# DEGREE SPECTRA OF ANALYTIC COMPLETE EQUIVALENCE RELATIONS 

DINO ROSSEGGER


#### Abstract

We study the bi-embeddability and elementary bi-embeddability relation on graphs under Borel reducibility and investigate the degree spectra realized by these relations. We first give a Borel reduction from embeddability on graphs to elementary embeddability on graphs. As a consequence we obtain that elementary bi-embeddability on graphs is a $\boldsymbol{\Sigma}_{1}^{1}$ complete equivalence relation. We then investigate the algorithmic properties of this reduction. We obtain that elementary bi-embeddability on the class of computable graphs is $\Sigma_{1}^{1}$ complete with respect to computable reducibility and show that the elementary bi-embeddability and bi-embeddability spectra realized by graphs are related.


## 1. Introduction

Equivalence relations on countable structures are among the most heavily studied objects in descriptive set theory and computability theory. In descriptive set theory, starting with the work of Friedman and Stanley [10], the complexity of equivalence relations on spaces of structures under Borel reducibility has seen much interest by experts, culminating in results by Louveau and Rosendal [16], who showed that, among others, the bi-embeddability relation on graphs is $\boldsymbol{\Sigma}_{1}^{1}$-complete. Since then there has been a constant stream of work on the complexity of the bi-embeddability relation, both on other classes of structures, see for instance [4], and refinements of completeness notions, e.g. in [5].

Equivalence relations are also one of the main objects of study in computability theory. Here, the equivalence relations are usually on the set of natural numbers and their complexity is established using computable reducibility. Identifying a computable structure with the index of the algorithm computing it, one can obtain completeness results like the ones in descriptive set theory for equivalence relations on computable structures [6, 7]. One object of study in computable structure theory which also takes non-computable structures into account are degree spectra of structures, introduced by Knight [15]. The degree spectrum of a given structure is the set of sets of natural numbers Turing equivalent to one of its isomorphic copies. They provide a measure of the algorithmic complexity of countable structures.

[^0]Recently, researchers initiated the study of degree spectra with respect to other model theoretic equivalence relations such as bi-embeddability 8], elementary biembeddability [20], elementary equivalence [3, 1, 2], or $\Sigma_{n}$ equivalence [9]. One of the main goals in this line of research is to distinguish these equivalence relations with respect to the degree spectra they realize. While for elementary equivalence and $\Sigma_{n}$ equivalence examples that separate them from each other and from isomorphism and elementary bi-embeddability are known, so far all attempts to separate isomorphism, bi-embeddability and elementary bi-embeddability have been unsuccessful.

There seem to be various reasons for this. That we can separate elementary equivalence and $\Sigma_{n}$ equivalence is the case because they have different levels in the Borel hierarchy while isomorphism and bi-embeddability are not even Borel. On the other hand bi-embeddability preserves very little structural properties and it is thus difficult to construct interesting examples. The aim of this article is to investigate the relationship between the degree spectra realized by the bi-embeddability relation and by the elementary bi-embeddability relation. First, we establish that elementary bi-embeddability on graphs is $\boldsymbol{\Sigma}_{1}^{1}$ complete with respect to Borel reducibility. We then proceed to establish a relationship between the degree spectra realized by the bi-embeddability and elementary bi-embeddability relation on graphs. Our main results are as follows.

Theorem 1. The elementary embeddability relation on graphs $\preccurlyeq_{\mathfrak{F}}$ is a complete $\boldsymbol{\Sigma}_{1}^{1}$ quasi-order. In particular, the elementary bi-embeddability relation on graphs $\widetilde{\approx}_{\mathfrak{G}}$ is a complete $\boldsymbol{\Sigma}_{1}^{1}$ equivalence relation.

As a corollary of Theorem 1 we obtain the corresponding result for elementary bi-embeddability on computable structures.

Theorem 2. The elementary embeddability relation on the class of computable graphs is a $\Sigma_{1}^{1}$ complete quasi-order with respect to computable reducibility. In particular, the elementary bi-embeddability relation on computable graphs is a $\Sigma_{1}^{1}$ complete equivalence relation with respect to computable reducibility.

The following result establishes a relationship between bi-embeddability spectyra of graphs and elementary bi-embeddability spectra of graphs.

Theorem 3. Let $\mathcal{G}$ be an automorphically non-trivial graph, then there is a graph $\hat{\mathcal{G}}$ such that

$$
D g S p_{\approx}(\hat{\mathcal{G}})=\left\{X: X^{\prime} \in D g S p_{\approx}(\mathcal{G})\right\}
$$

Note that in Theorem 3 we deal only with automorphically non-trivial graphs. This might seem like a shortcoming, however automorphically trivial structures are not interesting from a computability theoretic point of view. In particular, every structure bi-embeddable with an automorphically trivial graph is computable and thus both its bi-embeddability spectrum and elementary bi-embeddability spectrum is the set of all computable sets.

The proofs of Theorems 1 and 2 are the topic of Section 3 In Section 4 we build on these results to prove Theorem 3. In Section 2 we give the necessary background and definitions.

## 2. Background

Our definitions follow for the most part [12] and 19]. We assume that all structures have universe $\omega$ and are relational. Let $\mathcal{L}$ be a relational language $\left(R_{i}\right)_{i \in \omega}$ where without loss of generality $R_{i}$ has arity $i$. Then each element $\mathcal{A}$ of $\operatorname{Mod}(\mathcal{L})$ can be viewed as an element of the product space

$$
X_{\mathcal{L}}=\prod_{i \in \omega} 2^{\omega^{i}}
$$

and thus $\operatorname{Mod}(\mathcal{L})$ becomes a compact Polish space on which we can define the Borel and projective hierarchy in the usual way.

Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $\left(\varphi_{i}^{a t}\right)_{i \in \omega}$ be a computable enumeration of the atomic $\mathcal{L}$-sentences with variables in $\left\{x_{1}, x_{2}, \ldots\right\}$. The atomic diagram $D(\mathcal{A})$ of $\mathcal{A}$ is the element of Cantor space defined by

$$
D(\mathcal{A})(i)= \begin{cases}1 & \text { if } \mathcal{A} \models \varphi_{i}^{a t}\left[x_{j} \rightarrow j: j \in \omega\right] \\ 0 & \text { otherwise }\end{cases}
$$

The Turing degree of a structure $\mathcal{A}$ is the degree of $D(\mathcal{A})$. We will in general not distinguish between a structure as an element of $\operatorname{Mod}(\mathcal{L})$ and its atomic diagram and assume that what is meant is clear from the context.

Variations of the following definition were independently suggested in [18, 9, 22].

Definition 4. Let $E$ be an equivalence relation on $\operatorname{Mod}(\mathcal{L})$ and $\mathcal{A} \in \operatorname{Mod}(\mathcal{L})$. Then the degree spectrum of $\mathcal{A}$ with respect to $E$, or, short $E$-spectrum of $\mathcal{A}$, is the set

$$
D g S p_{E}(\mathcal{A})=\left\{X: \exists \mathcal{B} E \mathcal{A} D(\mathcal{B}) \equiv_{T} X\right\}
$$

We write $\mathcal{A} \hookrightarrow \mathcal{B}$ to say that $\mathcal{A}$ is embeddable in $\mathcal{B}$, and $\mathcal{A} \approx \mathcal{B}$ to say that $\mathcal{A}$ is bi-embeddable with $\mathcal{B}$, i.e., $\mathcal{A} \hookrightarrow \mathcal{B}$ and $\mathcal{B} \hookrightarrow \mathcal{A}$. Further, we write $\mathcal{A} \preccurlyeq \mathcal{B}$ to say that $\mathcal{A}$ is elementary embeddable in $\mathcal{B}$ and $\mathcal{A} \approx \mathcal{B}$ to say that $\mathcal{A}$ is elementary bi-embeddable with $\mathcal{B}$, i.e., $\mathcal{A} \preccurlyeq \mathcal{B}$ and $\mathcal{B} \preccurlyeq \mathcal{A}$.

Definition 5. Let $R, S$ be binary relations on a set $X$. The relation $R$ is reducible to $S$ if there is a function $f: X \rightarrow X$ such that for all $x, y \in X$

$$
x R y \Leftrightarrow f(x) S f(y)
$$

Assume $X=\operatorname{Mod}(\mathcal{L})$. Then
(1) $R$ is Borel reducible to $S$ if $f$ is Borel on $\operatorname{Mod}(\mathcal{L}) \times \operatorname{Mod}(\mathcal{L})$,
(2) $R$ is computably reducible to $S$ if there is a computable operator $\Phi$ such that for all $\mathcal{A} \in \operatorname{Mod}(\mathcal{L}), \Phi^{D(\mathcal{A})}=D(f(\mathcal{A}))$.
Assume $X$ is $\omega$ and that $\left(\mathcal{A}_{i}\right)_{i \in \omega}$ is a computable enumeration of all partial computable $\mathcal{L}$ structures. Then $R$ is computably reducible to $S$ if $f$ is a computable function.

We say that an equivalence relation (quasi-order) $R \in \Gamma$ is a $\Gamma$ complete equivalence relation (quasi-order) for a complexity class $\Gamma$ with respect to $x$-reducibility if all equivalence relations (quasi-orders) in $\Gamma$ are $x$-reducible to $R$.

A standard reference on Borel reducibility is 12]. Computable reducibility on the natural numbers can be seen as a natural effectivization of Borel reducibility where one only considers computable structures. Fokina and Friedman 6] showed
that bi-embeddability on trees and thus also graphs is $\Sigma_{1}^{1}$ complete with respect to computable reducibility, and in 7] it is shown that isomorphism on graphs is $\Sigma_{1}^{1}$ complete with respect to computable reducibility. This contrasts with Borel reducibility; it is well known that isomorphism on graphs is not $\boldsymbol{\Sigma}_{1}^{1}$ complete.

## 3. Elementary bi-Embeddability is analytic complete

In this section we prove Theorems 1 and 2 and some lemmas needed for Theorem 3. The section is structured as follows. In Section 3.1 we give a reduction from embeddability on the class of graphs $\mathfrak{G}$ to elementary embeddability on a Borel class $\mathfrak{C}$ of structures in an infinite relational language. In Section 3.2 we show that graphs are complete for elementary embeddability. That is, for every Borel class, elementary embeddability on this class can be reduced to elementary embeddability on graphs. Theorem then follows by composing the reductions given in Sections 3.1 and 3.2. For Theorem 2 we need a few more observations made at the end of this section.
3.1. The reduction from $\hookrightarrow_{\mathfrak{G}}$ to $\preccurlyeq_{\mathfrak{C}}$. The main idea of the construction is that for any given graph $\mathcal{G}$ we replace the edge relation with structures having the property that they are minimal under elementary embeddability.

Definition 6. A structure $\mathcal{A}$ is minimal if it does not have proper elementary substructures.

Minimal structures were investigated by Fuhrken 11] who showed that there is a theory with $2^{\aleph_{0}}$ minimal models, and Shelah [21] who showed that for every $n \leq \aleph_{0}$, there is a theory with $n$ minimal models. Later, Ikeda [14] investigated minimal models of minimal theories. Notice that a prime model is not necessarily minimal, as it might contain elementary substructures isomorphic to itself.

Given a graph $\mathcal{G}$, if $x, y \in G$ and $x E y$, then we associate a copy of a structure $\mathcal{A}$ with the pair $(x, y)$ and otherwise we associate a copy $\mathcal{B}$ with $(x, y)$. The structures $\mathcal{A}$ and $\mathcal{B}$ will be elementary equivalent and minimal.

Before we formally state the reduction let us describe $\mathcal{A}$ and $\mathcal{B}$. They will be models of the theory of the following structure studied by Shelah 21]. The language of the theory contains countably many unary functions $F_{\nu}$ and unary relation symbols $R_{\nu}$, one for each $\nu \in 2^{<\omega}$. Consider the structure

$$
\mathcal{S}=\left(2^{\omega},\left\langle F_{\nu}\right\rangle_{\nu \in 2^{<\omega}},\left\langle R_{\nu}\right\rangle_{\nu \in 2^{<\omega}}\right)
$$

where $F_{\nu}$ is defined by $F_{\nu}(\sigma)(x)=\sigma(x)+\nu(x) \bmod 2$ where we assume that $\nu(x)=0$ for $x \geq|\nu|$ and $R_{\nu}(\sigma)$ if and only if $\nu \prec \sigma$. Shelah showed that the theory of $\mathcal{S}$ has quantifier elimination and that each element of $\mathcal{S}$ generates an elementary substructure that is minimal.

Let $\hat{\mathcal{S}}_{0}$ be the substructure of $\mathcal{S}$ generated by $\overline{0}$, the constant string of 0 's and $\hat{\mathcal{S}}_{1}$ be the substructure generated by $\overline{1}$, the constant string of 1 's. These structures are countable and by Shelah's argument, $\hat{\mathcal{S}}_{0} \equiv \hat{\mathcal{S}}_{1} \equiv \mathcal{S}$. Furthermore, $\hat{\mathcal{S}}_{0}$ and $\hat{\mathcal{S}}_{1}$ are minimal models of $\operatorname{Th}(\mathcal{S})$. To see this, let $x \in \hat{S}_{0}$, then $x=F_{\nu}(\overline{0})$ and in particular, $\overline{0}=F_{\nu}(x)$ for some $\nu \in 2^{<\omega}$. So, the substructure of $\hat{\mathcal{S}}_{0}$ generated by $x$ is already $\hat{\mathcal{S}}_{0}$.

As we require our structures in $\mathfrak{C}$ to be of relational syntax we will let $\mathcal{S}_{0}$ and $\mathcal{S}_{1}$ be the structures corresponding to $\hat{\mathcal{S}}_{0}$, respectively $\hat{\mathcal{S}}_{1}$, after we replace each $F_{\nu}^{\mathcal{S}_{i}}$
by its graph $\operatorname{graph}_{F_{\nu}}^{\mathcal{S}_{i}}=\left\{\left(\sigma, F_{\nu}^{\mathcal{S}_{i}}(\sigma)\right): \sigma \in S_{i}\right\}$. We may assume without loss of generality that the universes of $\mathcal{S}_{0}$ and $\mathcal{S}_{1}$ are $\omega$ and let $\mathcal{A}=\mathcal{S}_{0}$ and $\mathcal{B}=\mathcal{S}_{1}$.

Let us describe the structures in the class $\mathfrak{C}$ more formally. The class of structures $\mathfrak{C}$ consists of all countable structures with universe $\omega$ in the language consisting of a unary relation $W$, binary relations $R_{\nu}$ and $\operatorname{graph}_{F_{\nu}}$ for all $\nu \in 2^{<\omega}$, and a ternary relation $O$. We are now ready to give the function $f: \mathfrak{G} \rightarrow \mathfrak{C}$ witnessing the reduction.

We formally describe how to obtain a structure in $\mathfrak{C}$ given a graph. Let $\mathcal{G}$ be a graph and partition $\omega$ into countably many infinite, coinfinite subsets $\left(A_{i}\right)_{i \in \omega}$. Then

- for every $a_{i} \in A_{0}, W^{f(\mathcal{G})}\left(a_{i}\right)$ (we will call elements of $A_{0}$ the vertices of $f(\mathcal{G}))$,
- for every $m, n \in \omega$, if $m E n$, then for all $\nu \in 2^{<\omega}$ define $R_{\nu}^{f(\mathcal{G})}$ and $\operatorname{graph}_{F_{\nu}}{ }^{f(\mathcal{G})}$ on $A_{\langle m, n\rangle+1}$ such that $\left(A_{\langle m, n\rangle+1},\left\langle\operatorname{graph}_{F_{\nu}}{ }^{f(\mathcal{G})}\right\rangle_{\nu \in 2<\omega}, R_{\nu}^{f(\mathcal{G})}\right) \cong$ $\mathcal{S}_{0}$,
- for every $m, n \in \omega$, if $\neg m E n$, then for all $\nu \in 2^{<\omega}$ define $R_{\nu}^{f(\mathcal{G})}$ and $\operatorname{graph}_{F_{\nu}}{ }^{f(\mathcal{G )}}$ on $A_{\langle m, n\rangle+1}$ such that $\left(A_{\langle m, n\rangle+1},\left\langle\operatorname{graph}_{F_{\nu}}{ }^{f(\mathcal{G})}\right\rangle_{\nu \in 2}<\omega, R_{\nu}^{f(\mathcal{G})}\right) \cong$ $\mathcal{S}_{1}$,
- for every $m, n \in \omega$, let $O^{f(\mathcal{G})}\left(a_{m}, a_{n}, j\right)$ for all $j \in A_{\langle m, n\rangle+1}$.

This finishes the construction of $f(\mathcal{G})$. We will refer to the substructure on the elements in $A_{\langle m, n\rangle+1}$ as the substructure associated to the pair $\left(a_{m}, a_{n}\right)$ and to $\left(A_{\langle m, n\rangle+1},\left\langle\operatorname{graph}_{F_{\nu}}{ }^{f(\mathcal{G})}\right\rangle_{\nu \in 2<\omega}, R_{\nu}^{f(\mathcal{G})}\right)$ as $\mathcal{S}_{\left(a_{m}, a_{n}\right)}$.

It is easy to see that the function $f$ so defined is Borel, indeed it is even computable. To see that $f$ is a reduction from $\hookrightarrow_{\mathfrak{G}}$ to $\preccurlyeq \mathfrak{C}$ it remains to prove the following.

Lemma 7. For $\mathcal{G}, \mathcal{H} \in \mathfrak{G}, \mathcal{G} \hookrightarrow \mathcal{H}$ if and only if $f(\mathcal{G}) \preccurlyeq f(\mathcal{H})$.
Proof. That $\mathcal{G} \hookrightarrow \mathcal{H}$ if $f(\mathcal{G}) \preccurlyeq f(\mathcal{H})$ follows trivially from the construction. To show the converse we will use the following model theoretic fact: For two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}, \mathcal{A}$ is an elementary substructure of $\mathcal{B}$ if and only if for every finite $\mathcal{R} \subseteq \mathcal{L}$, the $\mathcal{R}$ reduct of $\mathcal{A}$ is an elementary substructure of the $\mathcal{R}$ reduct of $\mathcal{B}$. Necessity follows trivially from the fact that $\mathcal{R} \subseteq \mathcal{L}$ and sufficiency is easily seen by noticing that every first order formula $\varphi$ is in a finite $\mathcal{R}_{\varphi} \subseteq \mathcal{L}$.

So, say $\mathcal{G} \hookrightarrow \mathcal{H}$ by $h$. We get an induced embedding $\hat{h}$ defined such that for all $i, i^{\prime} \in G$, if $h(i)=j$, then $\hat{h}\left(a_{i}\right)=a_{j}$ and $\hat{h}$ is the canonic isomorphism between the substructure associated to $\left(a_{i}, a_{i^{\prime}}\right)$ and the one associated to $\left(\hat{h}\left(a_{i}\right), \hat{h}\left(a_{i^{\prime}}\right)\right)$. Without loss of generality we may assume that $f(\mathcal{G})$ is a substructure of $f(\mathcal{H})$, i.e., that $\hat{h}$ is the identity. We use Ehrenfeucht-Fraïssé games to verify that in every finite $\mathcal{R} \subseteq \mathcal{L}, f(\mathcal{G})$ is an elementary substructure of $f(\mathcal{H})$. We assume without loss of generality that $\mathcal{R}=\left\{O, W, R_{\nu_{0}}, \ldots, R_{\nu_{k}}, \operatorname{Graph}_{F_{\nu_{0}}}, \ldots, \operatorname{Graph}_{F_{\nu_{k}}}\right\}$ where $\nu_{i}$ is the $i^{\text {th }}$ string in the lexicographical ordering of $2^{<\omega}$ and $k \in \omega$. Let us show that player II has a winning strategy in $G_{m}\left(\left(f(\mathcal{G}), g_{1}, \ldots, g_{n}\right),\left(f(\mathcal{H}), g_{1}, \ldots, g_{n}\right)\right)$ for arbitrary $m \in \omega$ played in $\mathcal{R}$. First, notice that since $\mathcal{S}_{0} \equiv \mathcal{S}_{1}$, II has a winning strategy for $G_{m}\left(\mathcal{S}_{0}, \mathcal{S}_{1}\right)$ in the reduct $\left\{R_{\nu_{0}}, \ldots, R_{\nu_{k}}, \operatorname{Graph}_{F_{\nu_{0}}}, \ldots, \operatorname{Graph}_{F_{\nu_{k}}}\right\}$. The following is a winning strategy for $G_{m}\left(\left(f(\mathcal{G}), g_{1}, \ldots, g_{n}\right),\left(f(\mathcal{H}), g_{1}, \ldots, g_{n}\right)\right)$ played in $\mathcal{R}$. Say that at turn $i$, the played substructures are $G_{i}$ and $H_{i}$ given by the partial isomorphism $h_{i}$. Assume we are on turn $i+1$.
(1) If I plays an element $c$ in the $O$-closure of $g_{1}, \ldots, g_{n}$, then let $h_{i+1}(c)=c$.
(2) If I plays an element $c$ in $f(\mathcal{G})$ not in the $O$-closure of $G_{i}$, say it is associated to $(a, b)$ where none of $a, b$ is in the $O$-closure of $G_{i}$, then pick vertices $\left(a^{\prime}, b^{\prime}\right)$ in $f(\mathcal{H})$. If $c=a$ or $b$, let $h_{i+1}(c)=a^{\prime}$, respectively, $h_{i+1}(c)=b^{\prime}$. Otherwise start running a $G_{m}\left(\mathcal{S}_{(a, b)}, \mathcal{S}_{\left(a^{\prime}, b^{\prime}\right)}\right)$ winning strategy $w_{(a, b)}^{\left(a^{\prime}, b^{\prime}\right)}$ and let $h_{i+1}(c)=w_{(a, b)}^{\left(a^{\prime}, b^{\prime}\right)}(c)$.
(3) If I plays an element $c$ in $f(\mathcal{G})$ not in the $O$-closure of $G_{i}$ but associated to $(a, b)$ where either $a$ or $b$ is in $G_{i}$, then pick $\left(a^{\prime}, b^{\prime}\right)$ such that $a^{\prime}$, respectively $b^{\prime}$, is the element corresponding to $a$, respectively $b$, in $H_{i}$ and continue as in (2), mutatis mutandis.
(4) If I plays an element in $f(\mathcal{H})$ not in the $O$-closure of $H_{i}$, then as $f(\mathcal{G})$ is infinite, II can play as in the cases (2) and (3), mutatis mutandis.
(5) If I plays an element $c$ in $f(\mathcal{G})$ that is in the $O$-closure of $G_{i}$ but not in the $O$-closure of $g_{1}, \ldots, g_{n}$, then it is associated to some $(a, b)$ in $f(\mathcal{G})$ and by induction there is a winning strategy $w_{(a, b)}^{\left(a^{\prime}, b^{\prime}\right)}$ that has already been used. If $c=a$ or $c=b$, let $h_{i+1}(c)=a^{\prime}$, respectively, $h_{i+1}(c)=b^{\prime}$. Otherwise let $h_{i+1}(c)=w_{(a, b)}^{\left(a^{\prime}, b^{\prime}\right)}\left(c_{1}, \ldots, c_{k}, c\right)$ where $c_{1}, \ldots, c_{k}$ are the elements from the structures associated with $(a, b)$ and $\left(a^{\prime}, b^{\prime}\right)$ played by $I$ so far.
(6) If I plays an element $c$ in $f(\mathcal{H})$ that is in the $O$-closure of $H_{i}$ but not in the $O$-closure of $g_{1}, \ldots, g_{n}$, then play as in (5), mutatis mutandis.
Since at each turn we play according to winning strategies for games of the form $G_{m}\left(\mathcal{S}_{i}, \mathcal{S}_{j}\right)$ where $i, j \in\{0,1\}$ we obtain that $h_{m}$ is a partial isomorphism between $\left(f(\mathcal{G}), g_{1}, \ldots, g_{n}\right)$ and $\left(f(\mathcal{H}), g_{1}, \ldots, g_{n}\right)$. We thus have given a winning strategy for $G_{m}\left(\left(f(\mathcal{G}), g_{1}, \ldots, g_{n}\right),\left(f(\mathcal{H}), g_{1}, \ldots, g_{n}\right)\right)$.
3.2. Graphs are complete for elementary embeddability. We will show that for every class of structures $\mathfrak{K}$, there is a computable reduction $\preccurlyeq_{\mathfrak{K}} \rightarrow \preccurlyeq_{\mathfrak{G}}$.

The result we are going to prove appeared in 20]. There, a proof sketch of the fact that the reduction preserves elementary bi-embeddability spectra was given. We will give a full proof of this fact in Section 4. Note that the coding used in the reduction is not new but was already used in 3] to show that graphs are universal for theory spectra. Let us first describe this coding.

We may assume without loss of generality that $\mathfrak{K}$ is a class of structures in relational language $\mathcal{L}=\left(R_{1}, \ldots\right)$ where each $R_{i}$ has arity $i$. Given $\mathcal{A} \in \mathfrak{K}$, the graph $g(\mathcal{A})$ has three vertices $a, b, c$ where to $a$ we connect the unique 3 -cycle in the graph, to $b$ the unique 5 -cycle, and to $c$ the unique 7 -cycle. For each element $x \in A$ we add a vertex $v_{x}$ and an edge $a \mapsto v_{x}$. For every $i$ tuple $x_{1}, \ldots, x_{i} \in A$ we add chains of length $i+k$ for every $k, 1 \leq k \leq i$ with common last elements $y$. We add an edge $v_{x_{k}} \hookrightarrow y_{1}$ only if $y_{1}$ is the first element of the chain of length of $i+k$. If $\mathcal{A} \models R_{i}\left(x_{1}, \ldots, x_{i}\right)$ we add an edge $y \mapsto b$ and otherwise add an edge $y \mapsto c$. This finishes the construction. See Fig. 1 for an example.

Let us fix the some notation for the following proofs. Given a structure $\mathcal{A}$ and $\bar{a} \in A^{<\omega}$ we let $\langle\bar{a}\rangle^{\mathcal{A}}$ be the substructure of $\mathcal{A}$ generated by $\bar{a}$.

Lemma 8. For $\mathcal{A}, \mathcal{B} \in \mathfrak{K}, \mathcal{A} \preccurlyeq \mathcal{B}$ if and only if $g(\mathcal{A}) \preccurlyeq g(\mathcal{B})$.
Proof. $(\Rightarrow)$. Assume that $\mathcal{A} \preccurlyeq \mathcal{B}$ and that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$. We may also assume without loss of generality that $g(\mathcal{A}) \subseteq g(\mathcal{B})$. We will show that


Figure 1. Part of the graph $F(\mathcal{A})$ coding that $\mathcal{A} \not \vDash R_{3}(3,2,1)$ and $\mathcal{A} \models R_{3}(1,2,3)$.
for all $n \in \omega$ and any $\bar{a} \in g(\mathcal{A})^{<\omega}$ player II has a winning strategy for the $n$ turn Ehrenfeucht Fraïssé game $G_{n}((g(\mathcal{A}), \bar{a}),(g(\mathcal{B}), \bar{a}))$. Assume that $n$ is the least such that player II has no winning strategy for $G_{n}((g(\mathcal{A}), \bar{a}),(g(\mathcal{B}), \bar{a}))$. Consider the set of partial isomorphisms from $(g(\mathcal{A}), \bar{a})$ to $(g(\mathcal{B}), \bar{a})$. This set can not have the back-and-forth property. In particular, the back-and-forth property fails already if we only consider partial isomorphisms with domain of size $n+|\bar{a}|$. Otherwise there would be a winning strategy for $G_{n}((g(\mathcal{A}), \bar{a}),(g(\mathcal{B}), \bar{a}))$. So, either there is $\bar{v} \in g(\mathcal{A})^{n}$ such that for all $\bar{u} \in g(\mathcal{B})^{n},\langle\overline{a v}\rangle^{g(\mathcal{A})} \neq\langle\overline{a u}\rangle^{g(\mathcal{B})}$ or there is $\bar{u} \in g(\mathcal{B})^{n}$ such that for all $\bar{v} \in g(\mathcal{B})^{n},\langle\overline{a u}\rangle^{g(\mathcal{B})} \not \not\left\langle\langle\overline{a v}\rangle^{g(\mathcal{A})}\right.$. We will derive a contradiction assuming the second case. Deriving one from the first case can be done in a similar fashion.

Notice that $\overline{a u}$ is in a substructure of $g(\mathcal{B})$ coding a finite substructure of $\mathcal{B}$ in a finite part $\mathcal{L}_{1}$ of the language of $\mathcal{B}$. Extend $\langle\overline{a u}\rangle^{g(\mathcal{B})}$ so that it codes such a substructure $\mathcal{B}_{1}$ of $\mathcal{B}$. Consider the conjunction $\varphi$ of atomic formulas, or negations thereof, true of $\mathcal{B}_{1}$ in $\mathcal{L}_{1}$. Let $\bar{a}^{\prime}$ be the elements in $B_{1} \cap A$ and $\bar{u}^{\prime}$ the elements in $B_{1} \backslash A$. Then $\mathcal{B} \models \varphi\left(\bar{a}^{\prime} \bar{u}^{\prime}\right)$ and the Tarski-Vaught test gives us elements $\bar{v}^{\prime}$ in $\mathcal{A}$ such that $\mathcal{A} \models \varphi\left(\bar{a}^{\prime} \bar{v}^{\prime}\right)$. It follows that we have a partial isomorphism between $\left\langle\bar{a}^{\prime} \bar{u}^{\prime}\right\rangle^{\mathcal{B}}$ and $\left\langle\bar{a}^{\prime} \bar{v}^{\prime}\right\rangle^{\mathcal{A}}$ in $\mathcal{L}_{1}$. This induces an isomorphism between the subgraph coding $\mathcal{B}_{1}$ and the subgraph coding $\left\langle\bar{a}^{\prime} \bar{v}^{\prime}\right\rangle^{\mathcal{A}}$. But $\langle\overline{a u}\rangle^{g(\mathcal{B})}$ is a subgraph of the graph coding $\mathcal{B}_{1}$ and thus it is isomorphic to a substructure $\langle\overline{a v}\rangle^{g(\mathcal{A})}$ of the structure coding $\left\langle\bar{a}^{\prime} \bar{v}^{\prime}\right\rangle^{\mathcal{A}}$, a contradiction.
$(\Leftarrow)$. An easy induction on the quantifier depth of formulas in $\mathcal{L}$ shows that for every $\mathcal{A} \in \mathfrak{K}$ and $\mathcal{L}$-formula $\varphi$ with $n$-free variables the set

$$
D_{\varphi}^{\mathcal{A}}=\left\{\left(v_{a_{1}}, \ldots, v_{a_{n}}\right):\left(\mathcal{A}, a_{1}, \ldots, a_{n}\right) \models \varphi\left(a_{1}, \ldots, a_{n}\right)\right\}
$$

is definable in $g(\mathcal{A})$. Now, assume that $g(\mathcal{A}) \preccurlyeq g(\mathcal{B})$ and without loss of generality that $g(\mathcal{A})$ is an elementary substructure of $g(\mathcal{B})$. Let $g_{\mathcal{B}}: \mathcal{B} \rightarrow g(\mathcal{B})$ be defined by $g_{\mathcal{B}}: b \mapsto v_{b}$. Notice that the map $a \mapsto g_{\mathcal{B}}^{-1}\left(v_{a}\right)$ is an embedding of $\mathcal{A}$ in $\mathcal{B}$. To see that this embedding is elementary assume that $(\mathcal{A}, \bar{a}) \models \varphi$, then $\bar{v}_{\bar{a}} \in D_{\varphi}^{\mathcal{A}}$ and by elementarity $\bar{v}_{\bar{a}} \in D_{\varphi}^{\mathcal{B}}$. So, $\left(\mathcal{B}, g_{\mathcal{B}}^{-1}\left(\bar{v}_{\bar{a}}\right)\right) \models \varphi\left(g_{\mathcal{B}}^{-1}\left(\bar{v}_{\bar{a}}\right)\right)$.

Concatenating the reductions $f$ and $g$ and from the fact that $\hookrightarrow_{\mathfrak{G}}, \approx_{\mathfrak{G}}$ are complete $\boldsymbol{\Sigma}_{1}^{1}$ quasi-orders, respectively equivalence relations, we obtain Theorem 1

To prove Theorem 2 notice that $f$ and $g$ are computable. Thus there is a Turing operator $\Phi$ such that $\Phi=g \circ f$. We can find a Turing machine $\varphi_{i}$ such that $\varphi_{i}(j, k)=\Phi^{\mathcal{A}_{j}}(k)$ for all $k \in \omega$ if $\mathcal{A}_{j}$ is a total computable structure. Using the s-m-n theorem we can then get a computable function $j \mapsto u(i, j)$ where $u(i, j)$ is an index for $\Phi^{\mathcal{A}_{j}}$. Thus $\hookrightarrow_{\mathfrak{G}}$ is computably reducible to $\preccurlyeq_{\mathfrak{G}}$ as a quasi-order on $\omega$. Fokina and Friedman [6] showed that $\hookrightarrow_{\mathfrak{G}}$ is $\Sigma_{1}^{1}$ complete. Thus, $\preccurlyeq_{\mathfrak{G}}$ is also $\Sigma_{1}^{1}$ complete and Theorem 2 follows.

## 4. Degree spectra

In this section we finish the proof of Theorem 3. As noticed before the two reductions $f: \mathfrak{G} \rightarrow \mathfrak{C}$ and $g: \mathfrak{C} \rightarrow \mathfrak{G}$ are computable. We will see that the two functions induce an even stronger notion of reduction that allows us to relate the degree spectra realized by $\approx_{\mathfrak{G}}$ and $\approx_{\mathfrak{G}}$.
Definition 9 (cf. 13, 17]). Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. A computable functor between $\mathfrak{C}$ and $\mathfrak{D}$ is a pair of computable operators $\left(\Phi, \Phi_{*}\right)$ such that
(1) for all $\mathcal{A} \in \mathfrak{C}_{1}, F(\mathcal{A})=\Phi^{\mathcal{A}}$,
(2) for all $f: \mathcal{A} \rightarrow \mathcal{B} \in \mathfrak{C}_{2}, F(f)=\Phi_{*}^{\mathcal{A} \oplus f \oplus \mathcal{B}}$.

Computable functors preserve many computability theoretic properties. One example are degree spectra: Recall that for $X, Y \subseteq \mathcal{P}(\omega), X$ is Medwedev reducible to $Y, X \leq_{s} Y$, if there is a Turing operator $\Phi$ such that for all $y \in Y$, there is $x \in X$ such that $\Phi^{y}=x$. We have in particular that if $F:\left(\mathfrak{C}, \rightsquigarrow_{1}\right) \rightarrow\left(\mathfrak{D}, \rightsquigarrow_{2}\right)$ is a computable functor, and $\sim_{i}$ is the equivalence relation given by

$$
\mathcal{A} \sim_{i} \mathcal{B} \Leftrightarrow \mathcal{A} \rightsquigarrow_{i} \mathcal{B} \wedge \mathcal{B} \rightsquigarrow_{i} \mathcal{A},
$$

then for all $\mathcal{A} \in \mathfrak{C}, D g S p_{\sim_{1}}(F(\mathcal{A})) \leq_{s} D g S p_{\sim_{2}}(\mathcal{A})$.
It is an easy exercise to see that $g \circ f$ induces a computable functor $H:(\mathfrak{G}, \hookrightarrow) \rightarrow$ $(\mathfrak{G}, \preccurlyeq)$ and thus for all $\mathcal{G} \in \mathfrak{G}, D g S p_{\approx}(H(\mathcal{G})) \leq_{s} D g S p \approx(\mathcal{G})$.

To get that every degree spectrum realized in $\mathfrak{C}$ is also realized in $\mathfrak{D}$ we need a stronger notion of reducibility. To define this we need an effectivization of the category theoretic notion of a natural isomorphism between functors.
Definition 10 (13]). A functor $F: \mathfrak{C} \rightarrow \mathfrak{D}$ is effectively isomorphic to $G: \mathfrak{C} \rightarrow \mathfrak{D}$ if there is a Turing operator $\Lambda$ such that for every $\mathcal{A} \in \mathfrak{C}, \Lambda^{\mathcal{A}}$ is an isomorphism from $F(\mathcal{A})$ to $G(\mathcal{A})$, and the following diagram commutes for all $\mathcal{A}, \mathcal{B} \in \mathfrak{C}_{1}$ and every $\gamma: \mathcal{A} \rightarrow \mathcal{B} \in \mathfrak{C}_{2}$.


Definition 11 (cf. [13]). We say that $\left(\mathfrak{C}, \rightsquigarrow_{1}\right)$ is CBF-reducible to ( $\mathfrak{D}, \rightsquigarrow_{2}$ ), $\left(\mathfrak{C}, \rightsquigarrow_{1}\right) \leq_{C B F}\left(\mathfrak{D}, \rightsquigarrow_{2}\right)$ if
(1) there is a computable functor $F: \mathfrak{C} \rightarrow \mathfrak{D}$ and a computable functor $G$ : $\mathfrak{D} \supseteq \hat{\mathfrak{D}} \rightarrow \mathfrak{C}$ where $\hat{\mathfrak{D}}$ is the $\sim_{2}$-closure of $F(\mathfrak{C})$,
(2) $F \circ G$ is effectively isomorphic to $I d_{\hat{\mathfrak{D}}}, G \circ F$ is effectively isomorphic to $I d_{\mathfrak{C}}$,
(3) and, if $\Lambda_{\mathfrak{C}}, \Lambda_{\mathfrak{D}}$ are the operators witnessing the effective isomorphism between $G \circ F$ and $I d_{\mathfrak{C}}$, respectively, $F \circ G$ and $I d_{\hat{\mathfrak{D}}}$, then for every $\mathcal{A} \in \mathfrak{C}, F\left(\Lambda_{\mathcal{C}}^{\mathcal{A}}\right)=\Lambda_{\mathfrak{D}}^{F(\mathcal{A})}: F(\mathcal{A}) \rightarrow F(G(F(\mathcal{A})))$ and every $\mathcal{B} \in \hat{\mathfrak{D}}$, $G\left(\Lambda_{\mathfrak{D}}^{\mathcal{B}}\right)=\Lambda_{\mathfrak{C}}^{G(\mathcal{B})}: G(\mathcal{B}) \rightarrow G(F(G(\mathcal{B})))$.
Consider two structures $\mathcal{A}$ and $\mathcal{B}$ and a morphism $f: \mathcal{A} \cong \mathcal{B}$. Then, clearly $\mathcal{A} \leq_{T}$ $\mathcal{B} \oplus f$; after all, we have that $R^{\mathcal{A}}\left(a_{1}, \ldots, a_{n}\right)$ if and only if $R^{\mathcal{B}}\left(f\left(a_{1}\right), \ldots, f\left(a_{n}\right)\right)$. The following definition generalizes this observation.

Definition 12. A category $\mathfrak{C}$ is degree invariant if for every $\mathcal{A}, \mathcal{B} \in \mathfrak{C}_{1}$ and every $f: \mathcal{A} \rightarrow \mathcal{B} \in \mathfrak{C}_{2}, f \equiv_{T} f^{-1}$ and $\mathcal{A} \leq_{T} \mathcal{B} \oplus f$.
Proposition 13. If $\mathfrak{C}$ and $\mathfrak{D}$ are degree invariant and $\mathfrak{C} \leq_{C B F} \mathfrak{D}$, then every set realized as $a \sim_{1}$-spectrum in $\mathfrak{C}$ is realized as $a \sim_{2}$-spectrum in $\mathfrak{D}$.

Proof. Say $F: \mathfrak{C} \rightarrow \mathfrak{D}$ and $G: \mathfrak{D} \rightarrow \mathfrak{C}$ witness that $\mathfrak{C} \leq_{C B F} \mathfrak{D}$. Fix $\mathcal{A} \in \mathfrak{C}$ and let $\Lambda$ be the Turing operator witnessing that $G \circ F$ is effectively isomorphic to the identity functor on $\mathfrak{C}$. Then, for $\hat{\mathcal{A}} \sim_{1} \mathcal{A}, \hat{\mathcal{A}} \geq_{T} F(\hat{\mathcal{A}}) \geq_{T} G(F(\hat{\mathcal{A}}))$ and by degree invariance $\hat{\mathcal{A}} \leq_{T} \Lambda^{\mathcal{A}} \oplus G(F(\hat{\mathcal{A}})) \equiv_{T} G(F(\hat{\mathcal{A}})) \leq_{T} F(\hat{\mathcal{A}})$. Thus, $D g S p_{\sim_{1}}(\mathcal{A}) \subseteq$ $D g S p_{\sim_{2}}(F(\hat{\mathcal{A}}))$. The proof that $D g S p_{\sim_{1}}(\mathcal{A}) \supseteq D g S p_{\sim_{2}}(F(\hat{\mathcal{A}}))$ is similar. So, if $X$ is a $\sim_{1}$ spectrum realized by $\mathcal{A}$ in $\mathfrak{C}$, then it is realized as a $\sim_{2}$ spectrum in $\mathfrak{D}$.

Notice that if $\mathfrak{K}$ is a class of relational structures, then whether $(\mathfrak{K}, \rightsquigarrow)$ is degree invariant only depends on $\rightsquigarrow$. Thus we might say that a relation on structures is degree invariant.
Definition 14. A class of structures $\mathfrak{C}$ is $C B F$-complete with respect to a degree invariant relation $\rightsquigarrow$, if for every class $\mathfrak{K},(\mathfrak{K}, \rightsquigarrow) \leq_{C B F}(\mathfrak{C}, \rightsquigarrow)$.

We showed in Section 3.2 that for any class $\mathfrak{K}$ equipped with the elementary embeddability relation there is a computable reduction $g$ from $(\mathfrak{K}, \preccurlyeq)$ to $(\mathfrak{G}, \preccurlyeq)$. We can now show that these reductions induce $C B F$-reductions $(\mathfrak{K}, \preccurlyeq) \leq_{C B F}(\mathfrak{G} \preccurlyeq)$ and that thus graphs are CBF-complete for elementary embeddability. Verifying the conditions of Definition 11 is quite technical, but the core ideas of the proof should not be too difficult.

Theorem 15. The class of graphs is CBF-complete for elementary embeddability.
Proof. Fix a class $\mathfrak{K}$. It is clear from the construction that $g$ induces a computable functor $F:(\mathfrak{K}, \cong) \rightarrow(\mathfrak{G}, \cong)$. We have to show that $F(\mathfrak{K})$ is closed under elementary bi-embeddability, that there is a functor $G: F(\mathfrak{K}) \rightarrow \mathfrak{K}$ such that $F \circ G$ and $G \circ F$ are effectively isomorphic to the identity on $\mathfrak{K}$, respectively, $F(\mathfrak{K})$ and that the witnesses of these effective isomorphisms agree.

Let $\mathcal{G} \approx F(\mathcal{A})$ for some $\mathcal{A} \in \mathfrak{K}$. We may assume without loss of generality that $\mathcal{G}$ is an elementary substructure of $F(\mathcal{A})$. For every $\bar{a} \in \mathcal{G}^{<\omega}$, $\operatorname{tp}_{\mathcal{G}}(\bar{a})=\operatorname{tp}_{\mathcal{A}}(\bar{a})$. Thus $\mathcal{G}$ must contain the elements $a, b, c$ of $F(\mathcal{A})$ with unique 3 -cycles, respectively, 5 -cycles and 7 -cycles connected to them. Furthermore, say $\bar{a} \in \mathcal{G}$ codes elements of $A$ in $F(\mathcal{A})$ such that $\mathcal{A} \models R_{i}(\bar{a})$, then this information must also be coded in $\mathcal{G}$ as it is preserved in the type of $\bar{a}$. We can compute a structure $G(\mathcal{G})$ as follows. Fix a $\mathcal{G}$ computable injective enumeration $f$ of the set $\{x: a \longmapsto x\}$. Notice that this can be done uniformly since the set $\{x: a \mapsto x\}$ is uniformly computable in
all structures in $F(\mathfrak{K})$. Let the universe of $G(\mathcal{G})$ be the pull-back along $f$. Then for all $a_{1}, \ldots, a_{i}=\bar{a} \in \omega^{i}, G(\mathcal{G}) \models R_{i}(\bar{a})$ if for every $a_{j}, j<i$, there is a chain of $i+j$ connected elements $y_{1}, \ldots y_{i+j}$ with $f\left(a_{j}\right) \mapsto y_{1}$, all $j$ chains share the same last element $y$ and $y \mapsto b$. Likewise, $G(\mathcal{G}) \models \neg R_{i}(\bar{a})$ if there are chains satisfying the above conditions with $y \mapsto c$. This finishes the construction of $G(\mathcal{G})$.

Let $\mathcal{G}, \hat{\mathcal{G}} \in F(\mathfrak{K})$ and $g: \mathcal{G} \preccurlyeq \hat{\mathcal{G}}$. As both graphs are elementary bi-embeddable with images of structures in $\mathfrak{K}$, they have unique vertices $a$, respectively, $\hat{a}$ with 3 -cycles connected to them. Computably in $\mathcal{G}$ and $\hat{\mathcal{G}}$ find the vertices and enumerate the sets $\{x: a \longmapsto x\}$, and $\{x: \hat{a} \longmapsto x\}$ using the same procedure as in the construction of $G(\mathcal{G})$ above. Let $f$, respectively, $\hat{f}$ be these enumerations. Now let $G(g)=\hat{f}^{-1} \circ g \circ f$. By construction $G(g): G(\mathcal{G}) \hookrightarrow G(\hat{\mathcal{G}})$ and $G(g)$ is uniformly computable in $\mathcal{G} \oplus g \oplus \hat{\mathcal{G}}$. To see that $G(g)$ is elementary, assume towards a contradiction that it is not. Then there is $\bar{a} \in G(\mathcal{G})$ and $\varphi$ such that $G(\mathcal{G}) \models \varphi(\bar{a})$ but $G(\hat{\mathcal{G}}) \not \vDash \varphi(G(g)(\bar{a}))$. Recall that the atomic diagram of the tuple $\bar{a}$ is coded in the type of $f(\bar{a})$ in $\mathcal{G}$ and similarly, the atomic diagram of $G(g)(\bar{a})$ is coded in the type of $g(f(\bar{a}))$ in $\hat{\mathcal{G}}$. So, $g$ could not be elementary, a contradiction.

To see that $G \circ F$ and $F \circ G$ are effectively isomorphic to the identities on $\mathfrak{K}$ and $F(\mathfrak{K})$, respectively, first note that $G(F(\mathcal{A})) \cong \mathcal{A}$. There is a canonic isomorphism given by the composition of the maps $a \mapsto v_{a}$ and the enumeration $f$ of the set $\{x: a \longmapsto x\}$, i.e., the isomorphism is defined by $a \mapsto f^{-1}\left(v_{a}\right)$. It is clearly uniformly computable, say by $\Lambda_{\mathfrak{K}}$. On the other hand let $\mathcal{G} \in F(\mathfrak{K})$, then we can compute an isomorphism between $F(G(\mathcal{G}))$ and $\mathcal{G}$ by doing the following. Every $v \in G$ either defines a relation $R_{i}$ on some tuple, codes an element, or is used to define $a, b, c$. One can computably determine which of the three cases holds. In the second case simply map $v$ to $v_{f^{-1}(v)}$, in the third case one can computably determine whether $v$ is used to define $a, b, c$ and, using $F$ and $G$, computably find the corresponding element in $F(G(\mathcal{G}))$. In the first case, we have to find the tuple $\bar{w}$ such that $v$ is involved in the coding of the relation $R_{i}$ on $\bar{w}$. We then map $v$ to the corresponding element in the coding of $R_{i}$ on the tuple $v_{f^{-1}(\bar{w})}$. It is easy to see that one can define a Turing operator $\Lambda_{\mathfrak{F}(\mathfrak{K})}$ computing this isomorphism. The Turing operators $\Lambda_{\mathfrak{F}(\mathfrak{K})}$ and $\Lambda_{\mathfrak{K}}$ will witness the effective isomorphism between $F \circ G$ and the identity on $F(\mathfrak{K})$, respectively, the effective isomorphism between $G \circ F$ and the identity on $\mathfrak{G}$.

It remains to show that the diagrams of Definition 10 commute and that for all $\mathcal{A} \in \mathfrak{K}$ and $\mathcal{G} \in F(\mathfrak{K}), F\left(\Lambda_{\mathfrak{K}}^{\mathcal{A}}\right)=\Lambda_{F(\mathfrak{K})}^{F(\mathcal{A})}$ and $G\left(\Lambda_{F(\mathfrak{K})}^{\mathcal{G}}\right)=\Lambda_{\mathfrak{K}}^{G(\mathcal{G})}$. For the commutation of the diagrams, say first that $\mathcal{A}, \hat{\mathcal{A}} \in \mathfrak{K}$ with $\iota: \mathcal{A} \preccurlyeq \hat{\mathcal{A}}$. Let $h: \mathcal{A} \rightarrow$ $F(\mathcal{A})$ and $\hat{h}: \hat{\mathcal{A}} \rightarrow F(\mathcal{A})$ given by $h, \hat{h}: a \mapsto v_{a}$. We have not given an explicit definition of $F(\iota)$ yet. But notice that $F(\iota)$ is uniquely determined by the way it maps the elements $v_{a}$. In particular, if $\nu(x)=F(\iota)(x)$ on the elements with $a \longmapsto x$, then $\nu=F(\iota)$. Thus we have that

$$
G(F(\iota))=\hat{f}^{-1} \circ F(\iota) \circ f=\hat{f}^{-1} \circ \hat{h} \circ \iota \circ h^{-1} \circ f,
$$

and $\Lambda_{\mathfrak{K}}^{\mathcal{A}}=f^{-1} \circ h$, so

$$
\Lambda_{\mathfrak{K}}^{\hat{\mathcal{A}}} \circ \iota=\hat{f}^{-1} \circ \hat{h} \circ \iota=G(F(\iota)) \circ \Lambda_{\mathfrak{K}}^{\mathcal{A}}
$$

and thus $G \circ F$ is effectively isomorphic to $i d_{\mathfrak{K}}$. Now, say $\mathcal{G}, \hat{\mathcal{G}} \in F(\mathfrak{K})$ with $\eta: \mathcal{G} \preccurlyeq \hat{\mathcal{G}}$. First let $x \in \mathcal{G}$ with $a \longmapsto x$. Let $h$, and $\hat{h}$ be as above, then $\left(\Lambda_{F(\mathfrak{K})}^{\hat{\mathcal{G}}} \circ \eta\right)(x)=$

$$
\begin{aligned}
& \left(\hat{h} \circ \hat{f}^{-1} \circ \eta\right)(x) \text { and } F\left(\hat{f}^{-1} \circ \eta \circ f\right)=\hat{h} \circ \hat{f}^{-1} \circ \eta \circ f \circ h^{-1}, \text { so } \\
& \quad\left(F(G(\eta)) \circ \Lambda_{F(\mathfrak{K})}^{\mathcal{G}}\right)(x)=\left(F\left(\hat{f}^{-1} \circ \eta \circ f\right) \circ h \circ f^{-1}\right)(x)=\left(\hat{h} \circ \hat{f}^{-1} \circ \eta\right)(x)
\end{aligned}
$$

Having established that the diagram commutes on the restricted universes we use the fact that any embedding is determined by these parts of the universes to obtain that $F \circ G$ is effectively isomorphic to $i d_{F(\mathfrak{K})}$.

To verify the last condition in Definition 11 let $\mathcal{A} \in \mathfrak{K}$, then on $\{x: x \mapsto a\}$

$$
F\left(\Lambda_{\mathfrak{K}}^{\mathcal{A}}\right)(x)=\left(\hat{h} \circ f^{-1} \circ h \circ h^{-1}\right)(x)=\left(\hat{h} \circ f^{-1}\right)(x)=\Lambda_{F(\mathfrak{K})}^{F(\mathcal{A})}(x)
$$

and as there is a unique extension of this to a mapping $F(\mathcal{A}) \rightarrow F(G(F(\mathcal{A})))$ $F\left(\Lambda_{\mathfrak{K}}^{\mathcal{A}}\right)=\Lambda_{F(\mathfrak{K})}^{F(\mathcal{A})}$. At last, let $\mathcal{G} \in F(\mathfrak{K})$, then

$$
G\left(\Lambda_{F(\mathfrak{K})}^{\mathcal{G}}\right)=\hat{f}^{-1} \circ h \circ f^{-1} \circ f=\hat{f}^{-1} \circ h=\Lambda_{\mathfrak{K}}^{G(\mathcal{G})}
$$

where $f$ is the enumeration of $\{x: x \longmapsto a\}$ in $\mathcal{G}$ and $\hat{f}$ the one in $G(F(\mathcal{G}))$.
The following is a direct consequence of Proposition 13 and Theorem 15
Corollary 16. For every structure $\mathcal{A}$, there is a graph $\mathcal{G}_{\mathcal{A}}$ such that

$$
D g S p_{\approx}(\mathcal{A})=D g S p_{\cong}\left(\mathcal{G}_{\mathcal{A}}\right)
$$

Unfortunately, for the reduction from bi-embeddability on graphs to elementary bi-embeddability on $\mathfrak{C}$ given in Section 3.1 we can not deduce that $(\mathfrak{G}, \hookrightarrow) \leq_{C B F}$ $(\mathfrak{C}, \preccurlyeq)$. However, we can still establish a relationship between the degree spectra in these classes. Recall that $\mathcal{S}_{0}$ is the substructure of $\mathcal{S}$ generated by the constant string of 0 's and $\mathcal{S}_{1}$ is the substructure generated by the constant string of 1 's.
Lemma 17. Let $X$ be $\Delta_{2}^{0}(Y)$ for some set $Y$. Then there exist a sequence of structures $\left(\mathcal{C}_{i}\right)_{i \in \omega}$, uniformly computable in $Y$, such that for all $i \in \omega$

$$
\mathcal{C}_{i} \cong \begin{cases}\mathcal{S}_{0} & \text { if } i \in X \\ \mathcal{S}_{1} & \text { if } i \notin X\end{cases}
$$

Proof. As $X$ is $\Delta_{2}^{0}$ there is an $X$-computable two valued function $f$ such that

$$
\lim _{s \rightarrow \infty} f(i, s)= \begin{cases}0 & \text { if } i \in X \\ 1 & \text { if } i \notin X\end{cases}
$$

Define a structure $\mathcal{C}$ as follows. Fix an enumeration $g$ of $2^{<\omega}$. At stage 0 define $\mathcal{C}_{0}$ to be the partial structure containing one element $a$ on which no relation holds and leave all function symbols undefined. Say we have defined the structure $\mathcal{C}_{s}$. At stage $s+1$ we look at $f(i, j)$ for $j<s$ and define $\mathcal{C}_{s+1}$ as if $a$ was the finite string with $a(j)=f(j)$ for $j<s$. To be more precise:
(1) For all $k$, if $k \leq s$ and $|g(k)| \leq s$ then let $R_{g(k)}(a)$ if and only if $g(k) \preceq a$, and if $F_{g(k)}(a)$ has not been defined yet add a new element and set $F_{g(k)}(a)$.
(2) We may assume by induction that for all elements $b$ in $\mathcal{C}_{s+1}$ there is $k \leq s$ such that $b=F_{g(k)}(a)$. We set $R_{g(l)}(b)$ respecting this equation for all $l \leq s$.
It is easy to see that this procedure yields a computable sequence of structures $\mathcal{C}_{s}$ with $\mathcal{C}_{s} \subseteq \mathcal{C}_{s+1}$ and a computable structure as its limit. We let $\mathcal{C}$ be this structure. $\mathcal{C}$ contains an element $a$ such that $\mathcal{A} \models R_{\sigma}(a)$ if and only if $a \preceq f(i,-)$ and all other elements are equal to $F_{\tau}(a)$ for some $\tau \in 2^{<\omega}$. Thus, in particular if $\lim f(i, s)=0$,
then there is an element representing the constant string of 0 's in $\mathcal{A}$ and otherwise there is an element representing the constant string of 1's in $\mathcal{A}$. Let $\mathcal{C}_{i}=\mathcal{C}$, then $\mathcal{C}_{i} \cong \mathcal{S}_{0}$ if and only if $i \in X$ and $\mathcal{C}_{i} \cong \mathcal{S}_{1}$ if and only if $i \notin X$ as required.

We use the usual category theoretic definition of pseudo-inverse. Two functors $F: \mathfrak{C} \rightarrow \mathfrak{D}$ and $G: \mathfrak{D} \rightarrow \mathfrak{C}$ are pseudo-inverses if $F \circ G$ is naturally isomorphic to $i d_{\mathfrak{D}}$ and $G \circ F$ is naturally isomorphic to $i d_{\mathfrak{C}}$.

Recall that a structure $\mathcal{A}$ is automorphically trivial if there is a finite set $D \subseteq A$ such that every permutation of $A$ that fixes $D$ pointwise is an automorphism. Knight [15] showed that isomorphism spectra of automorphically trivial structures contain only one Turing degree and that the isomorphism spectra of automorphically non-trivial structures are upwards closed in the Turing degrees. In [8] the authors showed that if $\mathcal{A}$ is automorphically trivial and $\mathcal{B} \approx \mathcal{A}$, then $\mathcal{B} \cong \mathcal{A}$. Thus, as every bi-embeddability and elementary bi-embeddability spectrum is a union of isomorphism spectra, Knight's result carries over to this setting.

Lemma 18. For every automorphically non-trivial structure $\mathcal{G} \in \mathfrak{G}$ there is $\mathcal{A} \in \mathfrak{C}$ such that

$$
D g S p_{\cong}(\mathcal{A})=\left\{X: X^{\prime} \in D g S p_{\approx}(\mathcal{G})\right\}
$$

Proof. Recall the reduction from embeddability on graphs to elementary embeddability on $\mathfrak{C}$ given in Section 3.1. It is easy to see that it induces a computable functor $F:(\mathfrak{G}, \approx) \rightarrow(\mathfrak{C}, \approx)$. We show that the functor has a pseudo-inverse $G$ on the $\approx$-saturation of $F(\mathfrak{G})$ and then use Lemma 17 to obtain the lemma. The minimality of the submodels $\mathcal{S}$ used in the construction of $F$ will play a crucial role in the proof.

Say $\mathcal{B} \approx F(\mathcal{G})$ for $\mathcal{G} \in \mathfrak{G}$, that $x, y$ are vertices in $\mathcal{B}$ and $\mathcal{S}_{(x, y)}$ is the substructure on the elements satisfying $O(x, y,-)$ in the reduct to the language of $\mathcal{S}$. We have that either $\mathcal{S}_{(x, y)} \cong \mathcal{S}_{0}$ or $\mathcal{S}_{(x, y)} \cong \mathcal{S}_{1}$ since it elementary embeds into $\mathcal{S}_{(u, v)}$ for some $u, v \in F(\mathcal{G})$ and $\mathcal{S}_{(u, v)} \cong \mathcal{S}_{0}$ or $\mathcal{S}_{(u, v)} \cong \mathcal{S}_{1}$ by minimality. Thus, we get a graph $G(\mathcal{B})$ from $\mathcal{B}$ by defining an edge between two vertex variables $x, y$ from $\mathcal{B}$ if and only if $\mathcal{S}_{(u, v)} \cong \mathcal{S}_{0}$. Clearly every elementary embedding of $\mathcal{B}$ in $F(\mathcal{G})$ yields an embedding of $G(\mathcal{B})$ in $\mathcal{G} \cong G(F(\mathcal{G}))$ and the analogous fact is true for every elementary embedding of $F(\mathcal{G})$ in $\mathcal{B}$. Likewise, we can argue that $F(G(\mathcal{A})) \cong \mathcal{A}$ for every $\mathcal{A} \in F(\mathfrak{G})$. Thus $G$ and $F$ are pseudo-inverses.

However, notice that $G$ is not effective. Within one jump over the diagram of any $\mathcal{B} \in F(\mathfrak{G})$ we can compute $G(\mathcal{B})$ as the isomorphism types of $\mathcal{S}_{1}$ and $\mathcal{S}_{0}$ are definable by $\Sigma_{2}^{c}$ formulas in $\mathfrak{C}$. This implies that for all $\mathcal{A} \in \mathfrak{G}, F(\mathcal{A})^{\prime} \geq_{T} G(F(\mathcal{A})) \cong \mathcal{A}$. So, in particular,

$$
\begin{equation*}
D g S p_{\approx}(F(\mathcal{A})) \geq_{s}\left\{X: X^{\prime} \in D g S p_{\approx}(G(F(\mathcal{A})))=D g S p_{\approx(\mathcal{A})}\right\} \tag{1}
\end{equation*}
$$

On the other hand, let $X \in D g S p_{\approx}(\mathcal{A})$ and $\hat{\mathcal{A}} \approx \mathcal{A}$ such that $\hat{\mathcal{A}} \equiv_{T} X$. Then by Lemma 17 for every $Y$ with $Y^{\prime} \geq_{T} X$, there is $\mathcal{B} \cong F(\hat{\mathcal{A}})$ with $\mathcal{B} \equiv_{T} Y$. This process is uniform in $\mathcal{B}$ and $Y$. Thus

$$
\begin{equation*}
D g S p_{\approx}(F(\mathcal{A})) \leq_{s}\left\{X: X^{\prime} \in D g S p_{\approx}(\mathcal{A})\right\} \tag{2}
\end{equation*}
$$

As both bi-embeddability and elementary bi-embeddability spectra of automorphically non-trivial structures are upwards closed, Eqs. (1) and (2) imply that

$$
D g S p_{\approx}(F(\mathcal{A}))=\left\{X: X^{\prime} \in D g S p_{\approx}(\mathcal{A})\right\}
$$

Theorem 3 follows directly from Lemma 18 and Corollary 16
We note that Theorem 3 may not be optimal. Using a different proof one might be able to get an even stronger relationship between the spectra realized by biembeddability on graphs and elementary bi-embeddability on graphs. We thus ask.

Question 1. Is every bi-embeddability spectrum of a graph the elementary biembeddability spectrum of a graph and vice versa?

One way to answer this question positively is by showing that if $X$ is an elementary bi-embeddability spectrum then so is $X^{\prime}=\left\{x^{\prime}: x \in X\right\}$. This is true for isomorphism spectra and usually shown by considering an appropriate definition for the jump of a structure. However, all known definitions do not preserve elementary embeddability (and not even elementary equivalence). We thus ask.
Question 2. Let $X$ be the elementary bi-embeddability spectrum of a graph. Is $X^{\prime}$ the elementary bi-embeddability spectrum of a graph?
Question 3. Let $X$ be the theory spectrum of a graph. Is $X^{\prime}$ the theory spectrum of a graph?

Also, while Theorem [15 shows that graphs are complete for elementary biembeddability spectra, it is unknown whether the same is true for bi-embeddability.

Question 4. Is every bi-embeddability spectrum of a structure realized as the biembeddability spectrum of a graph?

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Department of Mathematics, University of California, Berkeley and Institute of Discrete Mathematics and Geometry, Technische Universität Wien

Email address: dino@math.berkeley.edu


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