below strengthens and simplifies his result in several ways. In particular, the model of Theorem 11: (1) satisfies PFA (and a "plus" version of a fragment of PFA); (2) can be forced over an arbitrary model of PFA<sup> $\omega_1$ </sup> in a single step; and (3) satisfies "diagonal internal reflection to guessing, internally stationary sets" (DRP<sub>internal, GIS</sub>), which is a highly simultaneous form of internal stationary reflection to the so-called guessing sets used by Viale-Weiss [33] to characterize the principle ISP (see Section 2).

In what follows, GIC refers to the class of guessing, internally club sets, and GIS refers to the class of guessing, internally stationary sets. We also introduce some fragments of standard forcing axioms.  $\mathbf{PFA}_{H_{\omega_2}^V \notin \mathbf{IC}}^{+\omega_1}$  denotes the forcing axiom  $\mathbf{FA}^{+\omega_1}(\Gamma)$ , where  $\Gamma$  is the class of proper posets that force  $H_{\omega_2}^V \notin \mathbf{IC}$ ;  $\mathbf{PFA}_{H_{\omega_2}^V \notin \mathbf{IA}}^{+\omega_1}$  is defined similarly (see Section 2.3 for the meaning of  $\mathbf{FA}^{+\omega_1}(\Gamma)$ ).  $\mathbf{MM}_{H_{\omega_2}^V \notin \mathbf{IS}}^{+\omega_1}$  denotes the forcing axiom  $\mathbf{FA}^{+\omega_1}(\Gamma)$ , where  $\Gamma$  is the class of posets that preserve stationary subsets of  $\omega_1$  and force  $H_{\omega_2}^V \notin \mathbf{IS}$ .

**Theorem 10.**  $PFA_{H_{\omega_2}^V \notin IC}^{+\omega_1}$  implies  $DRP_{internal,GIS}$  (see Section 2), a highly simultaneous version of internal stationary reflection to GIS sets (in particular, this implies  $RP_{IS}^{\omega_1}$ ).

**Theorem 11.** There is a  $<\omega_2$  strategically closed poset that preserves  $PFA_{H_{\omega_2}^{\vee} \notin IC}^{+\omega_1}$  and forces  $\neg RP_{IC}$ . Moreover, if PFA held in the ground model, then PFA is also preserved.

**Corollary 12.**  $DRP_{internal, GIS}$  does not imply  $RP_{IC}$ . In particular, the implication  $RP_{IC} \implies RP_{internal}$  from Theorem 9 cannot be reversed.

We also prove the following similar theorems:

**Theorem 13.**  $PFA_{H_{\omega_2}^V \notin IA}^{+\omega_1}$  implies  $DRP_{GIC}$  (which is equivalent to  $DRP_{internal, GIC}$ ; see Observation 23).

**Theorem 14.** There is a  $<\omega_2$  strategically closed poset that preserves  $PFA^{+\omega_1}_{H^V_{\omega_2} \notin IA}$  and forces  $\neg RP_{IA}$ . Moreover, if PFA held in the ground model, then PFA is also preserved.

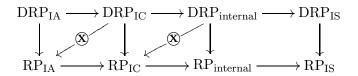
Corollary 15.  $DRP_{GIC}$  does not imply  $RP_{IA}$ . In particular, the implication  $RP_{IA} \implies RP_{IC}$  from (2) cannot be reversed.

Figure 2 summarizes the implications and non-implications. It shows that for the classes IA, IC, and IS, the maximal form of simultaneous reflection (i.e. DRP) for one class does not even imply RP¹ for the next "nicer" class. This contrasts greatly with the Foreman-Todorcevic result (Corollary 20 of [13]), that  $\text{RP}^3_{\wp_{\omega_2}^*(V)}$  implies  $\text{RP}^1_{\text{Unif}_{\omega_1}}$ , where  $\wp_{\omega_2}^*(V) := \{W : |W| = \omega_1 \subset W\}$  and

$$\mathrm{Unif}_{\omega_1} := \{W \in \wp^*(V) \ : \ \mathrm{cf}(\sup(W \cap \kappa)) = \omega_1 \ \text{for every regular uncountable} \ \kappa \in W\}.$$

We now address the implication  $RP_{IS} \implies RP_{IU}$  from (2). Whether this is reversible is closely related to Question 5.12 of Krueger [20], and to the question of whether the implication  $RP_{internal} \implies RP_{IS}$  from Theorem 9 can be reversed. We do not know the answer to any of those questions, but the following are some partial results that may shed light on this surprisingly difficult problem.

FIGURE 2. An arrow indicates an implication, and an arrow with an X indicates a non-implication. In order to simplify the figure, the non-implications shown do not incorporate the full strength of the theorems above.



In light of the way that we separated the various reflection principles above (Theorem 10 through Corollary 15), it is natural to conjecture that  $\mathrm{MM}^{+\omega_1}_{H^V_{\omega_2}\notin\mathrm{IS}}$  should imply  $\mathrm{DRP_{IU}}$ , and that  $\mathrm{MM}^{+\omega_1}_{H^V_{\omega_2}\notin\mathrm{IS}}$  should be preserved by a forcing that kills  $\mathrm{RP_{IS}}$ . This would have separated  $\mathrm{RP_{IS}}$  from  $\mathrm{RP_{IU}}$  in a manner similar to the earlier separation results. However,  $\mathrm{MM}^{+\omega_1}_{H^V_{\omega_2}\notin\mathrm{IS}}$  does not imply  $\mathrm{DRP_{IU}}$ , and in fact does not even imply the weakest generalized reflection principle of all, as the next theorem shows. The notation WRP is conventionally used to denote  $\mathrm{RP}_{\Gamma}$ , where  $\Gamma$  is the entire class  $\{W: |W| = \omega_1 \subset W\}$ .

**Theorem 16.**  $MM_{H_{\omega_{\gamma}}^{\downarrow} \notin IS}^{+\omega_{1}}$  is consistent with failure of  $WRP(\omega_{2})$ .

Recall that  $\mathrm{MM}_{H_{\omega_2}^V \notin \mathrm{IS}}^{+\omega_1}$  is (the  $+\omega_1$  version of) the forcing axiom for the class of posets that preserve stationary subsets of  $\omega_1$  and force  $H_{\omega_2}^V \notin \mathrm{IS}$ . A familiar poset in this class is Namba forcing. Martin's Maximum implies that Namba forcing is semiproper, which in turn implies  $\mathrm{WRP}(\omega_2)$ . Also, by Sakai [26],  $\mathrm{WRP}(\omega_2)$  is equivalent to the *semi*-stationary reflection principle for  $\omega_2$ , denoted  $\mathrm{SSR}(\omega_2)$ . The following corollary of Theorem 16 may be of independent interest:

Corollary 17. The axiom  $FA^{+\omega_1}(Namba\ forcing)$  does not imply  $SSR(\omega_2)$ .

Although  $\mathrm{MM}^{+\omega_1}_{H^V_{\omega_2}\notin\mathrm{IS}}$  does not imply even  $\mathrm{WRP}(\omega_2)$ , it does imply a diagonal kind of ordinal reflection. The principle  $\mathrm{DRP}^{\mathrm{cof}(\omega)}_{\mathrm{GIU}}$  is a weakening of  $\mathrm{DRP}_{\mathrm{GIU}}$  where one only asks for diagonal reflection of stationary subsets of  $\theta\cap\mathrm{cof}(\omega)$ , rather than stationary subsets of  $[\theta]^\omega$ . We prove:

Theorem 18.  $MM_{H_{\omega_0}^{\dagger} \notin IS}^{+\omega_1}$  implies  $DRP_{GIU}^{cof(\omega)}$ .

Corollary 19.  $DRP_{GIU}^{cof(\omega)}$  does not imply  $WRP(\omega_2)$ .

To aid in the proofs of most of the theorems above, we adapt some ideas of Woodin to prove a very general theorem (Theorem 20) about preservation of forcing axioms. Theorem 20 makes such preservation arguments more closely resemble arguments about lifting large cardinal embeddings. Theorem 20 consolidates a variety of results in the literature about forcing axiom preservation into a single framework (see examples in Section 4) and may be of use in other applications.

**Theorem 20** (Forcing Axiom Preservation Theorem). Assume  $\Gamma$  is a class of posets that is closed under restrictions (see Definition 39),  $\mathbb{P}$  is a partial order, and  $\dot{\Delta}$  is a  $\mathbb{P}$ -name for a (definable) class of posets that is also closed under restrictions. Assume  $\mu \leq \omega_1$  is a cardinal (possibly  $\mu = 0$ ), and that  $FA^{+\mu}(\Gamma)$  holds. Assume also that for every  $\mathbb{P}$ -name for a poset  $\dot{\mathbb{Q}} \in \dot{\Delta}$  and every  $\mathbb{P} * \dot{\mathbb{Q}}$ -name  $\langle \dot{S}_i : i < \mu \rangle$  for a sequence of  $\mu$  many stationary subsets of  $\omega_1$ , there exists a  $\mathbb{P} * \dot{\mathbb{Q}}$ -name  $\dot{\mathbb{R}}$  (possibly depending on  $\dot{\mathbb{Q}}$  and  $\dot{\vec{S}}$ ) such that:

<sup>&</sup>lt;sup>5</sup>By Shelah [27] and Todorcevic [29].

- (1)  $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}} \in \Gamma$
- (2)  $\mathbb{P} * \dot{\mathbb{Q}}$  forces that  $\dot{\mathbb{R}}$  preserves the stationarity of  $\dot{S}_i$  for every  $i < \mu$ ;
- (3) If  $j: V \to N$  is a generic elementary embedding,  $\theta \ge |\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}|^+$  is regular in V, and
  - (a)  $H_{\theta}^{V}$  is in the wellfounded part of N (which we assume has been transitivized);
  - (b)  $j[H_{\theta}^{V}] \in N$  and has size  $\omega_1$  in N;
  - (c)  $crit(j) = \omega_2^V$ ; and
  - (d) There exists a G \* H \* K in N such that:

(i) 
$$G * H * K$$
 is  $(H_{\theta}^{V}, \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}})$ -generic;

(ii) 
$$N \models \text{``}(\dot{S}_i)_{G*H}$$
 is stationary for all  $i < \mu$ ";

**then** N believes that j[G] has a lower bound in  $j(\mathbb{P})$ .

Then 
$$V^{\mathbb{P}} \models FA^{+\mu}(\dot{\Delta})$$
.

The theorem above is stated in a general form to accommodate both

- the "plus" versions of forcing axioms; and
- situations where one only wants to preserve a fragment of a forcing axiom—e.g. if  $\Gamma$  is the class of proper forcings, but  $\dot{\Delta}$  names the class of totally proper posets in  $V^{\mathbb{P}}$ .

In many situations, however, one wants for  $\mathbb{P}$  to just preserve, say, PFA. In this situation, the  $\mu$  from Theorem 20 is zero, and Theorem 20 tells us that it suffices to show that for every  $\mathbb{P}$ -name  $\dot{\mathbb{Q}}$  for a proper poset, there exists a  $\mathbb{P} * \dot{\mathbb{Q}}$ -name  $\dot{\mathbb{R}}$  (possibly depending on  $\dot{\mathbb{Q}}$ ) such that:

- (1)  $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$  is proper;
- (2) If  $j: V \to N$  is a generic elementary embedding,  $\theta \ge |\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}|^+$  is regular in V, and
  - (a)  $H_{\theta}^{V}$  is in the well founded part of N;
  - (b)  $j[H_{\theta}^{V}] \in N$  and has size  $\omega_1$  in N;
  - (c)  $\operatorname{crit}(j) = \omega_2^V$ ; and
  - (d) There exists a G\*H\*K in N that is  $(H^V_\theta, \mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{R}})$ -generic;

**then** N believes that j[G] has a lower bound in  $j(\mathbb{P})$ .

In addition to using Theorem 20 to prove our own results, Section 4.1 provides several examples from the literature that can be viewed as instances of Theorem 20.

Section 2 provides the relevant background. Section 3 provides some key theorems for precisely controlling the "internal part" of an elementary submodel of size  $\omega_1$ , while also ensuring that the elementary submodel will be a guessing set. These arguments frequently make use of Todorcevic's " $\in$ -collapse" poset of finite conditions (e.g. as in [28]), modified by a device introduced by Neeman [25] to ensure continuity of the generic chain. Section 4 proves the forcing axiom preservation theorem mentioned above. Section 5 proves Theorems 5, 6, and 7. Section 6 proves the remaining theorems from the introduction (about RP and the forcing axioms PFA $_{H_{\omega_2}^{V} \notin IC}^{+\omega_1}$  etc.). Section 7 includes some questions and closing remarks.

# 2. Preliminaries

Unless otherwise noted, all notation and terminology comes from Jech [18]. For  $m < n < \omega$ ,  $S_m^n$  denotes the set of  $\alpha < \omega_n$  such that  $\mathrm{cf}(\alpha) = \omega_m$ .

2.1. Classes of  $\omega_1$  sized sets. We will use  $\wp_{\omega_2}^*(V)$  to denote the class  $\{W: |W| = \omega_1 \subset W\}$  (the star superscript indicates that we are *not* including Chang-like structures, i.e.  $\wp_{\omega_2}^*(V)$  does not include W of size  $\omega_1$  such that  $|W \cap \omega_1| = \omega$ ). Foreman-Todorcevic [13] defined several weakenings of internal approachability; e.g. they called a  $W \in \wp_{\omega_2}^*(V)$  internally stationary iff  $W \cap [W]^\omega$  is stationary in  $[W]^\omega$ . For the proofs in this article it will be convenient to use the following equivalent definitions. If  $W \in \wp_{\omega_2}^*(V)$ , a filtration of W is any  $\subseteq$ -increasing and continuous sequence  $\vec{N} = \langle N_i : i < \omega_1 \rangle$  of countable sets such that  $W = \bigcup_{i < \omega_1} N_i$ . It is easy to see that if  $\vec{N}$  and  $\vec{M}$  are both filtrations of W, then  $N_i = M_i$  for all but nonstationarily many  $i < \omega_1$ . The internal part of W, denoted int(W), is the equivalence class

$$[\{i < \omega_1 : N_i \in W\}]$$

in the boolean algebra  $\wp(\omega_1)/\operatorname{NS}_{\omega_1}$ , where  $\vec{N}$  is any filtration of W. Since any two filtrations of W agree on a club subset of  $\omega_1$ , the choice of the filtration does not matter. We will often abuse terminology and refer to a subset  $T \subseteq \omega_1$  as the internal part of W, when really we mean that the equivalence class of T modulo  $\operatorname{NS}_{\omega_1}$  is the internal part of W. Similarly, when we say "the internal part of W contains T" we mean  $[T] \leq \operatorname{int}(W)$  in the boolean algebra  $\wp(\omega_1)/\operatorname{NS}_{\omega_1}$ . The **external part of** W is defined to be  $\omega_1 \setminus \operatorname{int}(W)$ . If  $W \in \wp_{\omega_2}^*(V)$ , W is called **internally approachable** if there exists a filtration  $\vec{N} = \langle N_i : i < \omega_1 \rangle$  of W such that  $\vec{N} \upharpoonright i \in W$  for every  $i < \omega_1$ ; **internally club** if  $\operatorname{int}(W)$  contains a club subset of  $\omega_1$ ; and **internally stationary** if  $\operatorname{int}(W)$  is a stationary subset of  $\omega_1$ . We use IA, IC, and IS, to denote the class of internally approachable, internally club, and internally stationary sets, respectively. We will also sometimes refer to the class IU of **internally unbounded** sets, which is the class of  $W \in \wp_{\omega_2}^*(V)$  such that  $W \cap [W]^\omega$  is  $\subseteq$ -cofinal in  $[W]^\omega$ .

**Lemma 21.** Suppose  $W \in \wp_{\omega_2}^*(V)$  has internal part T and external part  $T^c$  (we make no assumptions about stationarity or costationarity of T here). Then this remains true in any outer model where  $\omega_1$  is not collapsed.

*Proof.* Fix any filtration  $\vec{N} = \langle N_i : i < \omega_1 \rangle$  of W. Then by definition of internal and external parts, there is a club  $C \subseteq \omega_1$  such that

$$T \cap C \subseteq \{i < \omega_1 : N_i \in W\}$$

and

$$T^c \cap C \subseteq \{i < \omega_1 : N_i \notin W\}.$$

If V' is an outer model of V with the same  $\omega_1$ , then C is of course still club in V', so  $[T] = [T \cap C]$  and  $[T^c] = [T^c \cap C]$  in the  $\wp(\omega_1)/\mathrm{NS}_{\omega_1}$  of V'. So the above two containments witness that V' believes T is the internal part, and  $T^c$  is the external part, of W.

We also observe:

 $<sup>{}^6</sup>W \in \text{IU}$  can also be characterized by saying that there exists a filtration  $\vec{N}$  of W such that  $N_i \in W$  for unboundedly many  $i < \omega_1$ . If  $W \in \text{IU} \setminus \text{IS}$ , the set of such i is nonstationary in  $\omega_1$  for any filtration (and even empty for some filtrations of W), and hence  $\text{int}(W) = [\emptyset]$  in the boolean algebra  $\wp(\omega_1)/\text{NS}_{\omega_1}$ . However, the assertion  $\text{int}(W) = [\emptyset]$  does not characterize " $W \in \text{IU} \setminus \text{IS}$ " even among those W of uniform cofinality  $\omega_1$ , because there are always stationarily many  $W \in \wp_{\omega_2}^*(H_{\aleph_{\omega+1}})$  that have uniform cofinality  $\omega_1$ , yet are not internally unbounded (this is due to Zapletal; see Foreman-Magidor [11]).

**Observation 22.** If M is a transitive ZF<sup>-</sup> model of size  $\omega_1$  with internal part T, and  $\mu \in [\omega_2^M, M \cap ORD]$  is regular in M, then the internal part of  $(H_{\mu})^M$  contains T.

Proof. Let  $\vec{Q} = \langle Q_i : i < \omega_1 \rangle$  be a filtration of M, and let C be club in  $\omega_1$  such that  $Q_i \in M$  for every  $i \in T \cap C$ . Then  $\langle Q_i \cap (H_\mu)^M : i < \omega_1 \rangle$  is a filtration of  $(H_\mu)^M$ , and if  $i \in T \cap C$  then  $Q_i \in M$  and hence  $Q_i \cap (H_\mu)^M \in M$ .

Although we will not use it in this paper, we provide a brief sketch of the proof of the folklore Lemma 1 from the introduction. Similar arguments are implicit in work of Shelah, and somewhat more explicit in Foreman-Magidor [10]. By the assumption  $2^{\omega_1} = \omega_2$ , we can fix a bijection  $\Phi: \omega_2 \to H_{\omega_2}$ . It is routine to check that

(3) 
$$\operatorname{IU} \cap \wp_{\omega_2}(H_{\omega_2}) =^* \{ \Phi[\gamma] : \gamma \in S_1^2 \}$$

where the  $=^*$  means "equal mod NS in  $\wp_{\omega_2}(H_{\omega_2})$ ". Assume first that  $\omega_2 \in I[\omega_2]$ ; this implies (see Foreman [9]) that for a sufficiently large regular  $\theta$ , there is a first order structure  $\mathfrak{A} = (H_{\theta}, \in, ...)$  in a countable language and an  $\omega_1$ -club  $D \subseteq S_1^2$  such that for every  $\gamma \in D$ ,  $\operatorname{Sk}^{\mathfrak{A}}(\gamma) \cap \omega_2 = \gamma$  and there exists a strictly increasing sequence  $\vec{\beta}^{\gamma} = \langle \beta_i^{\gamma} : i < \omega_1 \rangle$  that is cofinal in  $\gamma$ , and every proper initial segment of  $\vec{\beta}^{\gamma}$  is an element of  $W_{\gamma} := \operatorname{Sk}^{\mathfrak{A}}(\gamma)$ . We can without loss of generality assume that  $\mathfrak{A}$  includes a predicate for  $\Phi$ , and hence for  $W_{\gamma} \cap H_{\omega_2} = \Phi[\gamma]$  for every  $\gamma \in D$ . By (3), together with the assumption that D is almost all of  $S_1^2$ , to show that IA = IU it will suffice to show that  $W_{\gamma} \cap H_{\omega_2} \in IA$  for every  $\gamma \in D$ . And for any  $\gamma \in D$ ,  $\langle i \cup \Phi[\{\beta_j^{\gamma} : j < i\}] : i < \omega_1 \rangle$  can easily be shown to be a filtration of  $\Phi[\gamma] = W_{\gamma} \cap H_{\omega_2}$  witnessing its internal approachability. The other direction of Lemma 1 is easier; we leave this to the reader.

2.2. Stationary reflection principles. Given a regular cardinal  $\theta \geq \omega_2$ , a subclass  $\Gamma$  of  $\wp_{\omega_2}^*(V)$ , and a (possibly finite) cardinal  $\mu \leq \omega_1$ ,  $\mathbf{RP}^{\mu}_{\Gamma}(\theta)$  is the assertion that for every  $\mu$ -sized collection  $\mathcal{S}$  of stationary subsets of  $[H_{\theta}]^{\omega}$ , there is a  $W \in \Gamma$  such that  $S \cap [W]^{\omega}$  is stationary for every  $S \in \mathcal{S}$ . The principle  $\mathbf{RP}^{\mu}_{\mathbf{internal}, \Gamma}(\theta)$ —which was isolated, but not named, in Fuchino-Usuba [15]—asserts that for every  $\mu$ -sized collection  $\mathcal{S}$  of stationary subsets of  $[H_{\theta}]^{\omega}$ , there is a  $W \in \Gamma$  such that  $S \cap W \cap [W]^{\omega}$  (not just  $S \cap [W]^{\omega}$ ) is stationary for every  $S \in \mathcal{S}$ .

If  $\Gamma$  is not specified in the subscript of RP, it is understood to be  $\wp_{\omega_2}^*(V)$ . We also typically write RP instead of RP<sup>1</sup>. RP<sup> $\mu$ </sup> means that RP $^{\mu}_{\Gamma}(\theta)$  holds for every regular  $\theta \geq \omega_2$ . Similar notational shortcuts apply to all the reflection principles we mention. Notice that:

**Observation 23.**  $RP_{IC}$  is equivalent to  $RP_{internal, IC}$ , since if  $W \in IC$  and  $S \cap [W]^{\omega}$  is stationary, then  $S \cap W \cap [W]^{\omega}$  is stationary as well.

We now recall the diagonal versions of stationary reflection, as introduced in [5].  $\mathbf{DRP}_{\Gamma}(\theta)$  asserts that there are stationarily many  $W \in \wp_{\omega_2}(H_{(2^{\theta})^+})$  such that  $W \cap H_{\theta} \in \Gamma$  and  $S \cap [W \cap H_{\theta}]^{\omega}$  is stationary in  $[W \cap H_{\theta}]^{\omega}$  whenever  $S \in W$  and S is stationary in  $[H_{\theta}]^{\omega}$ . We also define a diagonal version of Fuchino-Usuba's internal reflection:  $\mathbf{DRP_{internal}}, \Gamma(\theta)$  is defined the same way as  $\mathbf{DRP}_{\Gamma}(\theta)$ , except we require that  $S \cap W \cap [W \cap H_{\theta}]^{\omega}$ , not just  $S \cap [W \cap H_{\theta}]^{\omega}$ , is stationary (for every  $S \in W$  that is stationary in  $[H_{\theta}]^{\omega}$ ).

**Remark 24.** The principle DRP easily implies  $RP^{\omega_1}$ , and in Cox [5] it was asked if they are equivalent. The author subsequently noticed that Larson [23] gives a model where  $RP^{\omega_1}$  holds, but DRP fails. So DRP is strictly stronger than  $RP^{\omega_1}$ .

We also define a weaker kind of diagonal reflection. If  $\Gamma$  is a subclass of  $\wp_{\omega_2}^*(V)$ ,  $\mathrm{DRP}_{\Gamma}^{\mathrm{cof}(\omega)}$  asserts that for every regular  $\theta \geq \omega_2$ , there are stationarily many  $W \in \wp_{\omega_2}^*(H_{(2^{\theta})^+})$  such that for every  $R \in W$  that is a stationary subset of  $\theta \cap \mathrm{cof}(\omega)$ ,  $R \cap \mathrm{sup}(W \cap \theta)$  is stationary in  $\mathrm{sup}(W \cap \theta)$ . This is a consequence of "weak  $\mathrm{DRP}$ " (wDRP $_{\Gamma}$ ) introduced in [5]; whether these two principles are equivalent is not known.

We now mention a few facts that, although we will not use them in this paper, illustrate that the notion of internal stationary reflection (and its diagonal version) are quite natural and related to several other well-studied topics. The following version of Strong Chang's Conjecture was isolated (but not named) in Fuchino-Usuba [15]. It is a stronger version of the principle  $SCC_{gap}^{cof}$  considered in Cox [3].

**Definition 25.** Global  $SCC_{gap}^{cof}$  is the following assertion: for all sufficiently large regular  $\theta$  and all wellorders  $\Delta$  on  $H_{\theta}$  and every countable  $M \prec (H_{\theta}, \in, \Delta)$ , there are  $\subseteq$ -cofinally many  $W \in [H_{\theta}]^{\omega_1}$  such that:

- (1)  $\omega_1 \subset W$ ;
- (2) Letting M(W) denote  $Sk^{(H_{\theta}, \in, \Delta)}(M \cup \{W\})$ , we have

$$M(W) \cap W = M$$

We note that the version of Strong Chang's Conjecture isolated by Doebler-Schindler [7] has a similar characterization, where the  $M(W) \cap W = M$  requirement from Definition 25 is weakened to the requirement that  $M(W) \cap W \supseteq M$  (equivalently, that  $M(W) \cap \omega_1 = M \cap \omega_1$ ). Doebler-Schindler's version is equivalent to the † principle, which in turn is equivalent to Semistationary set reflection (see [7]).

The relevance of Global  $SCC_{gap}^{cof}$  to the current paper is the following theorem:

**Theorem 26** (Fuchino-Usuba [7]).  $RP_{internal}$  is equivalent to Global  $SCC_{gap}^{cof}$ .

The following lemma characterizes internal, diagonal reflection. It is a situation one often encounters when generically lifting a large cardinal embedding  $j:V\to M$  to domain  $V^{\mathbb{P}}$ , when  $j(\mathbb{P})$  is proper in M (but not necessarily in V).

**Lemma 27.** Let  $\Gamma$  be a subclass of  $\wp_{\omega_2}^*(V)$ . The following are equivalent:

- (1)  $DRP_{internal, \Gamma}$
- (2) For every regular  $\theta \geq \omega_2$  there are stationarily many  $W \in \wp_{\omega_2}^*(H_{(2^{\theta})^+})$  such that:
  - (a)  $W \cap H_{\theta} \in \Gamma$ ; and
  - (b)  $H_W$  is correct about stationary subsets of  $[\sigma_W^{-1}(\theta)]^{\omega}$ , where  $\sigma_W: H_W \to W \prec H_{(2^{\theta})^+}$  is the inverse of the Mostowski collapse of W.
- (3) For every regular  $\theta \geq \omega_2$  there is a generic elementary embedding  $j: V \to N$  such that:
  - (a)  $H_{(2^{\theta})^{+}}^{V}$  is in the wellfounded part of N;
  - (b)  $j \upharpoonright H_{(2^{\theta})^+}^V \in N;$
  - (c)  $N \models j[H_{\theta}^{V}]$  is a member of  $j(\Gamma)$ ;
  - (d)  $crit(j) = \omega_2^V$

(e)  $N \models \text{``}H^V_{(2^{\theta})^+}$  is correct about stationary subsets of  $[\theta]^{\omega}$ ''.

The equivalence of 1 with 2 is almost identical to the proof of Theorem 3.6 of Cox [5], and the equivalence of 2 with 3 is a generic ultrapower argument, closely resembling the proof of Theorem 8 of Cox [4]. We refer the reader to those sources.<sup>7</sup>

2.3. Forcing Axioms and partially generic filters. Given a class  $\Gamma$  of partial orders, and a (possibly finite) cardinal  $\mu \leq \omega_1$ ,  $\mathbf{FA}^{+\mu}(\Gamma)$  is the assertion: whenever  $\mathbb{P} \in \Gamma$ ,  $\mathcal{D}$  is an  $\omega_1$ -sized collection of dense subsets of  $\mathbb{P}$ , and  $\langle \dot{S}_i : i < \mu \rangle$  is a  $\mu$ -length list of  $\mathbb{P}$ -names for stationary subsets of  $\omega_1$ , then there is a filter  $g \subset \mathbb{P}$  such that  $g \cap D \neq \emptyset$  for every  $D \in \mathcal{D}$ , and for every  $i < \mu$  the following set is stationary in  $\omega_1$ :

$$(\dot{S}_i)_g := \{ \xi < \omega_1 : \exists p \in g \ p \Vdash \check{\xi} \in \dot{S}_i \}.$$

In the literature  $FA^{+\omega_1}(\Gamma)$  is often denoted  $FA^{++}(\Gamma)$ , but we do not use that convention (since we will need to deal both with the case  $\mu = 2$  and the case  $\mu = \omega_1$ ).  $FA(\Gamma)$  is as defined above, but without mentioning the names for stationary sets (so  $FA(\Gamma)$  is the same as  $FA^{+0}(\Gamma)$ ).

If  $\mathbb{Q}$  is a poset and  $\mathbb{Q} \in W \prec H_{\theta}$ , we say that g is a  $(W, \mathbb{Q})$ -generic filter iff g is a filter on  $W \cap \mathbb{Q}$  and  $g \cap D \cap W \neq \emptyset$  for every  $D \in W$  that is dense in  $\mathbb{Q}$ . If  $\mu \leq \omega_1$  and  $\dot{\vec{S}} = \langle \dot{S}_i : i < \mu \rangle$  is a sequence of  $\mathbb{Q}$ -names for stationary subsets of  $\omega_1$  such that  $\dot{\vec{S}} \in W$ , we say that g is an  $\dot{\vec{S}}$ -correct,  $(W, \mathbb{Q})$ -generic filter if g is a  $(W, \mathbb{Q})$ -generic filter as defined above, and for every  $i < \mu$ ,

$$(\dot{S}_i)_q := \{ \alpha < \omega_1 : \exists q \in g \ q \Vdash \check{\alpha} \in \dot{S}_i \}$$

is (in V) a stationary subset of  $\omega_1$ .

Let  $\sigma_W: H_W \to W \prec H_\theta$  be the inverse of the Mostowski collapse of W, and suppose  $g \subset W \cap \wp(\mathbb{Q})$ . It is easy to see that g is  $(W, \mathbb{Q})$ -generic if and only if  $\sigma_W^{-1}[g]$  is an  $(H_W, \sigma^{-1}(\mathbb{Q}))$ -generic filter in the usual sense. Furthermore, if  $\mu \subset W$ ,  $\dot{\vec{S}} \in W$ , and g is a  $(W, \mathbb{Q})$ -generic filter, then  $\dot{S}_i \in W$  for every  $i < \mu$ , and g is  $\dot{\vec{S}}$ -correct iff for every  $i < \mu$ , the evaluation of  $\sigma_W^{-1}(\dot{S}_i)$  by  $\sigma_W^{-1}[g]$  is (in V) a stationary subset of  $\omega_1$ .

2.4. Guessing models and strongly proper club shooting. A pair of transitive ZF<sup>-</sup> models (M, N) has the  $\omega_1$ -approximation property iff for every  $X \in M$  and every  $A \in \wp(X) \cap N$ , if  $A \cap z \in M$  for every  $z \in M$  such that  $M \models$  "z is countable", then  $A \in M$ . We say that (M, N) has the  $\omega_1$ -covering property iff for every  $z \in N$  that is countable in N, there is a  $z' \in M$  that is countable in M such that  $z \subseteq z'$ . A poset  $\mathbb{P}$  has the  $\omega_1$  approximation property iff it forces that  $(V, V^{\mathbb{P}})$  has the  $\omega_1$ -covering property. The following fact is often used:

**Fact 28** (Viale-Weiss [33]). If  $\mathbb{P}$  has the  $\omega_1$  approximation and covering properties, and  $\Vdash_{\mathbb{P}}$  " $\dot{\mathbb{Q}}$  has the  $\omega_1$  approximation and covering properties", then  $\mathbb{P} * \dot{\mathbb{Q}}$  has the  $\omega_1$  approximation and covering properties.

A set W is called an  $\omega_1$ -guessing model if  $W \in \wp_{\omega_2}^*(V)$ ,  $(W, \in \cap(W \times W)) \models \mathrm{ZF}^-$ , and  $(H_W, V)$  has the  $\omega_1$ -approximation property, where  $H_W$  is the transitive collapse of W. We will often just say "guessing model" instead of " $\omega_1$ -guessing model", and use either G or (when there is risk of confusion)  $G_{\omega_1}$  to denote the class of guessing models. Viale-Weiss [33] proved that Weiss'

<sup>&</sup>lt;sup>7</sup> Those two cited proofs deal with the particular class  $\Gamma = IC$ , but go through for internal reflection. The key use of the class IC in those proofs was the above Observation 23.

<sup>&</sup>lt;sup>8</sup>This is equivalent to the original definition of guessing model from Viale-Weiss [33].

generalized tree property principle ISP( $\omega_2$ ) is equivalent to the assertion that for every regular  $\theta \geq \omega_2$ , the set of guessing models is stationary in  $\wp_{\omega_2}^*(H_{\theta})$ . We say that W is an **indestructible** guessing model (in V) if, letting  $\theta_W$  be the least regular cardinal such that  $W \in H_{\theta_W}^V$ , then W remains a guessing model in any ZF<sup>-</sup> model  $(N, \in_N)$  such that  $H_{\theta_W}^V$  is an element of the wellfounded part of N and  $\omega_1^V = \omega_1^N$  (here N may be external to V, and we even allow that V can be a set from N's point of view.). This of course isn't a first order statement over V, but in typical contexts there are first-order substitutes for this notion, e.g. the presence of specializing functions in  $H_{\theta}^V$  that guarantee such indestructibility (see Proposition 4.4 of [6]).

We let GIU denote the (possibly empty) class of guessing models that are also internally unbounded.

**Theorem 29** (Viale-Weiss [33]; see also Proposition 4.4 of [6]). Assume  $\theta \geq \omega_2$  is regular, and  $\mathbb{P}$  is a poset that forces  $H_{\theta}^V \in GIU$ . Then there is a  $\mathbb{P}$ -name  $\dot{\mathbb{S}}(H_{\theta}^V)$  for a c.c.c. poset such that  $\mathbb{P} * \dot{\mathbb{S}}(H_{\theta}^V)$  forces  $H_{\theta}^V$  to be **indestructibly** guessing.

Corollary 30. Suppose  $W \prec H_{(2^{\theta})^+}$ ,  $W \in \wp_{\omega_2}^*(V)$ , and  $\mathbb{P} \in W$  is a poset forcing  $H_{\theta}^V \in GIU$ . Let  $\dot{\mathbb{S}}(H_{\theta}^V)$  be as in the conclusion of Theorem 29. If there exists a  $(W, \mathbb{P} * \dot{\mathbb{S}}(H_{\theta}^V))$ -generic filter, then  $W \cap H_{\theta}$  is indestructibly guessing.

Proof. Suppose g \* h is such a W-generic filter. Let  $\pi : W \to H_W$  be the transitive collapse of W,  $\theta_W := \pi(\theta)$ , and  $\bar{g} * \bar{h} := \pi[g * h]$ ; clearly  $\bar{g} * \bar{h}$  is generic over  $H_W$ . Then by Theorem 29,  $H_W[\bar{g} * \bar{h}] \models \text{``}(H_{\bar{\theta}})^{H_W}$  is indestructibly guessing." This statement is upward absolute to V (since V is an outer model of  $H_W[\bar{g} * \bar{h}]$  with the same  $\omega_1$ ). Finally, notice that  $(H_{\bar{\theta}})^{H_W} = \pi[W \cap H_{\theta}]$ . So the transitive collapse of  $W \cap H_{\theta}$  is indestructibly guessing, which implies that  $W \cap H_{\theta}$  is indestructibly guessing.

Given a partial order  $\mathbb{P}$ , a regular  $\theta \geq |\mathbb{P}|^+$ , and a condition  $p \in \mathbb{P}$ , we say that p is an  $(M, \mathbb{P})$ strong master condition iff there is a condition  $p|M \in M \cap \mathbb{P}$  such that for every  $r \in M$  that
is stronger than p|M, r is compatible in  $\mathbb{P}$  with p (the condition p|M is not typically unique, and
is called a reduction of p into M). A partial order  $\mathbb{P}$  is strongly proper on a stationary
set iff there is a stationary  $S \subseteq [H_{|\mathbb{P}|}^+]^\omega$  such that for all but nonstationarily many  $M \in S$  and
every  $p \in M$ , there is a  $p' \leq p$  such that p' is an  $(M, \mathbb{P})$ -strong master condition. This concept was
isolated by Mitchell, and its main use is:

**Fact 31** (Mitchell [24]). If  $\mathbb{P}$  is strongly proper on a stationary set, then it has the  $\omega_1$  covering and approximation properties.

There are two kinds of posets that are strongly proper on a stationary set that we will use in this paper: adding a Cohen real, and the following club-shooting poset, which is the "continuous" version of Todorcevic's  $\in$ -collapse. This poset is a special case of Neeman's "decorated" poset from [25]. The role of the f in Definition 32 is just to ensure that the  $\in$ -chain of elementary submodels added by the first coordinate is  $\subseteq$ -continuous.<sup>10</sup>

**Definition 32.** Let  $\theta \geq \omega_2$  be regular, and X a stationary subset of  $[H_{\theta}]^{\omega}$ , which we will without loss of generality assume consists only of elementary submodels of  $(H_{\theta}, \in)$ . The poset  $\mathbb{C}^{fin}_{dec}(X)$  is the collection of all pairs of the form  $(\mathcal{M}, f)$  where:

<sup>&</sup>lt;sup>9</sup>A sufficient condition for this is if  $\mathbb{P}$  collapses  $H_{\theta}^{V}$  to size  $\omega_{1}$ , and has the  $\omega_{1}$  covering and approximation properties.

<sup>&</sup>lt;sup>10</sup>Neeman's forcing is designed to preserve  $\omega_1$  and  $\theta$ ; we do not need preservation of  $\theta$  for our purposes.

- $\mathcal{M}$  is a finite subset of X, and for every  $M, N \in \mathcal{M}$  with  $M \neq N$ , either  $M \in N$  or  $N \in M$ . We let  $M_0, M_1, M_2, \ldots, M_k$  be the unique enumeration of  $\mathcal{M}$  such that  $M_i \in M_{i+1}$  for every i < k.
- f is a function from  $\mathcal{M} \to H_{\theta}$ , and has the property that  $f(M_i) \in M_{i+1}$  for every i < k.

The ordering is defined by:  $(\mathcal{N}, h) \leq (\mathcal{M}, f)$  iff  $\mathcal{N} \supseteq \mathcal{M}$  and  $h(M) \supseteq f(M)$  for every  $M \in \mathcal{M}$ .

**Fact 33** (similar to Neeman [25], Section 4). Suppose X is a stationary subset of  $[H_{\theta}]^{\omega}$  where  $\theta \geq \omega_2$  is regular. Then:

- (1)  $\mathbb{C}_{dec}^{fin}(X)$  is strongly proper on a stationary set. In particular, if  $N \prec H_{(2^{\theta})^+}$  is countable,  $N \cap H_{\theta} \in X$ , and  $(\mathcal{M}, f)$  is a condition in N, then  $(\mathcal{M} \cup \{N \cap H_{\theta}\}, f)$  is an  $(N, \mathbb{C}_{dec}^{fin}(X))$ -strong master condition.
- (2) If G is generic over V for  $\mathbb{C}^{fin}_{dec}(X)$ , then  $\bigcup G$  is an  $\in$ -increasing filtration of  $H^V_{\theta}$  of length  $\omega_1$  consisting entirely of members of X. Moreover, if  $X_0 \in V$  was a stationary subset of X and  $\langle Q_i : i < \omega_1 \rangle \in V[G]$  is the generic filtration, then there are stationarily many  $i < \omega_1$  such that  $Q_i \in X_0$ .

In particular,  $\mathbb{C}^{fin}_{dec}(X)$  has the  $\omega_1$  covering and approximation properties, and forces  $|H^V_{\theta}| = \omega_1$ .

We will also need:

**Lemma 34.** If  $\mathbb{P}$  is proper and  $\theta$  is a regular uncountable cardinal, then  $\mathbb{P} * \dot{\mathbb{C}}^{fin}_{dec}(V \cap [\theta]^{\omega})$  is proper.

Proof. Fix a regular  $\Omega$  with  $\mathbb{P}, \theta \in H_{\Omega}$ . Let  $N \prec (H_{\Omega}, \in, \theta, \mathbb{P})$  be countable, and  $(p, (\dot{\mathcal{M}}, \dot{f})) \in N$  be a condition in the two-step iteration. Since  $\mathbb{P}$  is proper, there is a  $p' \leq p$  that is an  $(N, \mathbb{P})$ -master condition, so in particular p' forces  $N[\dot{G}_{\mathbb{P}}] \cap \theta = N \cap \theta \in V \cap [\theta]^{\omega}$ . Then by Fact 33, p' forces that  $(\dot{\mathcal{M}} \cup \{N \cap \theta\}, \dot{f})$  is an  $(N[\dot{G}_{\mathbb{P}}], \dot{\mathbb{C}}_{dec}^{fin}(V \cap [\theta]^{\omega}))$  (strong) master condition. Then  $(p', (\dot{\mathcal{M}} \cup \{N \cap \theta\}, \dot{f}))$  is a master condition for N that is stronger than  $(p, (\dot{\mathcal{M}}, \dot{f}))$ .

2.5. A theorem of Gitik and Velickovic. For any set R and any  $T \subseteq \omega_1$ ,  $R \setminus T := \{z \in R : z \cap \omega_1 \in T\}$ . A set  $R \subset [H_\theta]^\omega$  is called **projective stationary** iff  $R \setminus T$  is stationary for every stationary  $T \subseteq \omega_1$  (this concept was isolated in Feng-Jech [8]). We make heavy use of the following theorem of Velickovic, which slightly improved an earlier theorem of Gitik [16]:

**Theorem 35** (Velickovic [30], Lemma 3.15). Suppose  $V \subset W$  are transitive ZFC models,  $\mathbb{R}^V \neq \mathbb{R}^W$ , and that  $\lambda$  is a W-regular cardinal with  $\lambda \geq \omega_2^W$ . Then in W, for every stationary  $R \subseteq \lambda \cap cof(\omega)$  and every stationary  $T \subseteq \omega_1$ , there are stationarily many  $z \in [\lambda]^{\omega} \setminus V$  such that  $z \cap \omega_1 \in T$  and  $sup(z) \in R$ . In particular,  $[\lambda]^{\omega} \setminus V$  is projective stationary.

The statement of Theorem 35 is slightly stronger than the version in [30], which didn't mention the set R (just the projective stationarity of  $[\lambda]^{\omega} \setminus V$ ). But a close examination of Velickovic's proof shows the statement above, because he proved that (in W) given any stationary  $T \subseteq \omega_1$  and any function  $F: [\lambda]^{<\omega} \to \lambda$ , for all but nonstationarily many members of the set

$$\{X \subset H_{(2^{\lambda})} : X \cap \lambda \in \lambda \cap \operatorname{cof}(\omega)\},\$$

there is a  $z \in [X \cap \lambda]^{\omega}$  such that z is closed under F,  $\sup(z) = X \cap \lambda$ ,  $z \notin V$ , and  $z \cap \omega_1 \in T$ . This will be the case for any X in the displayed set that has, as an element, Player II's winning strategy in the game from page 272 of [30]. In particular, this is true for some X such that  $X \cap \lambda \in R$ .

We also will use:

**Lemma 36.** Suppose  $\lambda$  is an uncountable ordinal, S is a stationary subset of  $[\lambda]^{\omega}$ ,  $\theta$  is a regular uncountable cardinal strictly larger than  $\lambda$ , and R is a stationary subset of  $\theta \cap cof(\omega)$ . Then

$$\{u \in [\theta]^{\omega} : sup(u) \in R \text{ and } u \cap \lambda \in S\}$$

is stationary in  $[\theta]^{\omega}$ .

Proof. Let  $\mathfrak{A}=(\theta,\ldots)$  be any first order structure on  $\theta$  in a countable language; we need to find a countable elementary substructure of  $\mathfrak{A}$  whose supremum is in R and whose intersection with  $\lambda$  is in S. Fix an  $M \prec \mathfrak{A}$  of size  $<\theta$  such that  $\lambda \subseteq M$  and  $M \cap \theta \in R$ . This is possible because  $\lambda < \theta$  and R is a stationary subset of  $\theta \cap \operatorname{cof}(\omega)$ . Fix a Skolemized structure  $\mathfrak{B}$  with universe M in a countable language extending  $\mathfrak{A} \upharpoonright M$  such that any elementary substructure of  $\mathfrak{B}$  is cofinal in  $M \cap \theta$ ; this is possible because  $M \cap \theta$  is  $\omega$ -cofinal. Since S is stationary in  $[\lambda]^{\omega}$  and  $\lambda \subseteq M$ , Fodor's lemma implies that there is a  $z \in S$  such that

$$\widetilde{z} \cap \lambda = z$$
.

where  $\widetilde{z}$  is the  $\mathfrak{B}$ -Skolem hull of z. Then  $\widetilde{z}$  is elementary in  $\mathfrak{B}$  and hence  $\sup(\widetilde{z}) = M \cap \theta \in R$ . Since  $\widetilde{z} \prec \mathfrak{B}$ ,  $\mathfrak{B}$  extends  $\mathfrak{A} \upharpoonright M$ , and  $M \prec \mathfrak{A}$ , then  $\widetilde{z} \prec \mathfrak{A}$ . And  $\widetilde{z} \cap \lambda = z \in S$ .

## 3. Controlling the internal part of a guessing model

In this section we prove some facts that will be used in most of the proofs of the paper. The tight control over the internal and external parts is mainly needed for Theorem 7. A poset is called  $\omega_1$ -SSP if it preserves all stationary subsets of  $\omega_1$ . Given uncountable sets  $H \subset H'$  and a subset  $S \subseteq [H]^{\omega}$ , Lift<sup>H'</sup>(S) denotes the set  $\{z \in [H']^{\omega} : z \cap H \in S\}$ ; it is a standard fact that if S is stationary, then so is its lifting.

**Theorem 37.** Fix a (possibly nonstationary) subset T of  $\omega_1$ , and a regular  $\theta \geq \omega_2$ .

- (1) There is an  $\omega_1$ -SSP poset  $\mathbb{Q}_{T,\theta}^{\omega_1-SSP}$  that forces the following for every V-regular cardinal  $\mu \in [\omega_2^V, \theta]$ :  $H_{\mu}^V$  has size  $\omega_1$ , is indestructibly guessing, internally unbounded, and its internal part is **exactly** (mod  $NS_{\omega_1}$ ) the set T. Also, every stationary subset of  $\theta \cap cof(\omega)$  from the ground model remains stationary in the extension.
- (2) There is a proper poset  $\mathbb{Q}_{T,\theta}^{proper}$  that forces the following for every V-regular cardinal  $\mu \in [\omega_2^V, \theta]$ :  $H_{\mu}^V$  has size  $\omega_1$ , is indestructibly guessing, and its internal part contains T. Moreover, if T was costationary in V, then both the internal and external parts of  $H_{\mu}^V$  are forced to have stationary intersection with  $T^c$ .

In particular: if T was stationary and costationary, then  $\mathbb{Q}_{T,\theta}^{proper}$  forces "int $(H_{\mu}^{V})$  contains the stationary set T, and  $ext(H_{\mu}^{V})$  is a stationary subset of  $T^{c}$ " for every V-regular  $\mu \in [\omega_{2}^{V}, \theta]$ .

*Proof.* For part 1:  $\mathbb{Q}_{T,\theta}^{\omega_1-\mathrm{SSP}}$  is defined to be the poset

(4) 
$$\operatorname{Add}(\omega) * \dot{\mathbb{C}}_{\operatorname{dec}}^{\operatorname{fin}}(\dot{X}_T^{\theta}) * \dot{\mathbb{S}}(H_{\theta}^V)$$

where, letting  $\dot{\sigma}$  be the Add( $\omega$ )-name for its generic object,

$$\dot{X}_{T}^{\theta} := \operatorname{Lift}^{H_{\theta}^{V}[\dot{\sigma}]} \left( \left( [H_{\theta}^{V}]^{\omega} \searrow T \right)^{V} \cup \left( [\omega_{2}]^{\omega} \searrow T^{c} \right) \backslash V \right) \\
= \left\{ z \in \left[ H_{\theta}[\dot{\sigma}] \right]^{\omega} : (z \cap H_{\theta}^{V} \in V \text{ and } z \cap \omega_{1} \in T) \text{ or } (z \cap \omega_{2} \notin V \text{ and } z \cap \omega_{1} \in T^{c}) \right\},$$

 $\dot{\mathbb{C}}^{\mathrm{fin}}_{\mathrm{dec}}(\dot{X}_T^{\theta})$  is the poset from Definition 32, and  $\mathbb{S}(H_{\theta}^V)$  is the poset given by Theorem 29, assuming that the assumptions of that theorem hold, which we verify first. For Theorem 29 to be applicable, we need to check that the first two steps of (4) force  $H_{\theta}^V \in \mathrm{IU}$ . This is part of the following claim:

Claim 37.1. The first two steps of the poset (4) force that

- (a)  $H_{\theta}^{V} \in G_{\omega_{1}} \cap IU$ ;
- (b) all V-stationary subsets of  $\theta \cap cof(\omega)$  remain stationary;
- (c) all V-stationary subsets of  $\omega_1$  remain stationary; and
- (d) for all  $\mu \in [\omega_2^V, \theta]$  that are regular in V,  $H_{\mu}^V$  has internal part exactly T.

*Proof.* (of Claim 37.1): the proofs of parts (a) and (b) break into cases, depending on whether or not T is stationary in V. Though the proofs of these parts could be combined, we find it conceptually simpler to prove each part separately, at the expense of some slight repetition.

For part (a): the poset  $Add(\omega)$  is strongly proper, and the 2nd step is forced to be strongly proper on a stationary set by Fact 33, provided that the set  $\dot{X}_T^{\theta}$  is forced by  $Add(\omega)$  to be a stationary subset of  $[H_{\theta}[\dot{\sigma}]]^{\omega}$ . We verify this now. Let  $\sigma$  be  $Add(\omega)$ -generic; we break into cases depending on whether T is stationary in V. If T is stationary in V, then  $[H_{\theta}]^{\omega} \searrow T$  is stationary, so properness of  $Add(\omega)$  implies that the set

$$([H_{\theta}^V]^{\omega} \searrow T)^V$$
,

remains stationary in  $V[\sigma]$ , and hence its lifting to  $H^V_{\theta}[\sigma]$ —which is a subset of  $(\dot{X}^{\theta}_T)_{\sigma}$ —is stationary from the point of view of  $V[\sigma]$ . On the other hand, if T is nonstationary in V, then  $T^c$  contains a club subset of  $\omega_1$ . By Gitik-Velickovic's Theorem 35 (noting that  $\omega_2^V = \omega_2^{V[\sigma]}$ ), the set  $([\omega_2]^{\omega}) \setminus V$  is (projective) stationary in  $V[\sigma]$ . So in particular  $([\omega_2]^{\omega} \setminus T^c) \setminus V$  is stationary, and hence its lifting to  $H^V_{\theta}[\sigma]$  is stationary. And this lifting is a subset of  $(\dot{X}^{\theta}_T)_{\sigma}$ .

In summary, whether T is stationary or not, the set  $\dot{X}_T^{\theta}$  is forced by  $\mathrm{Add}(\omega)$  to be stationary, and hence the first two steps of (4) are of the form "strongly proper, followed by strongly proper on a stationary set". So this 2-step iteration has the  $\omega_1$  approximation and cover properties by Facts 31 and 28. This, in turn, ensures that  $H_{\mu}^V$  will be  $\omega_1$ -guessing and internally unbounded after the first two steps of the iteration (4), for all V-regular  $\mu \in [\omega_2, \theta]$ . This completes the proof of part (a) of the claim.

Next we verify part (b) of the claim. Let R be a stationary subset of  $\theta \cap \operatorname{cof}(\omega)$  from the ground model. Consider two cases, depending on whether or not T is stationary in V. If T is stationary in V, then

$$V \models \ \widetilde{R}^V_T := \left\{z \in \left([H_\theta]^\omega\right)^V \ : \ \sup(z) \in R \ \text{and} \ z \cap \omega_1 \in T\right\} \ \text{is stationary in} \ [\theta]^\omega.$$

By properness of  $\mathrm{Add}(\omega)$ ,  $\widetilde{R}_T^V$  remains stationary in  $V[\sigma]$ , and the lifting of  $\widetilde{R}_T^V$  to  $H_{\theta}[\sigma]$  is a stationary subset of  $(\dot{X}_T^{\theta})_{\sigma}$ . By Fact 33, the second step of the forcing (4) preserves the stationarity of  $\widetilde{R}_T^V$ . Since the supremum of every element of  $\widetilde{R}_T^V$  lies in R, it follows that R is still stationary as well.

Now suppose T is nonstationary in V. By Fact 33 and the definition of  $\dot{X}_T^{\theta}$ , it suffices to prove that

$$V[\sigma] \models \{z \in [\theta]^{\omega} : \sup(z) \in R, \ z \cap \omega_2 \notin V \text{ and } z \cap \omega_1 \in T^c\} \text{ is stationary in } [\theta]^{\omega}.$$

Now  $\omega_2^V = \omega_2^{V[\sigma]}$ , and  $T^c$  contains a club subset of  $\omega_1$  by our case. So, in turn, it suffices to show that

(5) 
$$V[\sigma] \models \{z \in [\theta]^{\omega} : \sup(z) \in R \text{ and } z \cap \omega_2 \in [\omega_2]^{\omega} \setminus V\} \text{ is stationary in } [\theta]^{\omega}.$$

Since  $\omega_2^V = \omega_2^{V[\sigma]}$ , Theorem 35 implies that  $[\omega_2]^\omega \setminus V$  is stationary in  $V[\sigma]$  (where the  $\lambda$  from the statement of that theorem is taken to be  $\omega_2$ ). By properness of  $\mathrm{Add}(\omega)$ , R is still a stationary subset of  $\theta \cap \mathrm{cof}(\omega)$  in  $V[\sigma]$ . If  $\theta = \omega_2$  then (5) follows from Theorem 35 directly; if  $\theta > \omega_2$  then (5) follows from Theorem 35 (again with  $\lambda = \omega_2$ ) together with Lemma 36.

Next we verify part (c) of the claim. Let S be a stationary subset of  $\omega_1$ , and let  $\sigma$  be  $(V, \operatorname{Add}(\omega))$ -generic. Then at least one of  $S \cap T$  or  $S \cap T^c$  is stationary. Suppose first that  $S \cap T$  is stationary. Then  $([H^V_\theta]^\omega)^V \searrow (S \cap T)$  is stationary in V, and remains so in  $V[\sigma]$  by properness of  $\operatorname{Add}(\omega)$ ; then its lifting to  $H^V_\theta[\sigma]$  is a stationary subset of  $X^\theta_T := (\dot{X}^\theta_T)_\sigma$ . So by Fact 33 (viewing  $V[\sigma]$  as the ground model),  $\mathbb{C}^{\operatorname{fin}}_{\operatorname{dec}}(X^\theta_T)$  preserves the stationarity of  $\operatorname{Lift}^{H^V_\theta[\sigma]}\left(([H^V_\theta]^\omega)^V \searrow (S \cap T)\right)$ , and in particular the stationarity of  $S \cap T$ . Now suppose  $S \cap T^c$  is stationary. Then it is still stationary in  $V[\sigma]$ , so by Gitik-Velickovic's Theorem 35,  $\left([\omega_2]^\omega \searrow (S \cap T^c)\right) \setminus V$  is stationary, and hence its lifting to  $H^V_\theta[\sigma]$  is stationary. Moreover, this lifting is a subset of  $X^\theta_T$ . Then by Fact 33 (again viewing  $V[\sigma]$  as the ground model), the poset  $\mathbb{C}^{\operatorname{fin}}_{\operatorname{dec}}(X^\theta_T)$  preserves stationarity of  $\operatorname{Lift}^\theta\left(\left([\omega_2]^\omega \searrow (S \cap T^c)\right) \setminus V\right)$ , and hence stationarity of  $S \cap T^c$ .

Finally we verify part (d) of the claim. The poset  $\mathbb{C}(X_T^{\theta})$  adds a filtration  $\langle Q_i : i < \omega_1 \rangle$  of  $H_{\theta}^V[\sigma]$  with the property that whenever i is in the club  $C := \{i < \omega_1 : Q_i \cap \omega_1 = i\}$ , then:

- if  $i \in T$  then  $Q_i \cap H_{\theta}^V \in V$ , which implies that  $Q_i \cap H_{\mu}^V \in V$  for all  $\mu \in [\omega_2, \theta]$ ; and
- if  $i \in T^c$  then  $Q_i \cap \omega_2 \notin V$ , which implies that  $Q_i \cap H^V_\mu \notin V$  for all  $\mu \in [\omega_2, \theta]$ .

It follows that for each V-regular  $\mu \in [\omega_2, \theta]$ , the filtration  $\langle Q_i \cap H^V_{\mu} : i < \omega_1 \rangle$  has the property that for every  $i \in C$ ,  $Q_i \cap H^V_{\mu}$  is an element of  $H^V_{\mu}$  if and only if  $i \in T$ . This shows that the internal part of  $H^V_{\mu}$  is forced to be exactly T.

Then by Claim 37.1 and Theorem 29, the third step  $\dot{\mathbb{S}}(H_{\theta}^{V})$  of the poset (4) forces  $H_{\theta}^{V}$  to be indestructibly guessing (and still internally unbounded). Also, Claim 37.1 and Lemma 21 imply that for every V-regular  $\mu \in [\omega_{2}^{V}, \theta]$ ,  $H_{\mu}^{V}$  still has internal part exactly T after forcing with  $\dot{\mathbb{S}}(H_{\theta}^{V})$ , since the latter preserves  $\omega_{1}$  (it is c.c.c.). Since  $\dot{\mathbb{S}}(H_{\theta}^{V})$  is forced to be c.c.c., in particular it preserves all stationary subsets of  $\omega_{1}$  and of  $\theta \cap \operatorname{cof}(\omega)$  that lie in  $V^{\operatorname{Add}(\omega)} * \dot{\mathbb{C}}^{\operatorname{fin}}_{\operatorname{dec}}(\dot{X}_{T}^{\theta})$ . Together with Claim 37.1, it follows that all stationary subsets of  $\theta \cap \operatorname{cof}(\omega)$ , and of  $\omega_{1}$ , from V are preserved by the 3-step iteration (4). This completes the proof of part 1 of the theorem.

Part 2 is similar:  $\mathbb{Q}_{T,\theta}^{\text{proper}}$  is defined similarly to the poset  $\mathbb{Q}_{T,\theta}^{\omega_1\text{-SSP}}$  defined in (4), except that instead of the  $Add(\omega)$ -name  $\dot{X}_T^{\theta}$ , we use the  $Add(\omega)$ -name

$$\dot{Y}_{T}^{\theta} := \operatorname{Lift}^{H_{\theta}^{V}[\sigma]} \left( ([H_{\theta}^{V}]^{\omega})^{V} \cup ([\omega_{2}]^{\omega} \searrow T^{c}) \setminus V \right) \\
= \left\{ z \in \left[ H_{\theta}[\sigma] \right]^{\omega} : \left( z \cap H_{\theta}^{V} \in V \right) \text{ or } \left( z \cap \omega_{1} \in T^{c} \text{ and } z \cap \omega_{2} \notin V \right) \right\}.$$

The proof is similar to the proof of part 1, so we only briefly sketch it. Roughly, the fact that  $\dot{Y}_T^{\theta}$  contains the lifting of all of  $([H_{\theta}^{V}]^{\omega})^V$ —rather than just of  $([H_{\theta}^{V}]^{\omega})^V \searrow T$  as was the case with the club shooting portion of  $\mathbb{Q}_{T,\theta}^{\omega_1\text{-SSP}}$ —ensures that the iteration  $\mathbb{Q}_{T,\theta}^{\text{proper}}$  will indeed be proper. Now suppose  $\sigma$  is  $(V, \text{Add}(\omega))$ -generic and  $Y_T^{\theta} := (\dot{Y}_T^{\theta})_{\sigma}$ . Let  $\langle Q_i : i < \omega_1 \rangle$  be a generic filtration of  $H_{\theta}^{V}[\sigma]$  added by  $\mathbb{C}_{\text{dec}}^{\text{fin}}(Y_T^{\theta})$  over  $V[\sigma]$ . By definition of  $Y_T^{\theta}$ , if i is such that  $Q_i \cap \omega_1 = i$  and  $i \in T$ , then  $Q_i \cap H_{\theta}^{V} \in V$ , and hence  $Q_i \cap H_{\mu}^{V} \in V$  for every V-regular  $\mu \in [\omega_2^{V}, \theta]$ . It follows that the internal part of any such  $H_{\mu}^{V}$  contains (mod  $NS_{\omega_1}$ ) the set T. In particular, if T is stationary, then  $H_{\mu}^{V} \in IS^{V[\sigma*\bar{Q}]}$ . Now for the "moreover" clause of part 2, assume that T is costationary; we want to show that both the internal and external part of each  $H_{\mu}^{V}$  have stationary intersection with  $T^c$ . That the internal part of each  $H_{\mu}^{V}$  has stationary intersection with  $T^c$  follows from Fact 33 together with the fact that  $([H_{\theta}^{V}]^{\omega})^{V} \searrow T^c$  is, in  $V[\sigma]$ , a stationary subset of  $Y_T^{\theta}$  (more precisely, its lifting to  $H_{\theta}[\sigma]$  is a stationary subset of  $Y_T^{\theta}$ ). For the external part, the Gitik-Velickovic Theorem 35 ensures that, in  $V[\sigma]$ , the lifting of  $([\omega_2]^{\omega} \searrow T^c) \setminus V$  is a stationary subset of  $Y_T^{\theta}$ , and hence by Fact 33 there are stationarily many  $i < \omega_1$  such that  $Q_i \cap \omega_1 = i \in T^c$  and  $Q_i \cap \omega_2^{V} \notin V$ . For such  $i, Q_i \cap H_{\mu}^{V} \notin H_{\mu}^{V}$  for every V-regular  $\mu \in [\omega_2^{V}, \theta]$ .

The proof that  $H_{\theta}^{V}$  is forced by  $\mathbb{Q}_{T,\theta}^{\text{proper}}$  to be indestructibly guessing is almost identical to the proof from part 1.

Corollary 38. Assume  $\theta \geq \omega_2$  is regular and  $T \subseteq \omega_1$ . Let  $\mathbb{Q}_{T,\theta}^{proper}$  and  $\mathbb{Q}_{T,\theta}^{\omega_1-SSP}$  be the posets given by Theorem 37.

- (1) Assume  $W \prec (H_{(2^{\theta})^+}, \in, T)$ ,  $|W| = \omega_1 \subset W$ , and there exists a  $(W, \mathbb{Q}_{T,\theta}^{\omega_1 SSP})$ -generic filter. Then for every regular  $\mu \in [\omega_2, \theta] \cap W$ ,  $W \cap H_{\mu}$  is indestructibly guessing, has internal part exactly T, and external part exactly  $T^c$ . In particular, if T is nonstationary then  $W \cap H_{\mu} \in GIU \setminus IS$ , and if T is stationary and costationary then  $W \cap H_{\mu} \in GIS \setminus IC$ .
  - If the  $(W, \mathbb{Q}_{T,\theta}^{\omega_1-SSP})$ -generic filter is also stationary correct (with respect to names for stationary subsets of  $\omega_1$  that lie in W), then for every  $R \in W$  that is a stationary subset of  $\theta \cap cof(\omega)$ ,  $R \cap sup(W \cap \theta)$  is stationary in  $sup(W \cap \theta)$ .
- (2) Let  $\dot{S}_{external}^{\omega_2}$  be the  $\mathbb{Q}_{T,\theta}^{proper}$ -name for the external part of  $H_{\omega_2}^V$ . If T is costationary, then  $\dot{S}_{external}^{\omega_2}$  is forced to be a stationary subset of  $T^c$ . Furthermore, if  $W \prec (H_{(2^{\theta})^+}, \in, T)$ ,  $|W| = \omega_1 \subset W$ , and there exists a  $(W, \mathbb{Q}_{T,\theta}^{proper})$ -generic filter that interprets  $\dot{S}_{external}^{\omega_2}$  as a stationary set, then for every regular  $\mu \in [\omega_2, \theta] \cap W$ ,  $W \cap H_{\mu}$  is indestructibly guessing, its internal part contains T, and its external part is stationary and costationary in  $T^c$ . In particular, if T is stationary and costationary, then  $W \cap H_{\mu} \in G_{\omega_1} \cap IS \setminus IC$  for every regular  $\mu \in [\omega_2, \theta] \cap W$ .

*Proof.* For part 2, the fact that  $\dot{S}_{\rm external}^{\omega_2}$  is forced to be stationary follows immediately from Theorem 37. Also, Observation 22 implies that

(6) 
$$\Vdash \forall \mu \in \text{REG} \cap [\omega_2, \theta] \quad \text{ext}(H^V_\mu) \supseteq \text{ext}(H^V_{\omega_2}) = \dot{S}^{\omega_2}_{\text{external}}.$$

Now suppose W is as in the statement of part 2, and that g is a  $(W, \mathbb{Q}_{T,\theta}^{\text{proper}})$ -generic filter such that  $(\dot{S}_{\text{external}}^{\omega_2})_g$  is a stationary subset of  $\omega_1$  (in V). Let  $\sigma_W: H_W \to W \prec (H_{(2^\theta)^+}, \in, T)$  be the inverse of the transitive collapse of W, and  $b_W:=\sigma_W^{-1}(b)$  for every  $b \in W$ . Let  $\bar{g}$  be the pointwise image of g under the collapsing map. Note that  $T \in W$  and T is not moved by the collapsing map. It can also be easily verified that  $(\dot{S}_{\text{external}}^{\omega_2})_g$  is the same as  $(\sigma_W^{-1}(\dot{S}_{\text{external}}^{\omega_2}))_{\bar{g}}$ ; let S denote this set. Fix any regular  $\mu \in W \cap [\omega_2, \theta]$ . Then by (6) and Theorem 37,  $H_W[\bar{g}] \models \text{``}\sigma^{-1}(H_\mu)$ 

is indestructibly guessing, size  $\omega_1$ , its internal part contains  $\sigma^{-1}(T) = T$ , and its external part contains S". Since  $H_W[\bar{g}]$  is a transitive ZF<sup>-</sup> model and V is an outer model of  $H_W[\bar{g}]$  with the same  $\omega_1$ , those statements are upward absolute to V. Finally, notice that the statement "X is indestructibly guessing, its internal part contains T, and its external part contains S"—viewing T and S as fixed parameters—is invariant across sets that are  $\in$ -isomorphic to X. In particular, V believes the same statement about  $\sigma_W[\sigma_W^{-1}(H_\mu)]$ , which is just  $W \cap H_\mu$ . This completes the proof of part 2.

The proof of 1 is similar, using instead part 1 of Theorem 37. The only new thing to check is that if the W-generic filter g is stationary correct, then W is diagonally reflecting for stationary subsets of  $\theta \cap \operatorname{cof}(\omega)$  lying in W. To see this, note that  $\theta$  is collapsed to  $\omega_1$  and still has uncountable cofinality. Let  $\dot{f}$  be a  $\mathbb{Q}_{T,\theta}^{\omega_1-\operatorname{SSP}}$ -name for a strictly increasing, continuous, cofinal function from  $\omega_1 \to \theta$ . Suppose  $R \in W$  and R is stationary in  $\theta \cap \operatorname{cof}(\omega)$ . Let  $\dot{T}_R$  be the name for  $\{i < \omega_1 : \dot{f}(i) \in \check{R}\}$ . Since R remains stationary by Theorem 37, it follows that  $\dot{T}_R$  is forced to be a stationary subset of  $\omega_1$ , and hence its interpretation by g is really stationary in  $\omega_1$ . By Viale-Weiss [33],  $W \cap \theta$  is an  $\omega$ -closed set of ordinals. This, together with W-genericity of g, implies that  $(\dot{f})_g$  maps  $\omega_1$  continuously and cofinally into  $\sup(W \cap \theta)$ . Then the pointwise image of  $(\dot{T}_R)_g$  under  $(\dot{f})_g$  is a stationary subset of  $\sup(W \cap \theta)$ , and also contained in R.

## 4. Preservation of forcing axioms via generic embeddings

The material in this section is motivated by Woodin [34] and Viale ([32], [31]), where stationary tower forcing is used in conjunction with forcing axioms. Unlike those settings, which made use of large cardinals in the universe to ensure wellfounded generic ultrapowers via stationary tower forcing, wellfoundedness will not be important for the goals of this article. The main theorem in this section (Theorem 20) makes preservation arguments for forcing axioms very closely resemble arguments that involve lifting traditional large cardinal embeddings; this makes forcing axiom preservation arguments conceptually clearer, in the author's opinion. Theorem 20 is possibly folklore, but the author is not aware of any presentation of it in the literature. The section concludes with several preservation results from the literature which can be viewed as instances of Theorem 20.

We make the following definition, which holds of all standard classes of posets (proper, c.c.c., etc.):

**Definition 39.** We say that a class  $\Gamma$  of posets is **closed under restrictions** iff whenever  $\mathbb{Q} \in \Gamma$  and  $q \in \mathbb{Q}$ , then

$$\mathbb{Q} \upharpoonright q := \{ p \in \mathbb{Q} : p \le q \}$$

(with order inherited from  $\mathbb{Q}$ ) is in  $\Gamma$ .

Note that in the statement of Theorem 20, we do **not** require that  $\Gamma$  (or  $\Delta$ ) is closed under taking regular suborders; this will be convenient in the proof of the theorems about stationary reflection, since e.g. "being proper and and forcing  $H^V_{\omega_2} \notin IC$ " is not closed under regular suborders (the regular suborder will be proper, but may fail to force  $H^V_{\omega_2} \notin IC$ ). However, notice that if  $FA(\Gamma)$  holds and  $\widetilde{\Gamma}$  is the closure of  $\Gamma$  under taking regular suborders, then  $FA(\widetilde{\Gamma})$  also holds. This is not necessarily true of the "plus" versions, unless one also requires that the *quotient* of the regular suborder is  $\omega_1$ -SSP. We will use the following observation several times:

**Observation 40.** If  $FA(\Gamma)$  holds and  $\mathbb{P}$  is a regular suborder of some member of  $\Gamma$  (but possibly  $\mathbb{P} \notin \Gamma$ ), then  $\mathbb{P}$  preserves stationary subsets of  $\omega_1$ . This is because, by Foreman-Magidor-Shelah [12], every member of  $\Gamma$  must be  $\omega_1$ -SSP, and this property clearly is inherited by regular suborders.

The key to the preservation theorem is the following lemma of Woodin.

**Lemma 41** (Woodin [34], proof of Theorem 2.53). If  $\Gamma$  is a class of partial orders and  $\mu \leq \omega_1$  is a cardinal (possibly  $\mu = 0$ ), the following are equivalent:

- (1)  $FA^{+\mu}(\Gamma)$ ;
- (2) For every  $\mathbb{P} \in \Gamma$ , every  $\mu$ -sequence  $\dot{\vec{S}} = \langle \dot{S}_i : i < \mu \rangle$  of  $\mathbb{P}$ -names for stationary subsets of  $\omega_1$ , and every (equivalently, some) regular  $\theta > |\wp(\mathbb{P})|$ , there are stationarily many  $W \in \wp_{\omega_2}^*(H_\theta)$  such that there exists an  $\dot{\vec{S}}$ -correct,  $(W, \mathbb{P})$ -generic filter.

Remark 42. Suppose  $\mu = \omega_1$ , and  $\Gamma$  has the additional property that for every  $\mathbb{P} \in \Gamma$  and every cardinal  $\lambda$ , there exists a  $\mathbb{P}$ -name  $\dot{\mathbb{Q}}$  such that  $\mathbb{P} * \dot{\mathbb{Q}} \in \Gamma$  and  $\mathbb{P} * \dot{\mathbb{Q}} \Vdash |\lambda| \leq \omega_1$ . This holds, for example, if  $\Gamma$  is the class of proper posets, or  $\omega_1$ -SSP posets (but not c.c.c. posets). Then clause 2 of Lemma 41 can be replaced by: "For every  $\mathbb{P} \in \Gamma$  and every (equivalently, some) regular  $\theta > |\wp(\mathbb{P})|$ , there are stationarily many  $W \in \wp_{\omega_2}^*(H_\theta)$  such that there exists a  $(W, \mathbb{P})$ -generic filter g, such that for every  $\dot{S} \in W$  that names a stationary subset of  $\omega_1$ , the evaluation of  $\dot{S}$  by g is stationary (in V)."

Woodin's Lemma 41, together with basic theory of generic ultrapowers, yields the following Theorem 43, which is a variant of Theorem 2.53 of Woodin [34] with weaker hypotheses (e.g. no Woodin cardinals are needed) and weaker conclusion (e.g. the generic embeddings constructed in Theorem 43 are not even necessarily wellfounded, though the ones from Woodin [34] are generic almost huge embeddings). Essentially,  $FA(\Gamma)$  can be characterized by existence of generically "supercompact" elementary embeddings where the (possibly illfounded) target model has V-generics for the relevant forcing.

**Theorem 43.** [minor variant of Theorem 2.53 of Woodin [34]] Let  $\Gamma$  be a class of posets closed under restrictions, and let  $\mu \leq \omega_1$  be a cardinal. The following are equivalent:

- (1)  $FA^{+\mu}(\Gamma)$ ;
- (2) For every  $\mathbb{Q} \in \Gamma$ , every  $q \in \mathbb{Q}$ , every sequence  $\langle \dot{S}_i : i < \mu \rangle$  of  $\mathbb{Q}$ -name for stationary subsets of  $\omega_1$ , and every (equivalently, some) regular  $\theta > |\wp(\mathbb{Q})|$ , there is a generic elementary embedding  $j : V \to N$  such that:
  - (a)  $H_{\theta}^{V}$  is in the wellfounded part of N (we assume the wellfounded part of N has been transitivized), and  $|H_{\theta}^{V}|^{N} = \omega_{1}$ ;
  - $(b)\ j\restriction H^V_\theta\in N$
  - (c)  $crit(j) = \omega_2^V;$
  - (d) There exists some  $H \in N$  such that  $q \in H$ , H is  $(V, \mathbb{Q})$ -generic, and for every  $i < \mu$ ,  $N \models \text{``}(\dot{S}_i)_H$  is a stationary subset of  $\omega_1$ ".

We make a remark that is parallel to Remark 42:

**Remark 44.** Suppose  $\mu = \omega_1$ , and  $\Gamma$  has the additional property that for every  $\mathbb{Q} \in \Gamma$  and every cardinal  $\lambda$ , there exists a  $\mathbb{Q}$ -name  $\dot{\mathbb{R}}$  such that  $\mathbb{Q} * \dot{\mathbb{R}} \in \Gamma$  and  $\mathbb{Q} * \dot{\mathbb{R}} \Vdash |\lambda| \leq \omega_1$ . This holds, for example, if  $\Gamma$  is the class of proper posets, or  $\omega_1$ -SSP posets (but not c.c.c. posets). Then clause 2d of Theorem 43 can be replaced by: "There exists some  $H \in N$  such that  $q \in H$ , H is  $(V, \mathbb{Q})$ -generic filter, and every stationary subset of  $\omega_1$  in V[H] remains stationary in N."

*Proof.* (of Theorem 43): Since this is very similar to the proof of Theorem 2.53 of Woodin [34], we only briefly sketch the proof.

First assume  $\operatorname{FA}^{+\mu}(\Gamma)$ . Fix  $\mathbb{Q} \in \Gamma$ ,  $q \in \mathbb{Q}$ , a sequence  $\dot{S} = \langle \dot{S}_i : i < \mu \rangle$  of  $\mathbb{Q}$ -names for stationary subsets of  $\omega_1$ , and a large regular  $\theta$ . Since  $\Gamma$  is closed under restrictions,  $\mathbb{Q} \upharpoonright q$  is an element of  $\Gamma$ , and hence the forcing axiom holds for it. Let R be the set of  $W \in \wp_{\omega_2}(H_\theta)$  such that  $\omega_1 \subset W$ ,  $\dot{S} \in W$ , and there exists an  $\dot{S}$ -correct,  $(W, \mathbb{Q} \upharpoonright q)$ -generic filter  $h_W$ . By Lemma 41, R is stationary. For each  $W \in R$ , let  $\bar{W}$  be the transitive collapse of W, and  $\sigma_W : \bar{W} \to W \prec H_\theta$  be the inverse of the Mostowski collapsing map. Then  $\bar{h}_W := \sigma_W^{-1}[h_W]$  is an  $(\bar{W}, \sigma_W^{-1}(\mathbb{Q}))$ -generic filter,  $\bar{q}_W := \sigma_W^{-1}(q) \in \bar{h}_W$ ,  $\operatorname{crit}(\sigma_W) = \omega_2^{\bar{W}}$ , and the evaluation of  $\sigma_W^{-1}(\dot{S}_i)$  by  $\bar{h}_W$  is a stationary subset of  $\omega_1$  (in V), for all  $i < \mu$ . Let  $\mathcal{J}$  be the restriction of the nonstationary ideal to R. Let U be  $(V, \mathcal{J}^+)$ -generic and  $j : V \to_U N$  be the resulting generic elementary embedding. Then standard applications of Los' Theorem (see Foreman [9]) ensure that j has the desired properties mentioned above. In particular,  $[\operatorname{id}_R]_U = j[H_\theta^V]$ ,  $j \upharpoonright H_\theta^V = [W \mapsto \sigma_W]_U$ ,  $H_\theta^V = [W \mapsto \bar{W}]_U$  and is an element of the (transitivized) wellfounded part of N,  $\operatorname{crit}(j) = \omega_2^V$ , and  $H := [W \mapsto \bar{h}_W]_U$  is an  $(H_\theta^V, \mathbb{Q} \upharpoonright q)$ -generic filter such that from the point of view of the generic ultrapower N,  $(\dot{S}_i)_H$  is a stationary subset of  $\omega_1$  for all  $i < \mu = j(\mu)$ . Since  $\theta$  was chosen sufficiently large from the start, H is also V-generic.

Now we prove the converse. Assume 2, and that  $\mathbb{Q} \in \Gamma$ . Fix any  $\mu$ -sequence  $\vec{S} = \langle \dot{S}_i : i < \mu \rangle$  of  $\mathbb{Q}$ -names for stationary subsets of  $\omega_1$ . By Woodin's Theorem 43 it suffices to show that if  $F: [H_{\theta}]^{<\omega} \to H_{\theta}$  with  $F \in V$ , then there is a  $W \in \wp_{\omega_2}(H_{\theta})$  that is closed under F such that  $\omega_1 \subset W$  and there exists a  $\dot{S}$ -correct,  $(W, \mathbb{Q})$ -generic,  $\omega_1$ -stationary correct filter. Let  $j: V \to N$  and  $H \in N$  be as in assumption 2. Now  $j[H_{\theta}^V]$  is closed under j(F), and  $j[H_{\theta}^V] \in N$  by assumption. Also note that  $j[H] \in N$ , since  $H \in N$ ,  $j \upharpoonright H_{\theta}^V \in N$ , and  $\mathbb{Q} \in H_{\theta}^V$ . It is routine to check that  $j[H_{\theta}^V]$  and j[H] witness that  $N \models$  "there is a model containing  $\omega_1$  and closed under j(F) for which there exists a  $j(\dot{S}) = \langle j(\dot{S}_i) : i < \mu = j(\mu) \rangle$ -correct, generic filter for  $j(\mathbb{Q})$ ". Then elementarity of j yields the analogous statement in V, completing the proof.

We now prove Theorem 20 (from page 4), which is our main tool for preservation of forcing axioms. It makes preservation of forcing axioms closely resemble arguments where large cardinal embeddings are lifted after a "preparation on the j side", where the preparation is designed to provide a master condition. Both directions of Theorem 43 will be used in the proof of Theorem 20: the forward direction of Theorem 43 (using the class  $\Gamma$  in V) will be used to obtain a generic embedding  $j:V\to N$  with the relevant properties. We will generically extend the embedding j to domain  $V^{\mathbb{P}}$  (using the master condition provided by  $\dot{\mathbb{R}}$ ), and then apply the backwards direction of Theorem 43 (using the class  $\dot{\Delta}$  in  $V^{\mathbb{P}}$ ) to ensure that  $\mathrm{FA}^{+\mu}(\dot{\Delta})$  holds in  $V^{\mathbb{P}}$ .

*Proof.* (of Theorem 20): In order to show that  $\mathrm{FA}^{+\mu}(\dot{\Delta})$  holds in  $V^{\mathbb{P}}$ , it suffices to verify that clause 2 of Theorem 43 holds in  $V^{\mathbb{P}}$  (with respect to the class  $\dot{\Delta}$ ). More precisely, we prove that  $1_{\mathbb{P}}$  forces that "whenever  $\mathbb{Q}$  is a poset in  $\dot{\Delta}$ ,  $q \in \mathbb{Q}$ , and  $\dot{S}_i : i < \mu$  is a  $\mu$ -sequence of  $\mathbb{Q}$ -names for stationary subsets of  $\omega_1$ , then there exists a generic elementary embedding

$$\pi:V[\dot{G}]\to M$$

(where  $\dot{G}$  is the  $\mathbb{P}$ -name for its generic) such that  $\pi \upharpoonright H_{\theta}^{V[\dot{G}]} \in M$ ,  $\operatorname{crit}(\pi) = \omega_2^{V[\dot{G}]}$ ,  $H_{\theta}^{V[\dot{G}]}$  is in the wellfounded part of M, and there is an  $H \in M$  that is  $(V[\dot{G}], \mathbb{Q})$ -generic such that  $q \in H$  and for every  $i < \mu$ ,  $M \models (\dot{S}_i)_H$  is a stationary subset of  $\omega_1$ ."

So let p be an arbitrary condition in  $\mathbb{P}$ ,  $\dot{\mathbb{Q}}$  a  $\mathbb{P}$ -name for a poset in  $\dot{\Delta}$ ,  $\dot{q}$  a  $\mathbb{P}$ -name for a condition in  $\dot{\mathbb{Q}}$ , and  $\dot{\vec{S}} = \langle \dot{S}_i : i < \mu \rangle$  a  $\mu$ -sequence of  $\mathbb{P} * \dot{\mathbb{Q}}$ -names for stationary subsets of  $\omega_1$ . Let  $\dot{\mathbb{R}}$  be as in the hypotheses of Theorem 20. Fix a regular  $\theta$  such that all objects mentioned so far are in  $H_{\theta}$ . Note that by the assumptions about  $\dot{\mathbb{R}}$ , each  $\dot{S}_i$  remains stationary in  $V^{\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{R}}}$ ; so  $\dot{\vec{S}}$  can be regarded as a sequence of  $\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{R}}$ -names of stationary subsets of  $\omega_1$ . Since  $\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{R}}$  is an element of  $\Gamma$  by assumption, Theorem 43 ensures that there is, in some generic extension W of V, a generic elementary embedding  $j:V\to N$  such that

- (1)  $H_{(2\theta)+}^V$  is in the wellfounded part of N, and has size  $\omega_1$  in N;
- (2)  $j \upharpoonright H_{(2^{\theta})^+}^V \in N;$
- (3)  $\operatorname{crit}(j) = \omega_2^V$ ;
- (4) There is a  $G * H * K \in N$  that is V-generic for  $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ , and such that  $(p, \dot{q}, \dot{1}) \in G * H * K$  and  $N \models \text{``}(\dot{S}_i)_{G*H*K} = (\dot{S}_i)_{G*H}$  is a stationary subset of  $\omega_1$ , for all  $i < \mu$ ."

Note that  $j[G] \in N$  because  $G \in N$  and  $j \upharpoonright H^V_{(2^\theta)^+} \in N$ . So  $j: V \to N$  and G\*H\*K satisfy all the hypothesis in the "if-then" clause 3 of the theorem we are currently proving. So by that clause, there is some  $p' \in j(\mathbb{P})$  that N believes is a lower bound of j[G]. Now p' may be in the illfounded part of N, but this will not matter. Recall that W is the generic extension of V where the embedding  $j: V \to N$  resides. Work in W for the moment. Then even if N is illfounded, the fact that  $j(\mathbb{P})$  is a partial order is upward absolute to W, and W-generics are also N-generics. More precisely, in W consider the  $\in_N$ -extension  $\mathcal{P}'$  of  $j(\mathbb{P})$ —i.e.  $\mathcal{P}' := \{x \in N: x \in^N j(\mathbb{P})\}$ —and order  $\mathcal{P}'$  by  $x \leq y$  iff  $N \models x \leq_{j(\mathbb{P})} y$ ; it is routine to check that  $\mathcal{P}'$  is a partial order in W. Choose a  $(W, \mathcal{P}')$ -generic G' such that  $p' \in G'$ . Then again it is routine to see that G' is also  $(W, j(\mathbb{P}))$ -generic, in the sense that for every  $D \in N$  such that  $N \models "D$  is dense in  $j(\mathbb{P})$ " there is some  $q \in G'$  such that  $q \in N$  D.

Now consider the generic extension N[G']. A word is in order about what is meant by N[G'], since the standard recursive definition of name-evaluation by a generic does not make sense when the ground model is illfounded. One way to make sense of N[G'] is via quotients of Boolean-valued models.<sup>11</sup> In the current context, let  $\mathbb{B} := \operatorname{ro}^N(j(\mathbb{P}))$  and consider the quotient  $N^{\mathbb{B}}/G'$  of the  $\mathbb{B}$ -valued model  $N^{\mathbb{B}}$  (see Hamkins-Seabold [17]). Then since G' is generic over N, Lemma 13 and Theorem 16 of [17] ensure that there is a definable isomorphic copy of N inside  $N^{\mathbb{B}}/G'$ —which we identify with N—such that  $N^{\mathbb{B}}/G'$   $\models$  "I am the forcing extension of N by G'." Viewed in this way, N[G'] is simply  $N^{\mathbb{B}}/G'$ , and if  $\tau$  is a  $j(\mathbb{P})$ -name in N, then the name evaluation  $\tau_{G'}$  makes sense if computed from the point of view of  $N^{\mathbb{B}}/G'$  (which believes that N is wellfounded).

Since  $p' \in G'$  and p' was stronger than every element of j[G], elementarity of  $j: V \to N$  ensures that  $j[G] \subseteq G'$ . Then the standard argument shows that the map

$$\hat{j}:V[G]\to N[G']$$

defined by  $\sigma_G \mapsto j(\sigma)_{G'}$  is well-defined and elementary (here  $j(\sigma)_{G'}$  is computed in  $N^{\mathbb{B}}/G' = N[G']$ , which makes sense as described above). Note that  $\hat{j}$  is definable in the extension W[G'], which (since  $G \in W$  and W was a generic extension of V) is a generic extension of V[G].

<sup>&</sup>lt;sup>11</sup>Alternatively, one could use the fact that  $j \upharpoonright H_{\theta}^{V} \in N$  to work entirely within N—more precisely, with the forcing relation of N—for the remainder of the proof. This alternative would involve extending  $j \upharpoonright H_{\theta}^{V} : H_{\theta}^{V} \to N$  generically, rather than extending the entire map  $j : V \to N$  as is done below. But the result would be the same, namely, in this alternative context one would use generic liftings of  $j \upharpoonright H_{\theta}^{V}$  to prove that  $H_{\theta}^{V}[G] \models FA(\mathbb{Q})$ , which would imply that  $V[G] \models FA(\mathbb{Q})$ .

Next we observe that  $\mathbb{P}$  must have been  $<\omega_2$ -distributive; this will guarantee that  $\mathrm{crit}(\hat{j})=\mathrm{crit}(j)=\omega_2^V=\omega_2^{V[G]}$ . If  $\mathbb{P}$  were not  $<\omega_2$ -distributive, then there would be some name  $\dot{\vec{\alpha}}=\langle \dot{\alpha}_i:i<\omega_1^V\rangle\in H^V_\theta$  for a new  $\omega_1^V$ -length sequence of ordinals, and hence  $j(\dot{\alpha})=\langle j(\dot{\alpha}_i):i<\omega_1^V\rangle$  would be a  $j(\mathbb{P})$ -name for a new  $j(\omega_1^V)=\omega_1^V$ -length sequence of ordinals. Now N sees that the condition p' extends the  $(j[H^V_\theta],j(\mathbb{P}))$ -generic filter j[G], and hence that p' is a total master condition for the model  $j[H^V_\theta]$  (i.e. for every  $D\in j[H^V_\theta]$  such that D is dense in  $j(\mathbb{P})$ , there is a condition weaker than p' in  $D\cap j[H^V_\theta]$ ). It follows that p' already decides all values from the sequence  $\langle j(\dot{\alpha}_i):i<\omega_1^V\rangle$ . Since  $p'\in G'$ , N[G'] sees that  $\hat{j}(\dot{\alpha}_G)=(j(\dot{\alpha}))_{G'}$  is already an element of the ground model V, which is a contradiction.

Let  $\mathbb{Q} := \dot{\mathbb{Q}}_G$ . Since  $j \upharpoonright H_{\theta}^V \in N$  and  $G * H \in N$ , it follows that  $H \in N[G']$  and

$$\hat{j} \upharpoonright (H_{\theta}^{V}[G]) = \langle \sigma_G \mapsto j(\sigma)_{G'} : \sigma \in H_{\theta}^{V} \cap N^{j(\mathbb{P})} \rangle$$

is an element of N[G']. Furthermore,  $q := \dot{q}_G$  is an element of H, by (4).

Finally, we need to verify that from the point of view of N[G'], each  $(\dot{S}_i)_{G*H}$  is stationary. Indeed, each  $(\dot{S}_i)_{G*H}$  is stationary from the point of view of N, by (4). Furthermore, since  $\mathbb{P}$  is a regular suborder of  $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$  and the latter is in  $\Gamma$ , Observation 40 ensures that  $\mathbb{P}$  is  $\omega_1$ -SSP from the point of view of V. By elementarity of j,  $j(\mathbb{P})$  is  $\omega_1$ -SSP from the point of view of N. Hence, each  $(\dot{S}_i)_{G*H}$  remains stationary in N[G'].

Theorem 20 gives an alternative proof of several preservation results in the literature. We highlight a few below.

# 4.1. Examples from the literature.

4.1.1. The Larson example and generalizations. Larson's [23] theorem that MM is preserved by  $< \omega_2$ -directed closed forcing can be generalized to show the same is true for virtually any of the standard forcing axioms (MA $_{\omega_1}$ , PFA, MM, fragments of MM, etc.) and their "+ $\mu$ " versions:

**Theorem 45.** Suppose  $\Gamma$  is closed under restrictions and under 2-step iterations, and let  $\mu$  be any cardinal  $\leq \omega_1$ . Then  $FA^{+\mu}(\Gamma)$  is preserved by  $<\omega_2$ -directed closed forcing.

Proof. Suppose  $V \models \operatorname{FA}^{+\mu}(\Gamma)$ ,  $\mathbb{P}$  is  $< \omega_2$ -directed closed,  $\dot{\mathbb{Q}}$  is a  $\mathbb{P}$ -name for a poset in the class  $\dot{\Gamma}$ , and  $\dot{\vec{S}} = \langle \dot{S}_i : i < \mu \rangle$  is a  $\mu$ -sequence of  $\mathbb{P} * \dot{\mathbb{Q}}$ -names for stationary subsets of  $\omega_1$ . Since  $\Gamma$  is closed under 2-step iterations,  $\mathbb{P} * \dot{\mathbb{Q}} \in \Gamma$ . Let  $\dot{\mathbb{R}}$  be the  $\mathbb{P} * \dot{\mathbb{Q}}$ -name for the trivial forcing. Then clearly  $\mathbb{P} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}}) \in \Gamma$  also. Note that  $\dot{\mathbb{R}}$  trivially preserves the stationarity of each  $\dot{S}_i$ . Thus, clauses 1 and 2 of the hypotheses of Theorem 20 are satisfied.

We now verify that clause 3 of the assumptions of Theorem 20 holds. Suppose  $j:V\to N$  and G\*H(\*K) are as in the hypotheses of clause 3 of Theorem 20. Since  $|H^V_\theta|^N=\omega_1,\,\mathbb{P}\in H^V_\theta,\,G$  is a filter, and  $j\upharpoonright H^V_\theta\in N$ , then  $j[G]\in N$  and N believes that j[G] is a directed subset of  $j(\mathbb{P})$  of size  $<\aleph_2$ . Moreover, by elementarity of j,N believes that  $j(\mathbb{P})$  is  $<\aleph_2$ -directed closed. Hence N believes j[G] has a lower bound. So clause 3 of Theorem 20 is also satisfied. So, Theorem 20 ensures that  $V^\mathbb{P}\models \mathrm{FA}^{+\mu}(\dot{\Gamma})$ .

4.1.2. The Beaudoin-Magidor example. The classic Beaudoin-Magidor theorem that PFA is consistent with a nonreflecting stationary subset of  $S_0^2$  (see [2]) can be re-proved as follows (with  $\Gamma =$  the class of proper posets): assume PFA in V, and let  $\mathbb{P}$  be the forcing to add a nonreflecting stationary subset  $\dot{S}$  of  $S_0^2$  with initial segments. Let  $\dot{\mathbb{Q}}$  be a  $\mathbb{P}$ -name for a proper poset. Let  $\dot{\mathbb{R}}$  be the  $\mathbb{P}*\dot{\mathbb{Q}}$ -name for the poset to kill the stationarity of  $\dot{S}$  using countable conditions. Then  $\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{R}}$  is proper (see [2]). Moreover, if  $j:V\to N$  and  $S*H*K\in N$  are as in the hypotheses of clause 3 of Theorem 20, the presence of the  $\dot{\mathbb{R}}$ -generic club K ensures that S=j[S] is nonstationary in  $\omega_2^V$ , and hence is a condition in  $j(\mathbb{P})$  (and clearly a lower bound for j[S]). So clause 3 of Theorem 20 is satisfied, and so PFA holds in V[S].

The poset  $\mathbb{P}$  in this example is actually a member of the class of posets considered in the next example.

4.1.3. The Yoshinobu example. Yoshinobu's [35] theorem about preservation of PFA by  $\omega_1 + 1$ operationally closed forcings can also be viewed as a consequence of Theorem 20. Suppose PFA holds and  $\mathbb{P}$  is  $\omega_1 + 1$  operationally closed; roughly, this means that in the game  $\mathfrak{G}_{\mathbb{P}}$  of length  $\omega_1 + 1$ where the players create a descending chain of conditions with Player II playing at limit stages (and losing if she cannot play at some limit stage  $\alpha \leq \omega_1$ ), Player II has a winning strategy  $\sigma$  that only depends on the "current position" of the game (i.e. on the boolean infimum of the conditions played so far and the current ordinal stage, but not on the history of the game so far). Let  $\mathbb{Q}$ be a  $\mathbb{P}$ -name for a proper poset. Using properness of  $\mathbb{Q}$  and the fact that  $\sigma$  only depends on the current position, Yoshinobu designs a  $\mathbb{P} * \mathbb{Q}$ -name  $\mathbb{R}$  such that the 3-step iteration is proper, and if G\*H\*K is generic over V for  $\mathbb{P}*\dot{\mathbb{Q}}*\dot{\mathbb{R}}$ , then in V[G\*H\*K] there exists a play  $\mathcal{P}$  of length  $\omega_1$  such that all proper initial segments of  $\mathcal{P}$  are in V, player II used  $\sigma$  at every countable stage of  $\mathcal{P}$ , and the conditions played in  $\mathcal{P}$  generate G. Then, if  $G*H*K\in N$  and  $j:V\to N$  are as in clause 3 of Theorem 20—so in particular  $j \upharpoonright H_{\theta}^{V}$ ,  $j[\mathcal{P}]$ , and j[G] are elements of N—then N believes that  $j[\mathcal{P}]$  is a  $j(\omega_1) = \omega_1$ -length play of the game  $j(\mathfrak{G}_{\mathbb{P}})$  where player II used  $j(\sigma)$  at every countable stage, and that j[G] is generated by the conditions played in  $j[\mathcal{P}]$ . Hence, since  $j(\sigma)$  was used along the way, the conditions played in  $j[\mathcal{P}]$  have a lower bound, which (since j[G]is generated by  $j[\mathcal{P}]$  is also a lower bound for j[G]. Then  $V^{\mathbb{P}} \models PFA$  by Theorem 20.

#### 5. Separation of appproachability properties

The notions of **disjoint club** and **disjoint stationary** sequences on  $\omega_2$  were introduced in Friedman-Krueger [14] and Krueger [22], respectively. We say that  $\omega_2$  carries a disjoint club (resp. disjoint stationary) sequence iff there is a stationary  $S \subset S_1^2$  and a sequence  $\langle x_{\gamma} : \gamma \in S \rangle$  such that every  $x_{\gamma}$  is a club (resp. stationary) subset of  $[\gamma]^{\omega}$ , and  $x_{\gamma} \cap x_{\gamma'} = \emptyset$  whenever  $\gamma \neq \gamma'$  are both in S. Krueger proved:

**Theorem 46** (Theorems 6.3 and 6.5 of Krueger [22]). Assume  $2^{\omega_1} = \omega_2$ .

- The existence of a disjoint club sequence on  $\omega_2$  is equivalent to the assertion that  $IU \neq IS$  in  $\wp_{\omega_2}(H_{\omega_2})$ ; i.e. there are stationarily many  $W \in \wp_{\omega_2}^*(H_{\omega_2})$  that are internally unbounded, but not internally stationary.
- The existence of a disjoint stationary sequence on  $\omega_2$  is equivalent to the assertion that  $IU \neq IC$  in  $\wp_{\omega_2}(H_{\omega_2})$ ; i.e. there are stationarily many  $W \in \wp_{\omega_2}^*(H_{\omega_2})$  that are internally unbounded, but not internally club.

5.1. **Proof of Theorem 5.** In this section we show that PFA does not imply the existence of a disjoint stationary sequence on  $\omega_2$  (and thereby show that Krueger's Theorem 2 is sharp). By Theorem 46, it suffices to find a model of PFA that also satisfies

$$IC \cap \wp_{\omega_2}(H_{\omega_2}) =^* IU \cap \wp_{\omega_2}(H_{\omega_2}).$$

Assume V is a model of PFA; then  $2^{\omega_1} = \omega_2$  so we can fix a bijection  $\Phi : \omega_2 \to H_{\omega_2}$ . Let IC\* be the set of  $\gamma \in S_1^2$  such that  $\Phi[\gamma] \cap \omega_2 = \gamma$  and  $\Phi[\gamma]$  is an IC model. It is routine to check that all but nonstationarily many  $W \in IC \cap \wp_{\omega_2}(H_{\omega_2})$  are of the form  $\Phi[W \cap \omega_2]$  where  $W \cap \omega_2 \in IC^*$ .

Let  $\mathbb{C}(\mathrm{IC}^*)$  be the poset of closed, bounded  $c \subset \omega_2$  such that  $c \cap S_1^2 \subset \mathrm{IC}^*$ , ordered by end-extension. By an argument similar to Proposition 4.4 of Krueger [20],  $\mathbb{C}(\mathrm{IC}^*)$  is  $<\omega_2$  distributive. In particular  $H_{\omega_2}$  is unchanged. Then the club added by  $\mathbb{C}(\mathrm{IC}^*)$  witnesses that, in  $V^{\mathbb{C}(\mathrm{IC}^*)}$ , almost every  $\gamma \in S_1^2$  has the property that  $\Phi[\gamma] \in \mathrm{IC}$ , and hence that almost every element of  $\mathrm{IU} \cap \wp_{\omega_2}(H_{\omega_2})$  is internally club.

It remains to show that  $V^{\mathbb{C}(\mathrm{IC}^*)} \models \mathrm{PFA}$ . Let  $\dot{\mathbb{Q}}$  be a  $\mathbb{C}(\mathrm{IC}^*)$ -name for a proper poset. The poset  $\mathbb{C}(\mathrm{IC}^*)$  is  $\sigma$ -closed, and hence  $\mathbb{C}(\mathrm{IC}^*) * \dot{\mathbb{Q}}$  is proper. In particular,  $V \cap [\omega_2^V]^\omega$  remains stationary. Let  $\dot{\mathbb{R}}$  be the  $\mathbb{C}(\mathrm{IC}^*) * \dot{\mathbb{Q}}$ -name for the poset from Definition 32 that shoots a continuous  $\omega_1$ -chain through  $V \cap [\omega_2^V]^\omega$ . By Lemma 34, the poset  $\mathbb{C}(\mathrm{IC}^*) * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$  is proper.

Now suppose  $j: V \to N$  is a generic elementary embedding with the properties listed in clause 3 of Theorem 20 (with  $\mathbb{C}(\mathrm{IC}^*)$  playing the role of the  $\mathbb{P}$  from that theorem). More precisely, assume  $\mathrm{crit}(j) = \omega_2^V, j \upharpoonright H_\theta^V \in N, H_\theta^V$  is in the wellfounded part of N, and there is a  $C*H*K \in N$  that is V-generic for  $\mathbb{C}(\mathrm{IC}^*) * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ . The presence of K ensures that, in  $H_\theta^V[C*H*K], H_{\omega_2}^V \in \mathrm{IC}$ . By Lemma 21 this is upward absolute to N. Furthermore,  $j(\Phi)[\omega_2^V] = \mathrm{range}(\Phi) = H_{\omega_2}^V$ . So  $N \models \omega_2^V \in j(\mathrm{IC}^*)$ , and hence

(7) 
$$N \models C \cup \{\omega_2^V\} \in j(\mathbb{C}(\mathrm{IC}^*)).$$

and is clearly stronger than j[C] = C. By Theorem 20,  $V[C] \models PFA$ , completing the proof of Theorem 5.

5.2. **Proof of Theorem 6.** In this section we prove that PFA<sup>+ $\omega_1$ </sup> does not imply the existence of a disjoint club sequence on  $\omega_2$ , thereby showing that Krueger's Theorem 3 is sharp. By Theorem 46, it suffices to construct a model of PFA<sup>+ $\omega_1$ </sup> such that IS =\* IU in  $\wp_{\omega_2}(H_{\omega_2})$ .

Assume V is a model of PFA<sup> $+\omega_1$ </sup>. Fix a bijection  $\Phi: \omega_2 \to H_{\omega_2}$  and let IS\* be the set of  $\gamma \in S_1^2$  such that  $\Phi[\gamma] \in IS$ . Then almost every  $W \in IS \cap \wp_{\omega_2}(H_{\omega_2})$  is of the form  $\Phi[W \cap \omega_2]$  where  $W \cap \omega_2 \in IS^*$ .

Let  $\mathbb{C}(\mathrm{IS}^*)$  be the poset of closed, bounded  $c \subset \omega_2$  such that  $c \cap S_1^2 \subset \mathrm{IS}^*$ , ordered by end-extension. As in Section 5.1, this poset is  $< \omega_2$  distributive, and in particular  $H_{\omega_2}$  and  $\mathrm{IS} \cap H_{\omega_2}$  are computed the same in V and  $V^{\mathbb{C}(\mathrm{IS}^*)}$ . It is also routine to check that

(8) 
$$V^{\mathbb{C}(\mathrm{IS}^*)} \models \mathrm{IS} =^* \mathrm{IU in } \wp_{\omega_2}(H_{\omega_2}).$$

<sup>&</sup>lt;sup>12</sup>Briefly: if  $W \prec (H_{\omega_3}, \in, \Phi)$  and  $W \in IA$ , then  $W \cap H_{\omega_2} = \Phi[W \cap \omega_2]$ ,  $W \cap \omega_2 \in IC^*$  (since  $IA \subseteq IC$ ), and any condition in  $W \cap \mathbb{C}(IC^*)$  can be extended to a W-generic tower with supremum  $W \cap \omega_2$  (this argument uses internal approachability of W). Then since  $W \cap \omega_2 \in IC^*$ , that generic tower has a lower bound, obtained by placing  $W \cap \omega_2$  at the top of the tower. In summary, every condition in W can be extended to a condition whose upward closure is a  $(W, \mathbb{C}(IC^*)$ -generic filter. Since the set of such W is stationary, standard arguments then imply that  $\mathbb{C}(IC^*)$  is  $< \omega_2$  distributive.

<sup>&</sup>lt;sup>13</sup>For this application we could just as well have used a version of that poset using countable conditions, but we choose to stick with Definition 32 since it works just as well.

It remains to show that  $V^{\mathbb{C}(\mathrm{IS}^*)}$  is a model of PFA<sup> $+\omega_1$ </sup>. Let  $\dot{\mathbb{Q}}$  be a  $\mathbb{C}(\mathrm{IS}^*)$ -name for a proper poset, let  $\theta$  be a large regular cardinal with  $\mathbb{C}(\mathrm{IS}^*) * \dot{\mathbb{Q}} \in H_{\theta}$ .

By Theorem 20,<sup>14</sup> in order to show that PFA<sup>+ $\omega_1$ </sup> holds in  $V^{\mathbb{C}(\mathrm{IS}^*)}$  it suffices to prove that if  $j:V\to N$  is a generic elementary embedding with critical point  $\omega_2^V$  such that  $j\upharpoonright H_\theta^V\in N$ ,  $H_\theta^V$  is in the wellfounded part of N and has size  $\omega_1$  in N, and N has an  $\omega_1$ -stationary correct V-generic filter C\*H for  $\mathbb{C}(\mathrm{IS}^*)*\dot{\mathbb{Q}}$ , then N believes that j[C]=C has a lower bound in  $j(\mathbb{C}(\mathrm{IS}^*))$ . Now since  $j(\Phi)[\omega_2^V]=\mathrm{range}(\Phi)=H_{\omega_2}^V$ , it suffices to show that  $H_{\omega_2}^V\in\mathrm{IS}^N$  (since then  $C\cup\{\omega_2^V\}$  will be a condition in  $j(\mathbb{C}(\mathrm{IS}^*))$  below C=j[C]). We can without loss of generality assume that  $\dot{\mathbb{Q}}$  collapses  $\omega_2^V$ . Since  $\mathbb{C}(\mathrm{IS}^*)*\dot{\mathbb{Q}}$  is proper, it forces that  $V\cap[\omega_2^V]^\omega$  is stationary; and since it collapses  $\omega_2$ , it follows that  $V[C*H]\models H_{\omega_2}^V\in\mathrm{IS}$ . Let  $\dot{T}$  be the  $\mathbb{C}(\mathrm{IS}^*)*\dot{\mathbb{Q}}$ -name for the internal part of  $H_{\omega_2}^V$ ; then  $\dot{T}$  is forced to be a stationary subset of  $\omega_1$ . Since C\*H is an  $\omega_1$ -stationary correct filter,  $\dot{T}_{C*H}$  is stationary in N. Hence  $H_{\omega_2}^V$  is internally stationary from the point of view of N.

5.3. **Proof of Theorem 7.** We need to prove that the global separation of IC from IS: (1) follows from MM; (2) follows from PFA<sup>+</sup>; and (3) does not follow from PFA. That PFA does not imply separation of IC from IS already follows from our Theorem 5; recall in Section 5.1 we obtained a model of PFA where IC =\* IU in  $\wp_{\omega_2}(H_{\omega_2})$ . So it remains to show either MM or PFA<sup>+</sup> implies global separation of IS from IC.

We start with the MM proof. We prove a little more, namely:

**Theorem 47.** MM implies that  $G_{\omega_1} \cap IS \setminus IC$  is stationary for every regular  $\theta \geq \omega_2$ .

Theorem 47 appeared as Theorem 4.4 part (3) of Viale [32]; however the proof given there is incorrect (the proof given there appears to be implicitly using the stronger assumption MM<sup>+2</sup>, analogous to Krueger's proof in [20], and works fine under that stronger assumption.) We briefly summarize the argument from [32], and where the error occurs. A certain poset  $\mathbb{P}_2 * \mathbb{Q}_{\mathbb{P}_2}$  is defined that forces  $H_{\theta}^V$  to be a guessing model in IS \ IC. The third bullet at the very end of that proof (near the end of Section 4) claims that if  $W \prec H_{\theta}$  is such that  $|W| = \omega_1 \subset W \prec H_{(2^{\theta})^+}$  and there exists a  $(W, \mathbb{P}_2 * \mathbb{Q}_{\mathbb{P}_2})$ -generic filter, then W is guessing, and  $W \in IS \setminus IC$ . The guessing part is correct, but such a W is not necessarily in IS \ IC. To see why, let  $\mathbb{C} = \mathbb{C}_{\text{dec}}^{\text{fin}}(V \cap [\theta]^{\omega})$  be the  $\mathbb{P}_2 * \mathbb{Q}_{\mathbb{P}_2}$ -name for the poset to shoot a continuous  $\omega_1$ -chain through  $V \cap [\theta]^{\omega}$  as in Definition 32. Then  $\mathbb{P}_2 * \mathbb{Q}_{\mathbb{P}_2} * \mathbb{C}$  is proper, and forces  $H_{\theta}^V \in IC$ . Hence under MM (or just PFA) there are stationarily many W as above such that there exists a  $(W, \mathbb{P}_2 * \mathbb{Q}_{\mathbb{P}_2} * \mathbb{C})$ -generic filter; say g \* h \* c. So in particular (by projecting to the first 2 coordinates) there exists a W-generic for  $\mathbb{P}_2 * \mathbb{Q}_{\mathbb{P}_2}$ . But the 3rd coordinate c ensures that  $W \cap H_{\theta}$  is an element of IC, and hence not an element of IS \ IC.

(One could give a different counterexample under MM, by instead letting  $\mathbb{C}$  name the poset to shoot a continuous  $\omega_1$  chain through  $[\theta]^{\omega} \setminus V$ ; in that context one would be able to find W for which a  $\mathbb{P}_2 * \dot{\mathbb{Q}}_{\mathbb{P}_2}$ -generic exists, yet  $W \notin \mathrm{IS}$ ).

We now give a proof of Theorem 47. Most of the work was already done in Section 3 above.

*Proof.* (of Theorem 47): Assume MM, and fix a  $T \subset \omega_1$  that is stationary and costationary. Let  $\theta \geq \omega_2$  be regular, and let  $\mathbb{Q}_{T,\theta}^{\omega_1\text{-SSP}}$  be the poset given by Theorem 37. By Woodin's Lemma 41,

<sup>&</sup>lt;sup>14</sup>Viewing  $\mathbb{C}(IS^*)$  as the  $\mathbb{P}$  from that theorem, and taking  $\dot{\mathbb{R}}$  to be the  $\mathbb{C}(IS^*)*\dot{\mathbb{Q}}$ -name for the trivial poset (which is trivially forced by  $\mathbb{C}(IS^*)*\dot{\mathbb{Q}}$  to be  $\omega_1$ -SSP).

<sup>&</sup>lt;sup>15</sup>And is still  $\omega_1$ -guessing, since  $\mathbb{P}_2 * \dot{\mathbb{Q}}_{\mathbb{P}_2}$  forces  $H_{\theta}^V$  to be indestructibly guessing.

there are stationarily many  $W \prec H_{(2^{\theta})^+}$  for which there exists a  $(W, \mathbb{Q}_{T,\theta}^{\omega_1\text{-SSP}})$ -generic filter. By Corollary 38,  $W \cap H_{\theta}$  is an (indestructible) guessing model, and the internal part of W is exactly T. In particular,  $W \in \text{IS} \setminus \text{IC}$ , since T was stationary and costationary.

Finally, we show that the conclusion of Krueger's Theorem 4 follows from PFA<sup>+</sup>. Again we prove something a little stronger, namely:

**Theorem 48.** PFA<sup>+1</sup> implies that  $G_{\omega_1} \cap IS \setminus IC$  is stationary for every regular  $\theta \geq \omega_2$ .

Proof. Again, most of the work was done in Section 3 above. Assume PFA<sup>+</sup>, fix a regular  $\theta \geq \omega_2$ , and fix any stationary, costationary  $T \subset \omega_1$ . Let  $\mathbb{Q}_{T,\theta}^{\text{proper}}$  be the poset given by Theorem 37, and let  $\dot{S}_{\text{external}}$  be the  $\mathbb{Q}_{T,\theta}^{\text{proper}}$ -name for the external part of  $H_{\theta}^V$ . By Woodin's Lemma 41, there are stationarily many  $W \prec H_{(2^{\theta})^+}$  such that  $|W| = \omega_1 \subset W$  and there exists a  $(W, \mathbb{Q}_{T,\theta}^{\text{proper}})$ -generic filter g that interprets  $\dot{S}_{\text{external}}$  as a stationary subset of  $\omega_1$ . By Corollary 38,  $W \cap H_{\theta}$  is indestructibly  $\omega_1$ -guessing, its internal part contains the stationary set T (hence  $W \cap H_{\theta} \in IS$ ) and its external part is a stationary subset of  $T^c$  (hence  $W \cap H_{\theta} \notin IC$ ).

# 6. Separation of stationary set reflection

In this section we prove the various separations of stationary reflection from the introduction. The proofs of Theorems 10, 13, and 18 are similar, so we group those together. Most of the work for these theorems was done already in Section 3. The proofs of Theorems 11 and 14 are also similar, so those are grouped together; the main tool for those is Theorem 20. In the final subsection we prove Theorem 16.

# 6.1. Forcing axiom implications. Here we prove Theorems 10, 13, and 18.

6.1.1. Proof of Theorem 10. Assume  $\operatorname{PFA}_{H_{\omega_2}^V \notin \operatorname{IC}}^{+\omega_1}$ , and fix a regular  $\theta \geq \omega_2$ . Fix a costationary subset T of  $\omega_1$  ( $T = \emptyset$  will work). Let  $\mathbb{Q}_{\theta,T}^{\operatorname{proper}}$  be the poset from Theorem 37; that theorem (and costationarity of T) ensures that  $\mathbb{Q}_{\theta,T}^{\operatorname{proper}}$  is proper and forces  $H_{\omega_2}^V \in \operatorname{IS} \setminus \operatorname{IC}$ , so in particular is in the class of posets to which  $\operatorname{PFA}_{H_{\omega_2}^V \notin \operatorname{IC}}^{+\omega_1}$  applies. By Woodin's Lemma 41, there are stationarily many  $W \prec (H_{(2^{\theta})^+}, \in, \dots)$  such that  $|W| = \omega_1 \subset W$  and there exists a  $(W, \mathbb{Q}_{\theta,T}^{\operatorname{proper}})$ -generic filter g that is correct with respect to stationary subsets of  $\omega_1$  (that have names in W). By Corollary 38,  $W \cap H_{\theta}$  is a guessing set, and is in  $\operatorname{IS} \setminus \operatorname{IC}$ .

It remains to show that  $W \cap H_{\theta}$  is diagonally, internally reflecting, so let  $R \in W$  be a stationary subset of  $[H_{\theta}]^{\omega}$ . We need to prove that  $R \cap W \cap [W \cap H_{\theta}]^{\omega}$  is stationary in  $[W \cap H_{\theta}]^{\omega}$ . Let  $\langle \dot{Q}_i : i < \omega_1 \rangle$  be the name for the generic filtration of  $H_{\theta}^V$ , and  $\dot{S}_R$  be the name for

$$\{i<\omega_1\ :\ \dot{Q}_i\in \check{R}\}.$$

Since  $\mathbb{Q}_{\theta,T}^{\text{proper}}$  is proper, then R remains stationary, and it follows that  $\dot{S}_R$  names a stationary subset of  $\omega_1$ . Also notice that since  $R \in V$ , then in particular  $R \subset V$  and so

(9) 
$$\Vdash \forall i \in \dot{S}_R \ \dot{Q}_i$$
 is an element of the ground model.

Now the name  $\dot{S}_R$  can be taken to be an element of W, so  $S := (\dot{S}_R)_g$  is really stationary in V. Now  $\langle (\dot{Q}_i)_g : i < \omega_1 \rangle$  is a filtration of  $W \cap H_\theta$ . Then for every  $i \in S$ ,  $(\dot{Q}_i)_g$  is an element of R, and also (by (9)) an element of W. This completes the proof of Theorem 10.

- 6.1.2. Proof of Theorem 13. This is very similar to the proof of Theorem 10, so we only briefly sketch it. Assume  $\operatorname{PFA}_{H_{\omega_2}^V \notin \operatorname{IA}}^{+\omega_1}$ . Let  $\mathbb{Q}_{\theta,\omega_1}^{\operatorname{proper}}$  be the poset from Theorem 37, with  $T=\omega_1$ . By that theorem,  $H_{\omega_2}^V$  is forced to be indestructibly guessing, and have internal part containing T. Guessing models are never internally approachable, so in particular the poset forces  $H_{\omega_2}^V \notin \operatorname{IA}$ , and hence the forcing axiom  $\operatorname{PFA}_{H_{\omega_2}^V \notin \operatorname{IA}}^{+\omega_1}$  applies to it. So by Woodin's Lemma 41 there are stationarily many  $W \prec (H_{(2^\theta)^+}, \in)$  for which there exists a g that is W-generic and stationary correct, and Corollary 38 ensures that such W are indestructibly guessing and have internal part containing  $T = \omega_1$ , hence internally club. The proof that W is internally, diagonally reflecting is identical to the corresponding proof in Section 6.1.1.
- 6.1.3. Proof of Theorem 18. Assume  $\mathrm{MM}_{H^{V}_{\omega_{2}}\notin\mathrm{IS}}^{+\omega_{1}}$ , and consider the poset  $\mathbb{Q}_{\emptyset,\theta}^{\omega_{1}\text{-SSP}}$  from Theorem 37 (taking the T from that theorem to be the empty set); that theorem yields that  $H^{V}_{\omega_{2}}$  is forced to be indestructibly guessing and have internal part exactly  $T=\emptyset$ , and hence to not be in IS. So  $\mathbb{Q}_{\emptyset,\theta}^{\omega_{1}\text{-SSP}}$  is a poset to which  $\mathrm{MM}_{H^{V}_{\omega_{2}}\notin\mathrm{IS}}^{+\omega_{1}}$  is applicable. By Woodin's Lemma 41 there are stationarily many  $W\in \wp_{\omega_{2}}^{*}(H_{(2^{\theta})^{+}})$  such that there is some  $(W,\mathbb{Q}_{\emptyset,\theta}^{\omega_{1}\text{-SSP}})$ -generic filter that is stationary correct (about names for stationary subsets of  $\omega_{1}$  lying in W). By Corollary 38, it follows that  $R\cap\sup(W\cap\theta)$  is stationary in  $\sup(W\cap\theta)$  whenever  $R\in W$  and R is stationary in  $\theta\cap\operatorname{cof}(\omega)$ .
- 6.2. **Preservation results.** Here we prove Theorems 11 and 14.
- 6.2.1. Proof of Theorem 11. Assume V is a model of PFA $_{H_{\omega_2}^V \notin IC}^{+\omega_1}$ . By Theorem 10, DRP<sub>internal, GIS</sub> holds, so in particular WRP holds. Then by Foreman-Magidor-Shelah [12],  $2^{\omega_1} = \omega_2$ . Fix a bijection  $\Phi: \omega_2 \to H_{\omega_2}$  and define

$$\mathrm{IC}^* := \{ \gamma \in S_1^2 \ : \ \Phi[\gamma] \in \mathrm{IC} \}.$$

It is routine to see that for all but nonstationarily many  $W \in IC \cap \wp_{\omega_2}(H_{\omega_2})$ , W is of the form  $\Phi[W \cap \omega_2]$ .

As in Krueger [20], let  $\mathbb{P}_{nrIC}$  be the following partial order (nrIC stands for "nonreflecting in IC"): conditions are functions  $f: \alpha \cap cof(\omega) \to 2$  for some  $\alpha < \omega_2$  such that for every  $\gamma \in IC^*$ , the set

$$\{\xi < \gamma : f(\xi) = 1\}$$

is nonstationary in  $\gamma$ . The ordering is by function extension. Then:

(1)  $\mathbb{P}_{nrIC}$  is  $\sigma$ -closed and  $<\omega_2$  distributive (Lemma 4.1 of [20]), so in particular

$$H^V_{\omega_2} = H^{V^{\mathbb{P}_{\mathrm{nrIC}}}}_{\omega_2}$$

and

$$\Gamma := \left( \mathrm{IC} \cap \wp_{\omega_2}(H_{\omega_2}) \right)^V$$
 is equal to  $\left( \mathrm{IC} \cap \wp_{\omega_2}(H_{\omega_2}) \right)^{V^{\mathbb{P}_{\mathrm{nrIC}}}}$ .

- (2) If G is  $(V, \mathbb{P}_{nrIC})$ -generic, then by identifying G with  $\{\xi < \omega_1 : G(\xi) = 1\}$ , in V[G] we have:
  - G is a stationary subset of  $S_0^2$  (Lemma 4.2 of [20]);
  - $G \cap \gamma$  is nonstationary for all  $\gamma \in IC^*$ ;

<sup>&</sup>lt;sup>16</sup>One could probably modify the arguments of Velickovic and Todorcevic to prove  $2^{\omega_1} = \omega_2$  already follows from the "non-plus version"  $PFA_{H^V_{\omega_2} \notin IC}$ . We did not check this, however.

• RP<sub>IC</sub> fails in V[G]. We briefly sketch the argument. Set

$$\widetilde{G} := \{ z \in [\omega_2]^\omega : \sup(z) \in G \}.$$

Then  $\widetilde{G}$  is a stationary subset of  $[\omega_2]^{\omega}$ . If  $W \in \Gamma$  (as defined above) and

$$\operatorname{Sk}^{(H_{\omega_3}[G],\in,\Phi,G,\Delta)}(W)\cap H_{\omega_2}=W$$

where  $\Delta$  is some wellorder of  $H^{V[G]}_{\omega_3}$ , then since  $G \cap (W \cap \omega_2)$  is nonstationary and  $\mathrm{cf}(W \cap \omega_2) = \omega_1$ , it is straightforward to see that  $\widetilde{G} \cap [W]^{\omega}$  is nonstationary. This shows that  $\widetilde{G}$  can reflect to at most nonstationarily many members of  $\Gamma$ . But if  $\mathrm{RP}_{\mathrm{IC}}$  held, then any stationary subset of  $[\omega_2]^{\omega}$  would reflect to stationarily many members of  $\Gamma$  (see Foreman-Todorcevic [13], Lemma 8).

We use Theorem 20 to prove that  $\mathbb{P}_{\text{nrIC}}$  preserves  $\text{PFA}_{H_{\omega_2}^{V} \notin \text{IC}}^{+\omega_1}$ . Let  $\dot{\mathbb{Q}}$  be a  $\mathbb{P}_{\text{nrIC}}$ -name for a proper poset that forces "the ground model's  $H_{\omega_2}$  is not in IC"; i.e. that  $H_{\omega_2}^{V^{\mathbb{P}}_{\text{nrIC}}} \notin \text{IC}$ . Since  $H_{\omega_2}^{V} = H_{\omega_2}^{V^{\mathbb{P}}_{\text{nrIC}}}$ ,  $\mathbb{P}_{\text{nrIC}} * \dot{\mathbb{Q}}$  forces  $H_{\omega_2}^{V} \notin \text{IC}$ . Let  $\dot{\mathbb{R}}$  be the  $\mathbb{P}_{\text{nrIC}} * \dot{\mathbb{Q}}$  name for the trivial poset, which is trivially forced to be  $\omega_1$ -SSP. Then  $\mathbb{P}_{\text{nrIC}} * \dot{\mathbb{Q}}(*\dot{\mathbb{R}})$  is a poset to which the axiom  $\text{PFA}_{H_{\omega_2}^{V} \notin \text{IC}}^{+\omega_1}$  applies. This verifies clause 1 of Theorem 20.

We now verify clause 3 of Theorem 20. Suppose  $j:V\to N$  is a generic elementary embedding satisfying the assumptions of that clause, where in particular  $G*H(*K)\in N$  is generic over V for  $\mathbb{P}_{\text{nrIC}}*\dot{\mathbb{Q}}(*\dot{\mathbb{R}})$  and correct with respect to stationary subsets of  $\omega_1$ . In particular,  $H^V_{\theta}[G*H]$  is correct (from N's point of view) about the fact that the external part of  $H^V_{\omega_2}$  is stationary (recall from above that  $\mathbb{P}_{\text{nrIC}}*\dot{\mathbb{Q}}$  forces  $H^V_{\omega_2}\notin \text{IC}$ ). In other words,  $H^V_{\omega_2}\notin \text{IC}^N$ . Now  $j(\Phi)[\omega_2^V]=\text{range}(\Phi)=H^V_{\omega_2}$ , so  $\omega_2^V\notin j(\text{IC}^*)$ . Hence j[G]=G is a condition in  $j(\mathbb{P}_{\text{nrIC}})$ , since by the definition of the forcing  $\mathbb{P}_{\text{nrIC}}$ , there are no requirements whatsoever at points which are not in IC\*. This verifies clause 3 of Theorem 20, and hence by that theorem,  $\text{PFA}^{+\omega_1}_{H^V_{\omega_2}\notin \text{IC}}$  holds in  $V^{\mathbb{P}_{\text{nrIC}}}$ .

Finally, suppose in addition that PFA held in the ground model. The poset  $\mathbb{P}_{nrIC}$  is  $\omega_1+1$ -tactically closed, as in Yoshinobu [35]; the argument is virtually identical to the proof of Example 1 on page 751 of [35]. Hence,  $\mathbb{P}_{nrIC}$  preserves PFA by the main theorem of [35]. Alternatively, one could use Theorem 20 to run basically the same argument as the Beaudoin-Magidor example in Section 4.1.2 above.

6.2.2. Proof of Theorem 14. This proof is very similar to the proof of Theorem 11 above, so we only briefly sketch it. Assume that the ground model V satisfies  $PFA_{H_{\omega_2}^{V} \notin IA}^{+\omega_1}$ , and define  $IA^*$  and  $\mathbb{P}_{nrIA}$  similarly to the way that  $IC^*$  and  $\mathbb{P}_{nrIC}$  were defined in Section 6.2.1, with IA playing the role here that IC played there. Then  $\mathbb{P}_{nrIA}$  has the same properties that  $\mathbb{P}_{nrIC}$  had, with the obvious modifications. In particular, if G is generic for  $\mathbb{P}_{nrIA}$  then  $RP_{IA}(\omega_2)$  fails in V[G].

Again, use Theorem 20 to show that  $\mathbb{P}_{\text{nrIA}}$  preserves  $\text{PFA}_{H_{\omega_2}^V \notin \text{IA}}^{+\omega_1}$ . This is basically the same argument as above, except in the current situation, the reason that  $H_{\omega_2}^V \notin \text{IA}^N$  is because it is indestructibly guessing, and guessing models are never internally approachable. This ensures, just as in the proof above, that j[G] = G is a condition in  $j(\mathbb{P}_{\text{nrIA}})$ . The rest is identical to the proof above, so we omit the argument.

6.2.3. Proof of Theorem 16. Assume that V satisfies  $\mathrm{MM}_{H_{\omega_2}^{V} \notin \mathrm{IS}}^{+\omega_1}$ . Let  $\mathbb{P} = \mathbb{P}([\omega_2]^{\omega})$  be the poset from Section 1 of Aspero-Krueger-Yoshinobu [1]. Conditions are  $\omega_1$ -sized subsets p of  $[\omega_2]^{\omega}$  such

that  $p \cap [W]^{\omega}$  is nonstationary in  $[W]^{\omega}$  whenever  $W \in [\omega_2]^{\omega_1}$ . A condition q is stronger than p if  $q \supseteq p$  and for every  $y \in q \setminus p$ , y is not a subset of  $\bigcup p$ . Section 1 of [1] proves that:

- $\mathbb{P}$  is  $<\omega_2$  strategically closed; in particular it is  $\omega_1$ -SSP, and  $H_{\omega_2}^V = H_{\omega_2}^{V^{\mathbb{P}}}$ ;
- $\mathbb{P}$  adds a stationary subset of  $[\omega_2]^{\omega}$  that does not reflect to any set of size  $\omega_1$ ; hence  $RP(\omega_2)$  fails in the extension.

It remains to verify that  $\mathbb P$  preserves  $\mathrm{MM}_{H_{\omega_2}^V \notin \mathrm{IS}}^{+\omega_1}$ ; again we use Theorem 20. Let  $\dot{\mathbb Q}$  be a  $\mathbb P$ -name for an  $\omega_1$ -SSP poset that forces the ground model is not in IS, and let  $\dot{\mathbb R}$  be the  $\mathbb P * \dot{\mathbb Q}$  name for the trivial forcing, which is of course  $\omega_1$ -SSP. Since  $H_{\omega_2}^V = H_{\omega_2}^{V^{\mathbb P}}$ ,  $\mathbb P * \dot{\mathbb Q}(*\dot{\mathbb R})$  forces  $H_{\omega_2}^V \notin \mathrm{IS}$  and is  $\omega_1$ -SSP, so clause 1 of Theorem 20 is satisfied. Now suppose  $j:V\to N$  is as in the hypotheses of clause 3 of Theorem 20 (using the notation from that clause). Then  $V\cap [\omega_2^V]^\omega$  is nonstationary in  $H_{\theta}^V[G*H]$ , and so G, being the generic subset of  $V\cap [\omega_2^V]^\omega$  added by  $\mathbb P$ , is also nonstationary in  $H_{\theta}^V[G*H]$ . Hence j[G]=G is nonstationary in  $[\omega_2^V]^\omega$  from the point of view of N. Moreover, since G is nonreflecting in V[G], then  $G\cap [\gamma]^\omega$  is nonstationary for every  $\gamma<\omega_2^V$ , and this is upward absolute from  $H_{\theta}^V[G]$  to N. So j[G]=G is a condition in  $j(\mathbb P)$ . To see that it is stronger than every condition in j[G], consider some  $p\in G$ . Then  $j[G]=G\supset p=j(p)$ , and if  $y\in G\setminus p$  then there is some  $q\in G$  with  $y\in q$ , and hence  $y\in q\setminus p$ . Without loss of generality we can assume q is stronger than p, and so p0 is not a subset of p0. Without loss of clause 3 of Theorem 20 is satisfied, and so p0 satisfies p1.

# 7. Closing Remarks

As mentioned in the introduction, we do not know if the implication

$$RP_{internal} \implies RP_{IS}$$

from the Fuchino-Usuba Theorem 9, or the implication

$$RP_{IS} \implies RP_{III}$$

from (2), can be reversed. These questions are very similar, since separating either implication would require some stationary sets to reflect *only* to the external parts of members of  $\wp_{\omega_2}^*(V)$ . We suspect a solution to one of them would likely solve the other, and conjecture that neither can be reversed. The following theorem lends some support to this conjecture:

**Theorem 49.** Martin's Maximum implies that there is a stationary subset S of  $[H_{\omega_2}]^{\omega}$  such that for stationarily many  $W \in IS \cap \wp_{\omega_2}^*(H_{\omega_2})$ : S reflects to W, but not internally.<sup>17</sup>

Proof. Fix a stationary, costationary subset T of  $\omega_1$ , and set  $S_{T^c} := [H_{\omega_2}]^{\omega} \setminus T^c$ . Let  $\mathbb{Q}_T := \mathbb{Q}_{T,\omega_2}^{\omega_1\text{-SSP}}$  be the  $\omega_1\text{-SSP}$  poset from Theorem 37. By Woodin's Lemma 41, there are stationarily many  $W \in \wp_{\omega_2}^*(H_{\omega_3})$  for which there exists a  $(W, \mathbb{Q}_T)$ -generic filter. Let R be this stationary set. By Corollary 38, if  $W \in R$  then  $\bar{W} := W \cap H_{\omega_2}$  has internal part exactly T, which implies that  $\bar{W} \in \mathbb{R}$  and that  $S_{T^c} \cap \bar{W} \cap [\bar{W}]^{\omega}$  is nonstationary. On the other hand, stationarity of  $T^c$  ensures that  $S_{T^c} \cap [\bar{W}]^{\omega} = ([H_{\omega_2}]^{\omega} \setminus T^c) \cap [\bar{W}]^{\omega}$  is stationary, since in fact  $S_{T^c} \cap [Z]^{\omega}$  is stationary for any set  $Z \subseteq H_{\omega_2}$  such that  $\omega_1 \subset Z$ . Then  $R_1 := \{W \cap H_{\omega_2} : W \in R\}$  is as required.

<sup>&</sup>lt;sup>17</sup>I.e.  $W \in IS$  and  $S \cap [W]^{\omega}$  is stationary, but  $S \cap W \cap [W]^{\omega}$  is nonstationary.

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