Initial segments of the degrees of ceers

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INITIAL SEGMENTS OF THE DEGREES OF CEERS

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Abstract. It is known that every non-universal self-full degree in the structure of the degrees of computably enumerable equivalence relations (ceers) under computable reducibility has exactly one strong minimal cover. This leaves little room for embedding wide partial orders as initial segments using self-full degrees. We show that considerably more can be done by staying entirely inside the collection of non-self-full degrees. We show that the poset $\langle \omega^{<\omega}, \subseteq \rangle$ can be embedded as an initial segment of the degrees of ceers with infinitely many classes. A further refinement of the proof shows that one can also embed the free distributive lattice generated by the lower semilattice $\langle \omega^{<\omega}, \subseteq \rangle$ as an initial segment of the degrees of ceers with infinitely many classes.

1. Introduction

Computably enumerable equivalence relations appear quite often in mathematical logic and effective mathematics. For instance they appear as relations of provable equivalence of formal systems, and as word problems or isomorphism problems of effectively presented structures. A useful and natural way to compare the relative complexity of ceers is by computable reducibility (or, simply, reducibility) of equivalence relations on the set ω of natural numbers: If A, B are equivalence relations on ω, then A is *computably reducible* (or, simply, *reducible*) to B (notation: $A \leqslant B$) if there is a computable function f such that $x \ A \ y$ if and only if $f(x) \ B \ f(y)$, for all $x, y \in \omega$. Most of the initial investigations of ceers under the reducibility $\leqslant$ were oriented towards identifying universal ceers in logic and algebra. A ceer A is called *universal* if $B \leqslant A$ for every ceer B. For instance, provable equivalence in Peano Arithmetic, or in other related systems, gives a universal ceer [7]. C. F. Miller III [15] proved that there exists a finitely presented group G such that its equality is a universal ceer; he also proved that the isomorphism relation between finite presentations of groups is a universal ceer. A comprehensive survey on universal ceers can be found in [2]. There has also been study of the relationship between ceers and the c.e. algebraic structures which have the ceer as its domain, see, e.g., [8, 11, 10, 12].

When restricted to ceers, the reducibility $\leqslant$ gives rise to a degree structure called **Ceers** of which the degree of the universal ceers is the greatest element. We say that two equivalence relations A and B on ω are *equivalent* (denoted by $A \equiv B$) if $A \leqslant B$ and $B \leqslant A$. The *degree* of A is the set of the equivalence relations that are equivalent to A. The degrees are partially ordered by the partial ordering relation induced by computable reducibility. If

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P is a property of equivalence relations, then we say that a degree *has property* P if some member of the degree has property P. In the other direction, we will often transfer to ceers properties that should be more appropriately understood on degrees such as order-theoretic properties like the property of being a strong minimal cover, etc. In particular, given ceers A, B, we say that A is *a strong minimal cover* of B, if $B < A$ and for every $C \leqslant A$, either $C \equiv A$, or $C \leqslant B$.

The first paper explicitly directed to a systematic investigation of the above defined degree structure **Ceers** was [9]. Andrews and Sorbi provide in [5] a thorough investigation of this structure, with emphasis on existence and non-existence of meets and joins, minimal and strong minimal covers, definable classes of degrees, and automorphisms. They propose a partition of the ceers into three classes: the *finite* ceers (i.e. the ceers with finitely many equivalence classes), the *light* ceers (i.e. those ceers A such that $\mathrm{Id} \leqslant A$, where Id is the identity ceer), and the remaining ceers (called *dark* ceers). They show that no pair of incomparable dark ceers has join or meet. The same authors show in [4] that the first order theory of the poset **Ceers** as well as the theories of the sub-posets **Light** and **Dark**, comprised of the degrees of light and dark ceers, respectively, are computably isomorphic to true first order arithmetic.

For the convenience of the reader here we collect some definitions about ceers.

Definition 1.
- *The uniform join operation* $\oplus$ *is the operation on equivalence relations defined by:* $X \oplus Y = \{(2x, 2y) : x \; X \; y\} \cup \{(2x+1, 2y+1) : x \; Y \; y\}$.
- *A ceer* A *is* self-full *if whenever* f *is a reduction of* A *to* A *then* $\mathrm{range}(f)$ *intersects all* A*-equivalence classes; otherwise* A *is* non-self-full.
- *Equivalently (*[5, Observation 4.2]*), a ceer* A *is self-full if and only if* $A \oplus \mathrm{Id}_1 \not\leqslant A$*, where* Id_1 *is the ceer with only one equivalence class.*
- *The ceer* Id *is defined by equality, i.e.* $x \; \mathrm{Id} \; y$ *if and only if* $x = y$.
- *A ceer* A *is* finite *if it has only finitely many classes.*
- *A ceer* A *is* light *if* $\mathrm{Id} \leqslant A$.
- *A ceer* A *is* dark *if it is neither finite nor light.*
- *The posets* ***Fin***, ***Light*** *and* ***Dark*** *are the degree structures of the finite, light, and dark ceers. We write* $\boldsymbol{Ceers} \setminus \boldsymbol{Fin}$ *for the degree structure of ceers which have infinitely many classes.*

1.1. **Non-self-full strong minimal covers: Towards a theory of initial segments for the structure of ceers.** What is still missing is a satisfying theory of initial segments of **Ceers**, which leaves our understanding of the structure far behind our understanding of other familiar degree structures, possessing already well established theories of initial segments. Very little is indeed known in this regard about **Ceers**, besides the observations on minimal covers and strong minimal covers in [5], or the observation [3] that there is an initial segment I of the light degrees (namely those between the degree of Id, and the degree of R_K, where $x \; R_K \; y$ if and only if $x = y$, or x, y both belong to the halting set K) such that one can embed every finite distributive lattice as an initial segment I' of I. Note that I is not an initial segment in **Ceers**, but only in the light degrees, so this does not imply results about initial segments in **Ceers**. This follows from the fact that the 1-degrees of the non-simple c.e. sets can be isomorphically embedded onto I [3, Theorem 2.4] and that

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every finite distributive lattice can be embedded as an initial segment of these c.e. 1-degrees [13].

This paper aims at giving a first contribution to fill in this gap. For this it is very important to develop new techniques for building strong minimal covers. In the structure **Ceers** we know that every non-universal degree has a strong minimal cover. In particular, this splits into two cases, based on whether the non-universal degree is *self-full*.

If a degree $\mathbf{d}$ is self-full, then it has a strong minimal cover $\mathbf{e}$ such that, for any other degree $\mathbf{x}$, if $\mathbf{x} > \mathbf{d}$, then $\mathbf{x} \geqslant \mathbf{e}$ [5, Lemma 4.5]. Thus every self-full degree has exactly one strong minimal cover. On the other hand, we know [5, Corollary 7.11] that every non-universal non-self-full degree $\mathbf{d}$ has infinitely many incomparable self-full strong minimal covers $\mathbf{e}_1, \mathbf{e}_1, \ldots$. Since these strongly minimal covers are self-full, they each have exactly one strong minimal cover. With an eye towards understanding the initial segments of **Ceers**, unfortunately this does not help us embed wide posets as initial segments of the structure. In particular, while we have infinitely many strong minimal covers of the degree of Id (which, we recall, is the ceer defined by equality), each of these only has one strong minimal cover, which allows an embedding of the tree $\omega^{\leqslant 1}$, but not of $\omega^{\leqslant 2}$, where $\omega^{\leqslant n} = \{\alpha \in \omega^{<\omega} : |\alpha| \leqslant n\}$ (where $|\alpha|$ denotes the length of α).

We will show however (Theorem 6) that non-self-full ceers C with the extra property that $C \oplus \mathrm{Id} \equiv C$ each have infinitely many incomparable strong minimal covers A which are also non-self-full and $A \oplus \mathrm{Id} \equiv A$.

Thus, we get an embedding of the poset $\langle \omega^{<\omega}, \subseteq \rangle$ of the finite strings of numbers (where $\tau \subseteq \sigma$ if τ is an initial segment of σ) as an initial segment of $\mathbf{Ceers} \setminus \mathbf{Fin}$. Note that since **Fin** has order type ω and is bounded by every other ceer, classifying the initial segments in **Ceers** is equivalent to classifying the initial segments of $\mathbf{Ceers} \setminus \mathbf{Fin}$.

Further, in Corollary 33 we extend this embedding to an embedding of the free distributive lattice generated by $\omega^{<\omega}$ viewed as a lower semilattice as an initial segment J of $\mathbf{Ceers} \setminus \mathbf{Fin}$ (see Definition 25). We note in Observation 34 that the embedding we find is not a lattice-embedding (i.e. the degrees in question do not have joins in the ceers, though they do in J).

We leave the following questions open:

Question 1. *Does every non-self-full ceer have a non-self-full strong minimal cover?*

Note that it follows from [1, Corollary 3.3.4] that the assumption used in this paper that $C \oplus \mathrm{Id} \equiv C$ is strictly stronger than non-self-fullness.

Question 2. *If C is non-self-full and $C \oplus \mathrm{Id} \equiv C$, then does C have incomparable strong minimal covers A_1 and A_2 so that $A_i \oplus \mathrm{Id} \equiv A_i$ and A_0, A_1 have supremum in the structure of ceers?*

1.2. **Notations and terminology.** Our notations and terminology from computability theory are standard and can be found in [16] or [17]. If A is an equivalence relation on ω and $V \subseteq \omega$, then the *A-closure* of V is $[V]_A = \{x : \exists y \in V\ (x\ A\ y)\}$.

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We recall the notion of *restriction* of a ceer to a c.e. set, see [5, § 2.3]. If A is a ceer and W is a nonempty c.e. set then fix a computable surjection $\pi : \omega \to W$, and define $A{\restriction}W$ to be the ceer

$$x \; A{\restriction}W \; y \Leftrightarrow \pi(x) \; A \; \pi(y).$$

It is immediate to see that up to $\equiv$, $A{\restriction}W$ does not depend on the chosen computable surjection.

Lemma 2. *Let A, B be ceers:*

(1) For every nonempty c.e. set W, $A{\restriction}W \leqslant A$;
(2) $A \leqslant B$ if and only if there exists a nonempty c.e. set W such that $A \equiv B{\restriction}W$;
(3) If U, V are c.e. sets and for every $u \in U$, there is some $v \in V$ so that $u \; A \; v$, then $A{\restriction}U \leqslant A{\restriction}V$.

Proof. The second follows from the fact that if f is a reduction of A to B then $A \equiv B{\restriction}W$, where $W = \text{range}(f)$. For the last claim, define a map from $A{\restriction}U$ to $A{\restriction}V$ as follows. Fix $\pi : \omega \to U$ and $\varphi : \omega \to V$. Then for every V-class X, if the range of π intersects X then so does the range of φ. So, for every n, define $g(n)$ to be the first m seen so that $\pi(n) \; A \; \varphi(m)$. It is straightforward to check that this is a reduction of $A{\restriction}U$ (as defined using π) to $A{\restriction}V$ (as defined using φ). The first condition follows from the third with $V = \omega$. □

Definition 3. *We generalize the uniform join operation to finitely many summands. Let $(X_i)_{i<n}$ be equivalence relations, with $n \geqslant 1$. For each $i < n$, let $\omega_{i,n} = \{z \in \omega : z \equiv i \mod n\}$, and for $x \in \omega_{i,n}$ let $(x)_{i,n}$ be $\frac{x-i}{n}$. Then let*

$$x \; X_0 \oplus \cdots \oplus X_{n-1} \; y \Leftrightarrow (\exists i < n \,) \, [x, y \in \omega_{i,n} \,\&\, (x)_{i,n} \; X_i \; (y)_{i,n}].$$

If $f_0, \ldots f_{n-1}$ are computable functions from ω to ω, then define $\oplus_i f_i$ to be the function given by $\oplus_i f_i(x) = n \cdot f_i((x)_{i,n}) + i$ if $x \in \omega_{i,n}$.

For A any ceer and $\sim$ any c.e. subset of ω^2, define $A_{/\sim}$ to be the equivalence relation generated by A and $\sim$. Note that if X (here, $X = A \cup \sim$) is a c.e. set of pairs, then the equivalence relation E generated by X is defined by $x \; E \; y$ if and only if

$$\exists n \exists z_1, \ldots z_n \left(z_1 = x \wedge z_n = y \wedge \bigwedge_{i=1}^{n-1} (z_i, z_{i+1}) \in X \right)$$

so E is a ceer.

If A is any ceer and $\sim$ any c.e. subset of ω^2, we say that $\sim$ is A-closed if whenever $x \sim y$ and $x \; A \; x'$ and $y \; A \; y'$, then also $x' \sim y'$. That is, $\sim$ collapses whole A-classes together.

If f is a computable function from ω to ω, and $\sim$ is a c.e. subset of ω^2, define $\sim_f = \{(a, b) : (f(a), f(b)) \in \sim\}$.

Lemma 4. *If f is a reduction of A to B and $\sim$ is a transitive c.e. subset of ω^2 which is B-closed, then $A_{/\sim_f} \leqslant B_{/\sim}$.*

Proof. We will see that f is a reduction of $A_{/\sim_f}$ to $B_{/\sim}$.

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Suppose $a\ A_{/\sim_f}\ b$. Then there are $z_1, \dots z_n$ so that $z_1 = x \wedge z_n = y \wedge \bigwedge_{i=1}^{n-1}(z_i, z_{i+1}) \in A \cup \sim_f$. Then $f(z_1) = f(x) \wedge f(z_n) = f(y) \wedge \bigwedge_{i=1}^{n-1}(f(z_i), f(z_{i+1})) \in B \cup \sim$. So, $f(a) B_{/\sim} f(b)$.

Since $\sim$ is transitive and B-closed, $B \cup \sim$ is an equivalence relation, so $B_{/\sim} = B \cup \sim$. So, if $f(a)\ B_{/\sim}\ f(b)$, then either $f(a)\ B\ f(b)$ or $f(a) \sim f(b)$. In either case, we have $a\ A_{/\sim_f}\ b$. □

Applying the above to the case of a uniform join, we get:

Lemma 5. *If $f_0, \dots, f_{n-1}$ are reductions witnessing $A_i \leqslant B_i$ for $i < n$ and $\sim$ is a transitive c.e. subset of ω^2 which is $B_0 \oplus \cdots \oplus B_{n-1}$-closed, then*

$$(A_0 \oplus \cdots \oplus A_{n-1})_{/\sim_{\oplus_i f_i}} \leqslant (B_0 \oplus \cdots \oplus B_{n-1})_{/\sim}.$$

2. Non-Self-full Strong Minimal Covers

Since a ceer A is self-full if and only $A \oplus \mathrm{Id}_1 \not\leqslant A$, a ceer A satisfying $A \equiv A \oplus \mathrm{Id}$, such as the one we construct in the next theorem, is non-self-full.

Theorem 6. *Given a ceer C so that $C \equiv C \oplus \mathrm{Id}$ and a non-universal ceer $B \geqslant C$, there exists a ceer A which is a strong minimal cover of C so that $A \equiv A \oplus \mathrm{Id} \not\leqslant B$.*

Proof. Let C, B be as in the statement of the theorem. We want to build A so as to satisfy the following requirements:

$$\begin{aligned}
&\mathrm{NSF} : A \equiv A \oplus \mathrm{Id},\\
&\mathrm{D}_i : \varphi_i \text{ is not a reduction from } A \text{ to } B,\\
&\mathrm{R} : C \leqslant A,\\
&\mathrm{SMC}_i : A \leqslant A\upharpoonright W_i \vee A\upharpoonright W_i \leqslant C.
\end{aligned}$$

As already observed, NSF is a strictly stronger requirement than ensuring that A is non-self-full. Satisfaction of the D-requirements guarantees that $A \not\leqslant B$. The R requirement in conjunction with the D-requirements ensure that $C < A$. In particular, we cannot have $A \leqslant C$ since $C \leqslant B$ and the D-requirements ensure $A \not\leqslant B$. Finally if $X \leqslant A$, and by Lemma 2 (2) $X \equiv A\upharpoonright W_i$ for some i, then requirement SMC_i guarantees that either $A \leqslant X$ and thus $A \equiv X$, or $X \leqslant C$. Hence satisfaction of all SMC-requirements yields that A is a strong minimal cover of C.

As we construct the ceer A, we begin with $A_0 = \mathrm{Id}$ and as stages go by, we say we *A-collapse*, or often just say *collapse*, elements n and m. This means that at stage $s+1$, we let A_{s+1} be the equivalence relation generated by A_s along with the pairs that we collapse during stage s.

2.1. Informal description of the strategies to satisfy requirements.

The NSF *strategy*. We will fix a computable infinite set I_λ and we will ensure that if $x \in I_\lambda$, then $[x]_A = \{x\}$. This ensures that $A \equiv \mathrm{Id} \oplus (A\upharpoonright \overline{I_\lambda})$, so $A \oplus \mathrm{Id} \equiv A$.

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The D_i*-strategies*. We will have a c.e. set which we will call D_α for some node α, and we will cause collapse on this set only for the purpose of ensuring that φ_i is not a reduction of A to B. We will play a diagonalizing strategy which in a finitary way guarantees that φ_i is not a reduction witnessing $A \leqslant B$. To do this we carry out a finite amount of A-collapsing on the elements of D_α.

We now describe the strategy that α runs on the set D_α. We call this strategy the *finitary-diagonalization strategy*. We fix ahead of time a universal ceer U, with computable approximations $(U_s)_{s\in\omega}$ (namely, $U_0 = \mathrm{Id}$, $U_s \subseteq U_{s+1}$, $U_s \setminus \mathrm{Id}$ is a finite set of pairs of which we can compute the canonical index uniformly in s). Let also $(B_s)_{s\in\omega}$ be a computable approximation (defined in the same way) for the ceer B. We fix the enumeration $D_\alpha = \{a_j : j \in \omega\}$ of D_α. The strategy has a parameter n_α, which begins as $n_\alpha = 0$ and acts at each stage s when α is visited and $\varphi_i(a_j)\ B_s\ \varphi_i(a_k)$ if and only if $a_j\ A_s\ a_k$ for each $a_j, a_k \in D_\alpha$ with $j, k < n_\alpha$. In this case, we increment $n_\alpha = n_\alpha + 1$ and for each $a_j, a_k \in D_\alpha$ with $j, k < n_\alpha$, we collapse $a_j\ A_{s+1}\ a_k$ if and only if $j\ U_s\ k$. This is the only cause for collapse inside D_α. A priori, there are two possible outcomes of this strategy: In the first case, $\lim_s n_\alpha$ is finite, and thus we will never again see that $\varphi_i(a_j)\ B_s\ \varphi_i(a_k)$ if and only if $a_j\ A_s\ a_k$ for each $a_j, a_k \in D_\alpha$ with $j, k < n_\alpha$. Thus φ_i is not a reduction of A to B. In the second case, $\lim_s n_\alpha = \infty$. But then $a_j\ A\ a_k$ if and only if $j\ U\ k$. Thus $j \mapsto \varphi_i(a_j)$ is a reduction of U to B. Since B is non-universal by hypothesis of the theorem, this infinitary outcome is simply impossible. It is to emphasize this fact that we call this strategy the finitary-diagonalization strategy.

The R*-strategy*. We will fix a computable set $K_\lambda = \{x : x \equiv 0 \mod 3\}$ and we will directly encode C onto $A{\restriction}K_\lambda$. Then the map $f(x) = 3x$ will give a reduction of C to A. We note that, as opposed to the NSF-strategy, it will not be the case that $A \equiv A{\restriction}K_\lambda \oplus A{\restriction}\overline{K_\lambda}$. In fact, many nodes α on the true path will be building their own sets K_α which will necessarily have representatives of the same A-classes as K_λ. We will need to build these sets K_α in order to put a copy of C into $A{\restriction}W_i$ for SMC_i-strategies. Further, these must represent the same A-classes as K_λ, because we cannot afford to encode $C \oplus C$, which might be strictly above C.

The SMC_i*-strategies*. Here we use the Chinese boxes technique employed by Lachlan in the proof of [14, Theorem 2]. A node α on the true path will put numbers s into either $S_{\alpha^\frown f}$ or $S_{\alpha^\frown\infty}$. When we see a member of $S_{\alpha^\frown f}$ be A-equivalent to a number in W_i, α will collapse together every number (aside from those A-equivalent to a member of K_λ) in $S_{\alpha^\frown f}$ to a single class with s (the current stage) and put s into $S_{\alpha^\frown\infty}$. In this case, we then make $S_{\alpha^\frown f}$ empty.

Under the outcome that only finitely often puts numbers into $S_{\alpha^\frown\infty}$, the effect of this procedure is that almost every A-class (aside from some copies of Id held by higher-priority strategies, e.g. the NSF-strategy, or sets D_α being used for D_i-strategies, or sets I_β as described in the next paragraph) will be represented by members of $S_{\alpha^\frown f} \cup K_\lambda$, and $S_{\alpha^\frown f}$ has no member equivalent to a number in W_i. Then $A{\restriction}W_i$ will have to reduce to $C \oplus \mathrm{Id}$, coming from the elements A-equivalent to K_λ along with the finitely many copies of Id, and $C \oplus \mathrm{Id} \equiv C$.

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Under the outcome that infinitely often puts numbers into $S_{\alpha^\frown\infty}$, we have that the entire set $S_{\alpha^\frown\infty}$ is comprised of members which are A-equivalent to numbers in W_i. Thus $A{\upharpoonright}S_\alpha \leqslant A{\upharpoonright}W_i$. In this case, the goal is to ensure that $A \leqslant A{\upharpoonright}S_\alpha$. We do this by having a copy K_α of K_λ inside S_α and having a copy I_α of Id inside S_α. This suffices to give us a reduction of A to $A{\upharpoonright}S_\alpha$. We send each copy of K_λ into K_α (i.e. we send an x in some copy of K_λ to a representative of its own A-class which is in K_α), and each of the (finitely many) copies of Id being used by higher-priority strategies and I_α itself into I_α. This uses the immediate fact that $\mathrm{Id}\oplus\mathrm{Id}\equiv\mathrm{Id}$.

We now move onto a description of how these strategies fit together into the construction.

2.2. **Informal description of the construction.** We employ an infinite-injury priority construction on the priority tree $\mathrm{Tr} = \{\infty < f\}^{<\omega}$. We use standard notations and terminology about strings. In particular if $\alpha, \beta \in \mathrm{Tr}$ then we write $\alpha <_L \beta$ to mean that α is to the left of β and $\alpha \leqslant \beta$ to denote that either $\alpha <_L \beta$ or $\alpha \subseteq \beta$, the latter meaning that α is an initial segment of β. We write $\alpha \subset \beta$ if $\alpha \subseteq \beta$ and $\alpha \neq \beta$. The empty string is denoted by the symbol λ. If $\alpha \neq \lambda$ then α^- denotes the immediate predecessor of α along α. The symbol $|\alpha|$ denotes the length of α. As usual in computability constructions that use tree arguments, the construction will identify the true path Tp through Tr, that is the unique infinite path through Tr such that for every n, its restriction $\mathrm{Tr}{\upharpoonright}n$ is the leftmost string of length n which is visited in the construction infinitely many times.

2.2.1. *The parameters of α.* Each node $\alpha \in \mathrm{Tr}$ has parameters S_α, K_α, I_α, D_α, M_α. The values of these sets depend of course on the stage, and should therefore be denoted by $S_{\alpha,s}$, $K_{\alpha,s}$, $I_{\alpha,s}$, $D_{\alpha,s}$, $M_{\alpha,s}$, although we will omit specifying the approximating stage unless strictly necessary. If α is on the true path of the construction then the limit value S_α will be an infinite computable set consisting of the numbers which have been enumerated in S_α after the last stage s_α at which α has been initialized if there is any such stage, otherwise $s_\alpha = 0$. There is yet another parameter, a number n_α which pertains only to nodes $\alpha \neq \lambda$ such that $\alpha = (\alpha^-)^\frown f$, and is used in the finitary-diagonalization strategy.

A node $\alpha = (\alpha^-)^\frown\infty$ will be working towards satisfying $\mathrm{SMC}_{|\alpha|-1}$ and will have to ensure that $A \leqslant A{\upharpoonright}W_{|\alpha|-1}$. Such an α will be on the true path only if every element of S_α is A-equivalent to a member of $W_{|\alpha|-1}$. By injuring all strategies to the right, α will ensure that $A \equiv \mathrm{Id}\oplus A{\upharpoonright}S_\alpha \leqslant A{\upharpoonright}S_\alpha \leqslant A{\upharpoonright}W_{|\alpha|-1}$. For the equivalence $A \equiv \mathrm{Id}\oplus A{\upharpoonright}S_\alpha$, it will be essential that each equivalence class of the copy of C which is encoded in A has representatives in S_α. This is precisely the role of the set K_α. Similarly, it is needed that $\mathrm{Id}\oplus A{\upharpoonright}S_\alpha \leqslant A{\upharpoonright}S_\alpha$. This is the role of I_α. On I_α, we will encode a copy of Id which will be unrelated to the rest of S_α precisely to ensure $\mathrm{Id}\oplus A{\upharpoonright}S_\alpha \leqslant A{\upharpoonright}S_\alpha$. For $\alpha = (\alpha^-)^\frown\infty$, D_α will be empty. Finally, M_α is the stream of numbers given for $\alpha^\frown\infty$ and $\alpha^\frown f$ to work with.

A node $\alpha = (\alpha^-)^\frown f$ will automatically have $\mathrm{SMC}_{|\alpha|-1}$ satisfied (if this α is on the true path) and will instead work towards satisfying a D-requirement. This will be done by the finitary-diagonalization strategy on the set D_α. The parameter n_α will describe the progress of the finitary-diagonalization strategy. The sets K_α and I_α are empty, and once again M_α is the stream of numbers for $\alpha^\frown\infty$ and $\alpha^\frown f$ to work with.

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Partitioning S_α. At any stage s, we will have $S_\lambda = \{i : i \leqslant s\} \cup \{x : x \equiv 0 \mod 3\}$, and partition S_λ into $K_\lambda = \{x : x \equiv 0 \mod 3\}$, $I_\lambda = \{x \leqslant s : x \equiv 1 \mod 3\}$, $M_\lambda = \{x \leqslant s : x \equiv 2 \mod 3\}$, and $D_\lambda = \varnothing$. If $\alpha \neq \lambda$ then at stage s, if $s \in M_{\alpha^-}$ then s may enter S_α (in particular, $S_\alpha \subseteq \{x : x \leqslant s\}$). In fact, S_α will be the set of stages at which the node α is visited since its last initialization. At each stage, if $\alpha = (\alpha^-)^\frown\infty$ then $D_\alpha = \varnothing$ and S_α is partitioned by K_α, I_α, M_α. One out of every three elements which enter S_α will be put into K_α, one out of every three will be put into I_α, and one out of every three will be put into M_α. If $\alpha = (\alpha^-)^\frown f$, we define $K_\alpha = I_\alpha = \varnothing$, and S_α is partitioned by D_α, M_α. One out of every two elements which enter S_α will be put into D_α, and one out of every two elements which enter S_α will be put into M_α. The limit value S_α will then turn out to be partitioned by the (limit values of the) sets K_α, I_α, M_α if $\alpha = \lambda$ or $\alpha = (\alpha^-)^\frown\infty$, or D_α, M_α if $\alpha = (\alpha^-)^\frown f$. Moreover, every S_α with $\alpha \neq \lambda$ will be contained in M_{α^-}.

We will also have a single global set K (approximated by K_s at stage s), which is the set of elements which are A-equivalent to a member of K_λ. That is, $K = [K_\lambda]_A$. This will include K_α for every node α. This will even include numbers which enter a set K_α before α is injured. Moreover, K may contain numbers which themselves are never enumerated into any set K_α. This is a consequence of the fact that when we witness injury to α, we will collapse all of the numbers in $S_\alpha \smallsetminus K$ to a single class, and a representative of this class may enter K_β for some β. In this case, we will place the single representative of the class into K_β, but the remainder of the class will be in K, despite never having been enumerated into any set K_γ.

2.2.2. *More formal description of the strategy of a node α in isolation.* We now look at the strategies employed by the nodes on Tr (in fact, to describe the effects of the strategy employed by α, we assume that $\alpha \subset \mathrm{Tp}$ and α works in isolation), describing some procedures which will be used in the formal construction.

Each node α on the tree will be working with the c.e. set $W_{|\alpha|}$ to choose its outcome. Also, α builds S_α, K_α, I_α, D_α, M_α, and constructs particular ceers on K_α, I_α, D_α. We distinguish the three cases $\alpha = \lambda$, $\alpha = (\alpha^-)^\frown\infty$, and $\alpha = (\alpha^-)^\frown f$.

$\alpha = \lambda$*: Winning* R *and* NSF. We reserve K_λ and I_λ to meet the overall requirements R and NSF. Specifically, on K_λ (which will be exactly $\{x : x \equiv 0 \mod 3\}$) we place a copy of C. Thus $x \mapsto 3x$ will give a reduction witnessing $C \leqslant A$, and we let I_λ (which will limit to exactly $\{x : x \equiv 1 \mod 3\}$) be a copy of Id:

- (*Coding C in K_λ*) Towards making the reduction $C \leqslant A$, for every $n \in \omega$ let $x^\lambda_n = 3n$ be the nth element of K_λ. At every stage s, we will A-collapse x^λ_n A x^λ_m if and only if $x^\lambda_n, x^\lambda_m \leqslant s$ and n C_s m. The construction will guarantee that we cause no additional A-collapses on K_λ. The notation x^λ_n here refers to the nth element of K_λ. For nodes $\alpha = \alpha^{-\frown}\infty$, we will have x^α_n similarly refer to the nth element of K_α.
- We let I_λ be a copy of Id, by never A-collapsing throughout the construction any pair of distinct elements of I_λ. We will not collapse these elements with any other elements, so Lemma 19 will guarantee that $A \equiv A \oplus \mathrm{Id}$.

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$\alpha = (\alpha^-)^\frown\infty$. If we visit α^- at stage s and do not end the stage, it is because s enters M_{α^-}. Suppose that at infinitely many stages, α^- takes outcome ∞. At stage s, α^- takes outcome ∞ if and only if at that stage $W_{|\alpha^-|}$ contains an element $x \in [S_{\alpha^-{}^\frown f}]_A$ which is not in K. We will call such stages α^-*-expansionary*. In this case all elements of $S_{\alpha^-{}^\frown f}$ which are not in K will collapse together into the class of s, and we enumerate s into S_α. In this case, we will make $S_{\alpha^-{}^\frown f} = \varnothing$ as we injure $\alpha^-{}^\frown f$. This means that the x we found in $W_{|\alpha^-|} \cap [S_{\alpha^-{}^\frown f}]_A$ must be A-equivalent to a number which entered $S_{\alpha^-{}^\frown f}$ since the last time α was visited. This s then enters one of K_α, I_α, or M_α. If we take this outcome infinitely often, then S_α will be an infinite computable set.

- *K-procedure at α.* If s becomes the nth element of K_α (we express this by writing $s = x^\alpha_n$) then we A-collapse s A x^λ_n. Moreover we will cause no additional A-collapse on the elements of K_α apart from the ones inherited from C through K_λ.
- *The equivalence $A{\restriction}I_\alpha \equiv \mathrm{Id}$.* We will cause no additional A-collapse on the elements of I_α, so they will be pairwise non-A-equivalent as they were when first enumerated in I_α, and thus $A{\restriction}I_\alpha \equiv \mathrm{Id}$.
- *The reduction $A \leqslant A{\restriction}S_\alpha$.* Since every element of S_α is in a class which intersects $W_{|\alpha^-|}$, a reduction $A \leqslant A{\restriction}S_\alpha$ yields $A \leqslant A{\restriction}W_{|\alpha^-|}$ by Lemma 2(3), thus satisfying the requirement $\mathrm{SMC}_{|\alpha^-|}$. Assuming α is on the true path, ω will be partitioned into $[S_\beta]_A \smallsetminus K$ for $\beta <_L \alpha$, $[I_\gamma]_A, [D_\gamma]_A$ for $\gamma \subset \alpha$, and $[S_\alpha]_A$ (note that all of K, and thus $[K_\gamma]_A$ for all γ is contained in $[S_\alpha]_A$ as every element of K_γ is collapsed with an element of K_λ, and K_α, being infinite, contains an element x^α_n A-equivalent to x^λ_n for each n). We will show in the Disjointness Lemma below that these blocks are pairwise A-disjoint. Further, this is a computable partition (note that $[S_\beta]_A \smallsetminus K$ for $\beta <_L \alpha$ is a c.e. set since it contains only finitely many classes, since we are assuming that $\alpha \in \mathrm{Tp}$). So, to reduce A to $A{\restriction}S_\alpha$ comes down to reducing each of $[S_\beta]_A \smallsetminus K$ for $\beta <_L \alpha$, $[I_\gamma]_A, [D_\gamma]_A$ for $\gamma \subset \alpha$, and $[S_\alpha]_A$ A-disjointly into S_α. Each of $[S_\beta]_A \smallsetminus K$ for $\beta <_L \alpha$ will be finite ceers. Each of $[I_\gamma]_A, [D_\gamma]_A$ for $\gamma \subset \alpha$ will be either empty or equivalent to Id. Since I_α is a copy of Id, we build a reduction by reducing $[I_\alpha]_A$ along with each of $[S_\beta]_A \smallsetminus K$ for $\beta <_L \alpha$, $[I_\gamma]_A, [D_\gamma]_A$ for $\gamma \subset \alpha$ to I_α. The rest of $[S_\alpha]_A$ gets sent to a member of S_α in its own equivalence class.

$\alpha = (\alpha^-)^\frown f$. Suppose now that $\alpha = \alpha^-{}^\frown f$ is on the true path.

- *The finitary-diagonalization strategy at α.* Strategy α works towards satisfying D_i where i equals the number of bits f occurring in the string α^- (we let $\#(\alpha)$ be this number). That is, $\#(\alpha) = |\{\beta : \beta^\frown f \subset \alpha^-\}|$. Strategy α carries out the finitary-diagonalization strategy described in section 2.1, which we know will cause only finitely many collapses. Therefore, $A{\restriction}D_\alpha$ will be equivalent to Id.
- *The reduction $A{\restriction}W_{|\alpha^-|} \leqslant C$.* We see in this case that $A{\restriction}W_{|\alpha^-|} \leqslant C \oplus \mathrm{Id}$, meeting $\mathrm{SMC}_{|\alpha^-|}$. If $(\alpha^-)^\frown f \subset \mathrm{Tp}$ then $W_{|\alpha^-|} \cap [S_\alpha]_A \smallsetminus K$ is empty. Once again, ω is partitioned into $[S_\beta]_A \smallsetminus K$ for $\beta <_L \alpha$, $[I_\gamma]_A, [D_\gamma]_A$ for $\gamma \subset \alpha$, and $[S_\alpha]_A$. So, $A{\restriction}W_{|\alpha^-|}$ can be written as a uniform join of $A{\restriction}W_{|\alpha^-|} \cap [S_\beta]_A \smallsetminus K$ for $\beta <_L \alpha$, $A{\restriction}W_{|\alpha^-|} \cap [I_\gamma]_A$, $A{\restriction}W_{|\alpha^-|} \cap [D_\gamma]_A$, for $\gamma \subset \alpha$, and $A{\restriction}W_{|\alpha^-|} \cap K$, since $W_{|\alpha^-|} \cap [S_\alpha]_A \smallsetminus K$ is empty. The first of these is a finite ceer, the second reduces to $A{\restriction}[I_\gamma]_A \equiv \mathrm{Id}$,

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the third reduces to $A\upharpoonright[D_\gamma]_A \equiv \mathrm{Id}$, and $A\upharpoonright W_{|\alpha^-|}\cap K \leqslant A\upharpoonright K_\lambda \equiv C$. Thus, $A\upharpoonright W_{|\alpha^-|}$ reduces to a uniform join of finitely many copies of Id, some finite ceers, and C, which reduces to $C\oplus \mathrm{Id}\equiv C$.

2.3. **Formal Construction.** Let s be a stage. We proceed by substeps t at stage s. At substep t of s we define: (1) a string $\alpha_{s,t}\supset \alpha_{s,t-1}$, such that $|\alpha_{s,t}| = t$; (2) the values of the parameters relative to this string; and (3) a new value $A_{s,t}$ of the ceer A. To do this we may need to know the current values of the parameters relative to other strings β, and of A as well. We assume that these values are the ones assigned to them at the end of substep $t-1$ if $t>0$, or at the end of the previous stage if $t=0$. After completing substep t, we may end stage s and go on to stage $s+1$, or we may move on to substep $t+1$.

Remark 7 (Saving on notations)**.** *When describing the actions and the parameters at any substep t of any stage s, for simplicity we will omit specifying the subscript s,t. Thus for instance, we refer to S_α instead of $S_{\alpha,s,t}$ (being clear from the context whether we mean the value at the beginning of the substep, or the value which we define at the end of the substep), and for any set X, $[X]_A$ will stand for $[X]_{A_{s,t}}$, etc.*

A stage $s+1$ is *α-expansionary* if α is visited at that stage, does not end the stage, and $W_{|\alpha|}$ contains an element $x\in [S_{\alpha\hat{}f}]_A\setminus K$. Note that in this case, we will injure $\alpha\hat{}f$, in particular setting $S_{\alpha\hat{}f}=\varnothing$. Thus, x must be A-equivalent to an element which entered $S_{\alpha\hat{}f}$ since the last α-expansionary stage.

Stage 0. Let $\alpha_0=\lambda$. Let $S_\lambda = K_\lambda = \{x : x\equiv 0 \mod 3\}$. Define $x^\lambda_n = 3n$ for every n. All other sets are empty. Let $A_0=\mathrm{Id}$.

Initialize all $\beta\neq\lambda$ by setting $S_\beta = K_\beta = I_\beta = D_\beta = M_\beta = \varnothing$, and $n_\beta = 0$.

Stage $s+1$. Substep 0: Let $\alpha_{s+1,0}=\lambda$. If $s+1\equiv 0 \mod 3$, then we update K_λ by A-collapsing $3x$ with $3y$ if and only if $3x, 3y\leqslant s+1$ and $x\, C_{s+1}\, y$. We then end the stage. If $s+1\equiv 1 \mod 3$, then put $s+1$ into S_λ and I_λ. We then end the stage. Finally, if $s+1\equiv 2 \mod 3$, we put $s+1$ into S_λ and M_λ. We then proceed to the next substep.

Substep $t+1$: After completing stage $s+1$ substep t, having defined $\alpha=\alpha_{s+1,t}$ and the relevant parameters for α without having stopped the stage at t, we distinguish the following two cases:

- If $s+1$ is α-expansionary, then: Let $\alpha_{s+1,t+1}=\alpha\hat{}\infty$. Carry out the following procedures:
  - *Perform the dumping procedure.* A-collapse $S_{\alpha\hat{}f}\setminus K$ into the equivalence class of $s+1$. Enumerate $s+1$ into $S_{\alpha\hat{}\infty}$. We initialize all requirements $>_L \alpha\hat{}\infty$, and in particular we set $S_\beta = K_\beta = I_\beta = D_\beta = M_\beta = \varnothing$ and $n_\beta = 0$ for all $\beta\supseteq \alpha\hat{}f$.
  - *Perform the partition procedure.* If the cardinality $|S_{\alpha\hat{}\infty}|\equiv 1 \mod 3$ then we enumerate $s+1$ into $K_{\alpha\hat{}\infty}$. If $|S_{\alpha\hat{}\infty}|\equiv 2 \mod 3$ then we enumerate $s+1$ into $I_{\alpha\hat{}\infty}$. If $|S_{\alpha\hat{}\infty}|\equiv 0 \mod 3$ then we enumerate $s+1$ into $M_{\alpha\hat{}\infty}$.
  - *Perform the K-procedure.* If $s+1$ was enumerated into $K_{\alpha\hat{}\infty}$ and it is the nth element of $K_{\alpha\hat{}\infty}$, then we write $s+1 = x^{\alpha\hat{}\infty}_n$ and we collapse $s+1\ A\ x^\lambda_n$.

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  - If we put $s+1$ into $K_{\alpha^\frown\infty}$ or $I_{\alpha^\frown\infty}$, end the stage. If we put $s+1$ into $M_{\alpha^\frown\infty}$, then proceed to the next substep.
- If $s+1$ is not α-expansionary then let $\alpha_{s+1,t+1} = \alpha^\frown f$ and enumerate $s+1$ into $S_{\alpha^\frown f}$.
  - *Perform the partition procedure.* If $|S_{\alpha^\frown f}|$ is odd, then we put $s+1$ into $D_{\alpha^\frown f}$. Otherwise, we place $s+1$ into $M_{\alpha^\frown f}$.
  - *Perform the finitary-diagonalization-procedure.* For this, we refer to the informal description of the strategy given earlier as regards notations and terminology. In particular, a_j refers to the jth element in D_α. The finitary-diagonalization strategy at $\alpha^\frown f$ *requires attention* if $|D_\alpha| \geqslant n_\alpha$ and
$$\varphi_{\#\alpha,s}(a_j)\ B_s\ \varphi_{\#\alpha,s}(a_k) \Leftrightarrow a_j\ A\ a_k,$$
for each $a_j, a_k \in D_\alpha$ with $j,k < n_\alpha$. If this happens then we *act* by incrementing $n_\alpha = n_\alpha + 1$ and by A-collapsing $a_j\ A\ a_k$ if and only if $j\ U_s\ k$, for each $a_j, a_k \in D_\alpha$ with $j,k < n_\alpha$.
  - If we put $s+1$ into $D_{\alpha^\frown f}$, end the stage. If we put $s+1$ into $M_{\alpha^\frown f}$, then proceed to the next substep.

Finally let $\alpha_{s+1} = \alpha_{s+1,\hat{t}}$ and $A_{s+1} = A_{s+1,\hat{t}}$, where $\hat{t}$ is the last substep of stage $s+1$ (which we show exists in Lemma 8).

2.4. **Verification.** We first observe that every stage terminates. This is because every node ends the stage when it is visited for the first time.

Lemma 8. *Every stage has a last substep.*

Proof. By induction, we may assume that all previous stages have terminated, thus there are only finitely many α so that S_α is non-empty. Suppose towards a contradiction that stage s does not terminate. Then there is some t so that $S_{\alpha_{s,t}}$ is empty at the beginning of the substep of the stage. Then we make $|S_{\alpha_{s,t}}| = 1$, and therefore s is placed into K_α or D_α and the stage terminates. □

We say that α is *on the true path at stage* s, or s is an α*-true stage* if $\alpha \subseteq \alpha_s$.

Next we verify that we do not accidentally cause more collapse than intended. This will be necessary for instance to show that $A{\restriction}K_\lambda \equiv C$.

Lemma 9. *(1) At every substep t of every stage s and nodes $\gamma \neq \beta$, if $x\ A\ y$ for $x \in S_\gamma$, $y \in S_\beta$ then either $\gamma \subseteq \beta$ and $x \in M_\gamma$, $\beta \subseteq \gamma$ and $y \in M_\beta$, or $x, y \in K$.*
(2) At the beginning of stage $s+1$ substep t, if $s+1 \in M_{\alpha_{s+1,t}}$, and $y \in S_\beta$ for any β and $s+1\ A\ y$, then $\beta \subseteq \alpha_{s+1,t}$ and $y \in M_\beta$.
(3) For each α, $[I_\alpha]_A$, $[D_\alpha]_A$, $[M_\alpha]_A \setminus K$, K are always disjoint sets.

Proof. We prove all three claims by simultaneous induction on substeps of stages. They are clearly true at stage 0 since the only α with $S_\alpha \neq \varnothing$ is λ and no collapse has happened at stage 0.

We consider each of the actions taken during the construction and see that they maintain these conditions. At each step, the substep 0 only introduces a fresh number to the

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construction and can only cause collapse if both numbers are in K_λ. Thus it maintains all three conditions. We consider $\alpha = \alpha_{s+1,t}$ as we take the $t+1$th substep and look at each case. First suppose that $s+1$ is an α-expansionary stage.

During the *dumping procedure*, we put $s+1$ into $S_{\alpha^\frown\infty}$ and collapse all elements of $S_{\alpha^\frown f} \setminus K$ to be A-equivalent with $s+1$. We also make $S_\delta = \varnothing$ for every $\delta \supseteq \alpha^\frown f$. By inductive hypothesis, each element of $S_{\alpha^\frown f} \setminus K$ could only be A-equivalent to a $y \in S_\delta$ if $\delta \supseteq \alpha^\frown f$ or $\delta \subset \alpha^\frown f$ and $y \in M_\delta$. Also, by the inductive hypothesis on the second condition, $s+1$ itself is only A-equivalent to $y \in S_\delta$ if $\delta \subseteq \alpha$ and $y \in M_\delta$. Thus the first condition is maintained by the dumping procedure. The third condition is also preserved since no set I_α, D_α, M_α or K has been increased, and the collapse we caused did not cause collapse between any of these sets by the inductive hypothesis on the first condition. The second condition is changed slightly. Since now $s+1$ itself has entered $S_{\alpha^\frown\infty}$, but is the only representative of its class in $S_{\alpha^\frown\infty}$, we have the additional possibility that $y = s+1$ itself and $\delta = \alpha^\frown\infty$. The condition only talks about the beginning of the substep, so this is not a direct violation of the second condition, but this effects the possibilities we must consider if $s+1$ enters $M_{\alpha^\frown\infty}$ during the partition procedure.

Next, during the *partition procedure*, no collapse is caused and no S_β is changed, so the first condition is preserved. If $s+1$ is not placed into $M_{\alpha^\frown\infty}$, then the second condition holds vacuously. If it is placed in $M_{\alpha^\frown\infty}$, then we had $s+1$ only A-equivalent to $y \in S_\delta$ if $\delta \subseteq \alpha$ and $y \in M_\delta$ or if $y = s+1$ and $\delta = \alpha^\frown\infty$. In the latter case, since $s+1$ entered $M_{\alpha^\frown\infty}$, the condition still holds. Finally, the only new class in $[I_\beta]_A$, $[D_\beta]_A$, $[M_\beta]_A \setminus K$ is the class of $s+1$ itself, by the second condition, $s+1$ is not A-equivalent to any other member of $S_{\alpha^\frown\infty}$, so the third condition is preserved.

The *K-procedure* only occurs if $s+1$ entered $K_{\alpha^\frown\infty}$. In this case, the second condition holds vacuously after this as $s+1 \notin M_{\alpha^\frown\infty}$. In this case, $s+1$ collapses with an element of K_λ. Thus the first statement is preserved because this only collapses a class into K and the statement allows for two elements of K. The third statement is preserved as well, since the inductive hypotheses imply that both $s+1$ and each member of K_λ are A-non-equivalent to any member of $[I_\gamma]_A$, $[D_\gamma]_A$, $[M_\gamma]_A \setminus K$ for any γ.

Now we suppose that $s+1$ is not an α-expansionary stage. All three statements are clearly preserved by adding $s+1$ to $S_{\alpha^\frown f}$. During the *partition procedure* no collapse or entry into any S_β happens, so the first statement is preserved. Since $s+1$ is not A-equivalent to any member of $S_{\alpha^\frown f}$ except $s+1$ itself by the inductive hypothesis on the second condition, the third condition is also preserved. Again the second condition has the new possibility that $y = s+1$ itself and $\delta = \alpha^\frown f$, which maintains the second condition if $s+1$ enters $M_{\alpha^\frown f}$. If $s+1$ enters $D_{\alpha^\frown f}$, the condition holds vacuously.

During the *finitary-diagonalization-procedure*, collapse can happen only between members of $D_{\alpha^\frown f}$. Since the inductive hypothesis shows that each of these numbers can only be A-equivalent to $x \in S_\delta$ for $\delta \neq \alpha^\frown f$ if $\delta \subset \alpha^\frown f$ and $x \in M_\delta$, the first statement is preserved. This collapse can only involve $s+1$ if $s+1$ has entered $D_{\alpha^\frown f}$, in which case the second statement holds vacuously. Similarly, since each of these elements of $D_{\alpha^\frown f}$ are not A-equivalent to any

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member of $I_{\alpha^\frown f}$ or K or $M_{\alpha^\frown f}$ by inductive hypothesis, the third condition is maintained as well. □

We introduce some convenient notation:

Definition 10. *For every α such that there is a biggest stage s_α at which α is initialized (taking $s_\alpha = 0$ if $\alpha = \lambda$) , if $P_\alpha \in \{S_\alpha, K_\alpha, I_\alpha, D_\alpha, M_\alpha\}$ then let*

$$P_\alpha = \bigcup_{t \geqslant s_\alpha} P_{\alpha,t}.$$

For every string α, at any stage and substep, let

$$R_\alpha = \begin{cases} I_\alpha, & \text{if } \alpha = \lambda \text{ or } \alpha = (\alpha^-)^\frown\infty, \\ D_\alpha, & \text{if } \alpha = \alpha^-{}^\frown f, \end{cases}$$
$$S^-_{<_L\alpha} = \bigcup \{S_\beta \setminus K : \beta <_L \alpha\}.$$

Lemma 11 (Disjointness Lemma)**.** *For any $\alpha \subset \mathrm{Tp}$, the (limiting values of the) following sets are pairwise disjoint: $[R_\alpha]_A$, $[S^-_{<_L\alpha}]_A$, K, $[M_\alpha]_A \setminus K$. Further, for any $\beta \subset \alpha$, these sets are all disjoint from $[R_\beta]_A$.*

Proof. This follows immediately from Lemma 9 since each of these sets are disjoint at each stage. □

Lemma 12. *For every α on the true path, and for every x,*

$$x \in K \cup [S^-_{<_L\alpha}]_A \cup [\bigcup_{\beta\subseteq\alpha} R_\beta]_A \cup ([M_\alpha]_A \setminus K).$$

Proof. For $\alpha = \lambda$, the claim is trivial because every x lies in $K_\lambda \cup I_\lambda \cup M_\lambda$.

Suppose by induction that the claim is true of α on the true path. We distinguish as usual the two possible cases $\alpha^\frown\infty \subset \mathrm{Tp}$ or $\alpha^\frown f \subset \mathrm{Tp}$.

Assume first that $\alpha^\frown\infty$ is on the true path, and let x be any number. If $x \in K \cup [S^-_{<_L\alpha}]_A \cup [\bigcup_{\beta\subseteq\alpha} R_\beta]_A$ then the claim is trivial. Note that $S^-_{<_L\alpha} = S^-_{<_L(\alpha)^\frown\infty}$. So suppose that $x \in [M_\alpha]_A \setminus K$. Then by the dumping procedure, $x \in [S_{\alpha^\frown\infty}]_A$, which gives $x \in [R_{\alpha^\frown\infty}]_A \cup [M_{\alpha^\frown\infty}]_A$.

Assume now that $\alpha^\frown f$ is on the true path, and let x be any number. Again, the case which deserves some attention is when $x \in [M_\alpha]_A \setminus K$. Then either $x \in [S_{\alpha^\frown\infty}]_A$ or $x \in [S_{\alpha^\frown f}]_A$. In the former case, $x \in [S^-_{<_L\alpha^\frown f}]_A$. In the latter, $x \in [R_{\alpha^\frown f}]_A$ or $x \in [M_{\alpha^\frown f}]_A$. In any case, the statement is true for $\alpha^\frown f$. □

Lemma 13. *For every m, n, β, γ, if $x^\beta_m[s]$ and $x^\gamma_n[t]$ are defined, then $x^\beta_m[s]$ A $x^\gamma_n[t]$ if and only if m C n.*

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Proof. We A-collapse $x^\beta_m[s]\ A\ x^\lambda_m$ and $x^\beta_n[t]\ A\ x^\lambda_n$. Thus we cause the collapse $x^\beta_m[s]\ A\ x^\gamma_n[t]$ when we update K_λ during substep 0 of some stage if and only if we cause the collapse $x^\lambda_m\ A\ x^\lambda_n$ if and only if $m\ C\ n$. Note that we only ever cause A-collapse during the construction either during a dumping procedure, where we do not collapse elements in K, or during a finitary-diagonalization strategy, where we collapse elements of D_γ for some γ, which are not in K by Lemma 9, or during the K-procedure where we collapse an element which is not yet in K, by the second claim in Lemma 9, to an element of K_λ. Thus we never unintentionally collapse together distinct members of K. □

Lemma 14. *If $\alpha \leqslant \mathrm{Tp}$ then each set S_α, K_α, R_α, and M_α are computable (not uniformly). If $\alpha <_L \mathrm{Tp}$ then each of these sets is finite. If $\alpha \subset \mathrm{Tp}$ then each of these sets is infinite.*

Proof. We check the claim that refers to strings $\alpha \subset \mathrm{Tp}$, as the part that refers to strings $\alpha <_L \mathrm{Tp}$ is obvious.

If $\alpha \subset \mathrm{Tp}$ then let s_0 be the last stage at which S_α is initialized. Then $s > s_0$ enters S_α if and only if it enters at stage s. Thus S_α is computable. It is infinite since s enters S_α at every stage s where α is visited.[1] The rest of the claim is obvious by the way S_α is partitioned into the other relevant sets. □

Lemma 15. *$C \leqslant A$.*

Proof. For each pair n, m, we have ensured that $n\ C\ m \Leftrightarrow x^\lambda_n\ A\ x^\lambda_m$ by Lemma 13. □

Lemma 16. *For every k, there is an α on the true path with $\#(\alpha) = k$.*

Proof. For every $\alpha \neq \lambda$ on the true path such that $W_{|\alpha^-|} = \varnothing$ we have $\alpha = \alpha^{-\frown} f$. Thus there are infinitely many fs along the true path, so there is an α on the true path with $\#(\alpha) = k$. □

Lemma 17. *$A \not\leqslant B$. Thus $A \not\leqslant C$.*

Proof. Given any k, we want to show that φ_k is not a reduction of A to B. Let α be on the true path with $\alpha = (\alpha^-)\hat{}f$ and $\#(\alpha) = k$. Then D_α is infinite. On the set D_α, α runs the finitary-diagonalization strategy. As we have argued in the description of the finitary-diagonalization strategy, the only possible outcome is the finite diagonalization outcome which ensures that φ_k is not a reduction of A to B. □

Lemma 18. *If $\alpha \subset \mathrm{Tp}$ then $A{\restriction}R_\alpha \equiv \mathrm{Id}$ and $A{\restriction}(S^-_{<_L\alpha} \cup \bigcup_{\beta\subseteq\alpha} R_\beta) \equiv \mathrm{Id}$.*

Proof. First of all we show that if $\alpha \subset \mathrm{Tp}$ then $A{\restriction}R_\alpha \equiv \mathrm{Id}$. Note that we only ever cause A-collapse either during substep 0, where we collapse elements in K_λ thus we do not collapse elements in R_α; during a dumping procedure, where we do not collapse elements in R_α; or during a finitary-diagonalization strategy, where we collapse elements of D_γ for some γ, which are not A-equivalent to elements of R_α unless $R_\alpha = D_\gamma$, in which case this is a collapse for the sake of α's finitary-diagonalization strategy; or during a K-procedure where we collapse an element of K_α to an element of K_λ, neither of which can be equivalent

[1] The non-uniformity is because we cannot uniformly find s_0, the last stage at which α is initialized.

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to a member of R_α. Thus we never unintentionally collapse together members of R_α. So, we can focus on the strategy itself. If $R_\alpha = I_\alpha$, then we never collapse any elements, so I_α is comprised of distinct elements, so $A{\restriction}I_\alpha \equiv \mathrm{Id}$. In the case of $R_\alpha = D_\alpha$, note that we cause a total of finitely many collapses via the finitary-diagonalization strategy since $\lim_{s\to\infty} n_{\alpha,s} < \infty$. So $A{\restriction}D_\alpha \equiv \mathrm{Id}$.

For any α, $A{\restriction}(S^-_{<_L\alpha} \cup \bigcup_{\beta\subseteq\alpha} R_\beta)$ is equivalent to $A{\restriction}S^-_{<_L\alpha} \oplus \bigoplus_{\beta\subseteq\alpha} A{\restriction}R_\beta$ by the Disjointness Lemma and the fact that $S^-_{<_L\alpha}$ is finite (the latter being needed to see that the partition is computable). Each direct summand is finite or equivalent to Id, and one of them ($A{\restriction}R_\alpha$) is equivalent to Id, so $A{\restriction}(S^-_{<_L\alpha} \cup \bigcup_{\beta\subseteq\alpha} R_\beta) \equiv \mathrm{Id}$. □

Lemma 19. *$A \equiv A \oplus \mathrm{Id}$. In particular, A is non-self full.*

Proof. It is immediate from the previous Lemma that $A{\restriction}I_\lambda \equiv \mathrm{Id}$. Further, since I_λ is computable, the Disjointness Lemma implies that $A \equiv A{\restriction}I_\lambda \oplus A{\restriction}\overline{I_\lambda}$. Thus, $A \oplus \mathrm{Id} \equiv \mathrm{Id} \oplus A{\restriction}I_\lambda \oplus A{\restriction}\overline{I_\lambda} \equiv \mathrm{Id} \oplus \mathrm{Id} \oplus A{\restriction}\overline{I_\lambda} \equiv \mathrm{Id} \oplus A{\restriction}\overline{I_\lambda} \equiv A{\restriction}I_\lambda \oplus A{\restriction}\overline{I_\lambda} \equiv A$ □

Lemma 20. *If $\alpha^\frown\infty$ is on the true path then $A \leqslant A{\restriction}S_{\alpha^\frown\infty} \leqslant A{\restriction}W_{|\alpha|}$ and the requirement $\mathrm{SMC}_{|\alpha|}$ is satisfied.*

Proof. In view of Lemma 18, there exist a computable function providing a reduction from $A{\restriction}(S^-_{<_L\alpha^\frown\infty} \cup \bigcup_{\beta\subseteq\alpha^\frown\infty} R_\beta)$ to Id, and a computable reduction of Id to $A{\restriction}I_{\alpha^\frown\infty}$. This lets us build a partial computable function f which has domain $S^-_{<_L\alpha^\frown\infty} \cup \bigcup_{\beta\subseteq\alpha^\frown\infty} R_\beta$ and range $I_{\alpha^\frown\infty}$ and for x, y in the domain, $x\ A\ y$ if and only if $f(x)\ A\ f(y)$.

To define a reduction g witnessing $A \leqslant A{\restriction}S_{\alpha^\frown\infty}$, consider any number x. We use Lemma 12, and we search for a y in $K_\lambda \cup S^-_{<_L\alpha^\frown\infty} \cup \bigcup_{\beta\subseteq\alpha^\frown\infty} R_\beta \cup M_{\alpha^\frown\infty}$ so that $x\ A\ y$ using simultaneous effective listings of these four sets:

(1) if we first find $y = x^\lambda_n \in K_\lambda$ then let $g(x) = x^{\alpha^\frown\infty}_n$;
(2) if we first find $y \in S^-_{<_L\alpha^\frown\infty} \cup \bigcup_{\beta\subseteq\alpha^\frown\infty} R_\beta$ then let $g(x) = f(y)$;
(3) if we first find $y \in M_{\alpha^\frown\infty}$, then we let $g(x) = y$.

We will show that g is a function with domain ω and range $S_{\alpha^\frown\infty}$ so that $x\ A\ y$ if and only if $g(x)\ A\ g(y)$, showing that $A \leqslant A{\restriction}S_{\alpha^\frown\infty}$. By the Disjointness Lemma, the only thing to check is $x_0\ A\ x_1 \Leftrightarrow g(x_0)\ A\ g(x_1)$, for a pair $x_0, x_1 \in K$ such that we use (1) to define $g(x_0)$, but we use (3) to define $g(x_1)$. Note that in both case (1) and (3), we define $g(x)$ to be A-equivalent to x. So we have $g(x_0)\ A\ g(x_1)$ if and only if $x_0\ A\ x_1$.

But if $A \leqslant A{\restriction}S_{\alpha^\frown\infty}$ then by Lemma 2(3) $A \leqslant A{\restriction}W_{|\alpha|}$ since every member of $S_{\alpha^\frown\infty}$ is A-equivalent to a member of $W_{|\alpha|}$, as $s+1$ enters $S_{\alpha^\frown\infty}$ only at α-expansionary stages, where we see $s+1\ A\ x$ for some $x \in W_{|\alpha|}$. Thus the requirement $\mathrm{SMC}_{|\alpha|}$ is satisfied. □

Lemma 21. *If $\alpha^\frown f$ is on the true path, then $A{\restriction}W_{|\alpha|} \leqslant C \oplus \mathrm{Id}$ and requirement $\mathrm{SMC}_{|\alpha|}$ is satisfied.*

Proof. Suppose now that $\alpha^\frown f$ is on the true path. Since $\alpha^\frown f$ is on Tp, we have that $W_{|\alpha|} \cap [S_{\alpha^\frown f}]_A \setminus K$ is empty.

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Thus Lemmas 11 and 12 show that $A{\restriction}W_{|\alpha|}$ is partitioned by:

$$A{\restriction}W_{|\alpha|} \equiv A{\restriction}(K \cap W_{|\alpha|}) \oplus A{\restriction}([S^-_{<_L \alpha^\frown f}]_A \cap W_{|\alpha|}) \oplus A{\restriction}([\bigcup_{\beta \subset \alpha^\frown f} R_\beta]_A \cap W_{|\alpha|})$$

But the first summand reduces to $A{\restriction}K$, which is equivalent to C, the second is a finite ceer, the third reduces to $A{\restriction}\bigcup_{\beta \subseteq \alpha^\frown f} R_\beta$, which is equivalent to Id by Lemma 18. So, the uniform join of all of these reduces to $C \oplus \mathrm{Id}$. □

Lemma 22. *If $X < A$, then $X \leqslant C$.*

Proof. Let i be so $X \equiv A{\restriction}W_i$ by Lemma 2 and let α be so $|\alpha| = i$ and α is on the true path. We consider two cases:

Case 1: If $\alpha^\frown\infty$ is on the true path, then Lemma 20 shows that $A \leqslant A{\restriction}S_{\alpha^\frown\infty} \leqslant A{\restriction}W_i \equiv X$.

Case 2: If $\alpha^\frown f$ is on the true path, then Lemma 21 shows that $X \equiv A{\restriction}W_i \leqslant C \oplus \mathrm{Id} \equiv C$. □

This ends the proof of the theorem. □

3. Initial Segments in the Structure **Ceers** $\setminus$ **Fin**

Let $\langle \omega^{<\omega}, \subseteq \rangle$ be the poset with universe the set of finite strings of natural numbers partially ordered by the relation $\sigma \subseteq \tau$ if σ is an initial substring of τ. In the following we use notations and terminology about finite strings of numbers, similar to those introduced at the beginning of Section 2.2 for strings in the tree of strategies Tr. The following corollary is an application of Theorem 6:

Corollary 23. *There is an initial segment of* ***Ceers*** $\setminus$ ***Fin*** *isomorphic to $\langle \omega^{<\omega}, \subseteq \rangle$.*

Proof. We begin with Id, and note that $\mathrm{Id} \equiv \mathrm{Id} \oplus \mathrm{Id}$. Theorem 6 allows us to build infinitely many incomparable strong minimal covers each of which satisfies $A \equiv A \oplus \mathrm{Id}$. Repeating as such lets us embed $\omega^{<\omega}$ as an initial segment of **Ceers** $\setminus$ **Fin**.

In more details, we can refer to a linear ordering $\preceq$ of $\omega^{<\omega}$ of order type ω, so that if $\sigma \subseteq \tau$ then $\sigma \preceq \tau$. For instance define $\Gamma_n = \{\sigma \in \omega^{<\omega} : |\sigma| \leqslant n \,\&\, \forall i < n\, (\sigma(i) < n)\}$, and let $h(\sigma)$ be the least n so that $\sigma \in \Gamma_n$. Define $\sigma \preceq \tau$ if $h(\sigma) < h(\tau)$ or $h(\sigma) = h(\tau)$ and σ is quasi-lexicographically less than τ.

We start at "step λ" by setting $A_\lambda = \mathrm{Id}$. When time comes to build A_σ, with $\sigma \neq \lambda$ (this happens at "step σ", i.e. at step n, with σ the nth string in $\preceq$), then we use Theorem 6 to make A_σ a strong minimal cover of A_{σ^-}, and A_σ not reducible to $\bigoplus\{A_\tau : \tau \prec \sigma\}$ (the universal degree is join-irreducible, i.e., there is no pair of incomparable degrees b, c so that the universal degree is the least upper bound of b and c (see [3, Proposition 2.6]), so this uniform join is not universal). Remember that all these A_σs satisfy $A_\sigma \oplus \mathrm{Id} \equiv A_\sigma$. In particular, all these A_σs are non-self-full.

Let us check that the mapping $\sigma \mapsto A_\sigma$ provides in fact an embedding of $(\omega^{<\omega}, \subseteq)$. If $\sigma \subseteq \tau$ then either $A_\sigma = A_\tau$ or A_τ is built after A_σ so there is a computable function $f_{\sigma,\tau}$

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which reduces $A_\sigma \leqslant A_\tau$. More precisely, in the construction of A_τ using Theorem 6 we encode $C = A_{\tau^-}$ directly onto $K_\lambda = \{x : x \equiv 0 \mod 3\}$, so $f_{\sigma,\tau}(x) = 3^{|\tau|-|\sigma|} \cdot x$.

Suppose towards a contradiction that there are σ and τ with $\sigma \not\subseteq \tau$ and $A_\sigma \leqslant A_\tau$. Take such a pair with τ of minimal length. If the length of τ is 0, then A_σ is above $A_{\langle\sigma(0)\rangle}$, which is strictly above $\mathrm{Id} = A_\lambda = A_\tau$ by Theorem 6. So, we must have the length of τ is > 0. But then $A_\sigma \leqslant A_\tau$ implies that either $A_\sigma \equiv A_\tau$ or $A_\sigma \leqslant A_{\tau^-}$. The latter case contradicts the minimality of the length of τ. In the former case, let ρ be the $\preceq$-greater of τ and σ. Then A_ρ is constructed so that $A_\rho \not\leqslant \bigoplus\{A_\gamma : \gamma \prec \rho\}$, contradicting $A_\sigma \equiv A_\tau$.

Thus, we have an embedding of $\omega^{<\omega}$ as an initial segment of $\mathbf{Ceers} \setminus \mathbf{Fin}$. □

Corollary 24. *The partial order $\omega + \omega^{<\omega}$ obtained by placing $\langle \omega^{<\omega}, \subseteq\rangle$ on top of $\langle \omega, \leqslant\rangle$ is embeddable as an initial segment of* ***Ceers****.*

Proof. The finite ceers have order type ω and are initial in **Ceers**. Corollary 23 shows that $\omega^{<\omega}$ is an initial segment on top of that. □

Notice that $\langle \omega^{<\omega}, \subseteq\rangle$ is in fact a lower semilattice with least element. We now turn towards extending the embedding given by Corollary 23 to an embedding of the free distributive lattice generated by the lower semilattice $\langle \omega^{<\omega}, \wedge\rangle$ into $\mathbf{Ceers} \setminus \mathbf{Fin}$. To do this, we identify for each tuple $\sigma_0, \ldots, \sigma_{n-1}$ a ceer $B_{\sigma_0,\ldots,\sigma_{n-1}}$ to act as the join of the ceers $A_{\sigma_0}, \ldots, A_{\sigma_{n-1}}$. We begin by recalling the definition:

Definition 25. *The free distributive lattice generated by the lower semilattice $\langle Q, \wedge\rangle$ is a distributive lattice D which has a function $i : Q \to D$ which preserves meets, so that D, i satisfy the universal property: If L is any distributive lattice and $f : Q \to L$ preserves meets, then there is a unique lattice-homomorphism $h : D \to L$ so that $f = h \circ i$.*

We will recall a constructive lattice-theoretic characterization of the free distributive lattice generated by the lower semilattice $\langle \omega^{<\omega}, \wedge\rangle$ below in Definition 30 and Lemma 31.

Fix a sequence of ceers $(A_\sigma)_{\sigma\in\omega^{<\omega}}$ as built in Corollary 23. We define a map which assigns, to any finite subset $\sigma_0, \ldots \sigma_{n-1}$ of $\omega^{<\omega}$ a ceer $B_{\sigma_0,\ldots,\sigma_{n-1}}$ as follows:

Definition 26. *For each $\sigma_0, \ldots, \sigma_{n-1}$, we assign the ceer $B_{\sigma_0,\ldots,\sigma_{n-1}}$ which is the ceer generated by $A_{\sigma_0} \oplus \cdots \oplus A_{\sigma_{n-1}}$ plus the set $\sim$ of pairs defined by:*

$$\begin{aligned}\sim = \{(x, y) : (\exists i, j < n)\,(\exists u)\,[&x\ A_{\sigma_0} \oplus \cdots \oplus A_{\sigma_{n-1}}\ f_{\sigma_i\wedge\sigma_j,\sigma_i}(u)\\ &\&\ y\ A_{\sigma_0} \oplus \cdots \oplus A_{\sigma_{n-1}}\ f_{\sigma_i\wedge\sigma_j,\sigma_j}(u)]\},\end{aligned}$$

where $f_{\tau,\sigma}$ for $\tau \subseteq \sigma$ is the reduction from A_τ into A_σ as defined in the proof Corollary 23. (In other words, we mod out $A_{\sigma_0} \oplus \cdots \oplus A_{\sigma_{n-1}}$ so that for each $i, j < n$, we identify the copy of $A_{\sigma_i\wedge\sigma_j}$ in A_{σ_i} with the copy of $A_{\sigma_i\wedge\sigma_j}$ in A_{σ_j}.)

Remark 27. *We will also denote $B_{\sigma_0,\ldots,\sigma_{n-1}}$ by the expression $(A_{\sigma_0} \oplus \cdots \oplus A_{\sigma_{n-1}})_{/\sim}$.*

Remark 28. *If we did not mod out to identify the copies of $A_{\sigma_i\wedge\sigma_j}$ in A_{σ_i} and in A_{σ_j} in Definition 26, then we would be putting $A_{\sigma_i\wedge\sigma_j} \oplus A_{\sigma_i\wedge\sigma_j}$ below B_{σ_i,σ_j}, which we are constructing to be the join of A_{σ_i} and A_{σ_j}. But it is possible that $A_{\sigma_i\wedge\sigma_j} \oplus A_{\sigma_i\wedge\sigma_j} > A_{\sigma_i\wedge\sigma_j}$.*

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This would mean that we would not be constructing an embedding to an initial segment in ***Ceers*** $\setminus$ ***Fin****.*

Obviously, $B_{\sigma_0,\dots,\sigma_{n-1}} \equiv B_{\sigma_{p(0)},\dots,\sigma_{p(n-1)}}$ for every permutation p of the set $\{0,\dots,n-1\}$.

Theorem 29. *The following hold for $m, n > 0$:*

(1) If $\sigma_i \supseteq \sigma_j$ with $i, j < n$ and $i \neq j$, then $B_{\sigma_0,\dots,\sigma_{n-1}} \equiv B_{\sigma_0,\dots,\sigma_{j-1},\sigma_{j+1},\dots,\sigma_{n-1}}$. Hence,

$$B_{\sigma_0,\dots,\sigma_{n-1}} \equiv B_{\sigma_{i_0},\dots,\sigma_{i_{k-1}}},$$

where $\{\sigma_{i_0},\dots,\sigma_{i_{k-1}}\}$ is the $\subseteq$-antichain comprised of the $\subseteq$-maximal elements in $\{\sigma_0,\dots,\sigma_{n-1}\}$.

(2) If $\{\sigma_0,\dots,\sigma_{n-1}\}$ is a $\subseteq$-antichain and $X < B_{\sigma_0,\dots\sigma_{n-1}}$, then there exists $i < n$ so that $X \leqslant B_{\sigma_0,\dots,\sigma_i^-,\dots\sigma_{n-1}}$. Further, the only degrees of infinite ceers below $B_{\sigma_0,\dots\sigma_{n-1}}$ are degrees of the form $B_{\tau_0,\dots,\tau_{m-1}}$ where $(\forall j < m)(\exists i < n)[\tau_j \subseteq \sigma_i]$.

(3) If $\{\sigma_0,\dots,\sigma_{n-1}\}$ and $\{\tau_0,\dots,\tau_{m-1}\}$ are $\subseteq$-antichains, then

$$B_{\tau_0,\dots,\tau_{m-1}} \leqslant B_{\sigma_0,\dots,\sigma_{n-1}} \Leftrightarrow (\forall j < m)(\exists i < n)[\tau_j \subseteq \sigma_i].$$

Proof. (1) We have $\sigma_i \supseteq \sigma_j$, with $i, j < n$ and $i \neq j$. In the definition of $\sim$, we mod out the entirety of the copy of A_{σ_j} with a part of A_{σ_i}. It follows from Lemma 2(2) that

$$\begin{aligned} B_{\sigma_0,\dots,\sigma_{n-1}} &\equiv B_{\sigma_0,\dots,\sigma_{n-1}} \restriction (\omega \setminus \omega_{j,n}) \\ &\equiv B_{\sigma_0,\dots,\sigma_{j-1},\sigma_{j+1},\dots,\sigma_{n-1}}. \end{aligned}$$

(2) Suppose that $X < B_{\sigma_0,\dots,\sigma_{n-1}}$, and f is a reduction from X to $B_{\sigma_0,\dots,\sigma_{n-1}}$. Let $W_e = [\mathrm{range}(f)]_{B_{\sigma_0,\dots,\sigma_{n-1}}}$. Hence $X \equiv \left(A_{\sigma_0} \oplus \cdots \oplus A_{\sigma_{n-1}}\right)_{/\sim} \restriction W_e$ by Lemma 2(2).

Let $W_{e_i} = \{x \in W_e : x \equiv i \mod n\}$, and α_i be the node of length e_i on the true path of the construction of A_{σ_i} in Theorem 6.

Case 1: For each $i < n$, $\alpha_i \hat{} \infty$ is on the true path. Then (use the argument in the proof that A_{σ_i} is a strong minimal cover of $A_{\sigma_i^-}$ to give the reduction on each component individually) we get reductions f_i witnessing $A_{\sigma_i} \leqslant A_{\sigma_i} \restriction W_{e_i}$. Thus $\left(A_{\sigma_0} \oplus \cdots \oplus A_{\sigma_{n-1}}\right)_{/\sim_{\oplus_{i<n} f_i}} \leqslant \left(A_{\sigma_0} \oplus \cdots \oplus A_{\sigma_{n-1}}\right)_{/\sim} \restriction W_e$ by Lemma 5 as $\sim$ is transitive and $A_{\sigma_0} \oplus \cdots \oplus A_{\sigma_{n-1}}$-closed. Finally, note that $\sim$ includes relations on the set K_λ in each A_{σ_i}. But in the reductions f_i, each element in K_λ is sent to an image in its own equivalence class. Thus $\sim_{\oplus_{i<n} f_i} = \sim$. Thus

$$\begin{aligned} \left(A_{\sigma_0} \oplus \cdots \oplus A_{\sigma_{n-1}}\right)_{/\sim} &= \left(A_{\sigma_0} \oplus \cdots \oplus A_{\sigma_{n-1}}\right)_{/\sim_{\oplus_{i<n} f_i}} \\ &\leqslant \left(A_{\sigma_0} \oplus \cdots \oplus A_{\sigma_{n-1}}\right)_{/\sim} \restriction W_e. \end{aligned}$$

That is, $B_{\sigma_0,\dots,\sigma_{n-1}} \leqslant X$.

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Case 2: There is some $i < n$ so that $\alpha_i \hat{} f$ is on the true path. So $A_{\sigma_i} \upharpoonright W_{e_i} \leqslant A_{\sigma_i^-} \oplus \mathrm{Id}$. Let f be the reduction as such and note that $f(x_n^\lambda) = 2n$. It follows that

$$\begin{aligned} A_{\sigma_0} \upharpoonright W_{e_0} \oplus \cdots \oplus A_{\sigma_{n-1}} \upharpoonright W_{e_{n-1}} &\leqslant A_{\sigma_0} \ \oplus \cdots \oplus A_{\sigma_i^-} \oplus \cdots \oplus A_{\sigma_{n-1}} \oplus \mathrm{Id} \\ &\equiv A_{\sigma_0} \ \oplus \cdots \oplus A_{\sigma_i^-} \oplus \cdots \oplus A_{\sigma_{n-1}}. \end{aligned}$$

We note that in the latter reduction, we are collapsing the copy of Id with the copy of Id in any one of the A_{σ_i}, and that $\sim$ does not touch the sets I_λ in any of the A_{σ_j}. Since f sends K_λ in A_σ (which is a copy of A_{σ^-}) exactly to A_{σ^-}, Lemma 5 yields

$$\left(A_{\sigma_0} \oplus \cdots \oplus A_{\sigma_{n-1}}\right)_{/\sim} \upharpoonright W_e \leqslant \left(A_{\sigma_0} \ \oplus \cdots \oplus A_{\sigma_i^-} \oplus \cdots \oplus A_{\sigma_{n-1}}\right)_{/\sim'},$$

where $\sim'$ is as in the definition of $B_{\sigma_0,\ldots,\sigma_i^-,\ldots,\sigma_{n-1}}$. That is, $X \leqslant B_{\sigma_0,\ldots,\sigma_i^-,\ldots,\sigma_{n-1}}$.

Now, suppose X is an infinite ceer and $X \leqslant B_{\sigma_0,\ldots,\sigma_{n-1}}$. We can repeatedly apply the above condition until either we represent X as $B_{\tau_0,\ldots,\tau_{n-1}}$ or until we get to tuple of τs which is no longer an anti-chain. In this case, we can use the result of (1) and then repeat. This process is monotonically decreasing in the sum of the lengths of the strings, so it must either terminate with $X \equiv B_{\tau_0,\ldots,\tau_{m-1}}$ and every τ_j being an initial substring of some σ_i for $i < n$, or the sum of the lengths of the strings must go to 0, in which case we have $X \leqslant B_\lambda = A_\lambda \equiv \mathrm{Id}$. But since X is infinite, this latter case gives $X \equiv \mathrm{Id} \equiv B_\lambda$ and of course λ is an initial segment of σ_0.

(3) Let $\{\sigma_0, \ldots, \sigma_{n-1}\}$ and $\{\tau_0, \ldots, \tau_{m-1}\}$ be two $\subseteq$-antichains. It is immediate by definition that

$$(\forall j < m)(\exists i < n)[\tau_j \subseteq \sigma_i] \Rightarrow B_{\tau_0,\ldots,\tau_{m-1}} \leqslant B_{\sigma_0,\ldots,\sigma_{n-1}}.$$

Next, we suppose that $B_{\tau_0,\ldots,\tau_{m-1}} \leqslant B_{\sigma_0,\ldots,\sigma_{n-1}}$ and show τ_j is an initial segment of some σ_i. In particular, we have that $A_{\tau_j} \leqslant B_{\sigma_0,\ldots,\sigma_{n-1}}$. We use the previous result to see that $A_{\tau_j} \equiv B_{\rho_0,\ldots,\rho_k}$ for some antichain of ρs which are initial segments of the sequence of σs. But then $A_{\tau_j} \geqslant A_{\rho_i}$ for each i, which implies that $\rho_i \subseteq \tau_j$ for each $i < k$. Since the sequence of ρs forms an antichain, we conclude that $k = 0$ and $\rho_0 = \tau_j$. Then τ_j is an initial substring of some σ. □

We now recall some lattice-theoretic facts that will help us show that we indeed have an embedding of the free distributive lattice generated by the lower semilattice $\langle \omega^{<\omega}, \wedge \rangle$.

Definition 30. *Given a poset $Q = \langle Q, \leqslant \rangle$, and a subset $X \subseteq Q$, let $(X] = \{y \in Q : \exists x \in X\, (y \leqslant x)\}$.*

Let also $P_{<\infty}(Q)$ be the set of nonempty finite subsets of Q.

Let $\mathcal{L}(Q) = \langle \{(X] : X \in P_{<\infty}(Q)\}, \subseteq \rangle$, and let us use the symbol $(P_{<\infty}(Q)]$ to denote the universe of this poset.

Lemma 31. *If Q is a lower semilattice then $\mathcal{L}(Q)$ is the free distributive lattice generated by the lower semilattice Q. If Q is a lower semilattice with least element then $\mathcal{L}(Q)$ has least element.*

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Proof. Suppose that $Q = \langle Q, \wedge \rangle$ is a lower semilattice. Clearly $\mathcal{L}(Q)$ is an upper semilattice, with join operation $\vee$ given by $(X] \vee (Y] = (X \cup Y]$, and having inclusion as the partial ordering relation. To show that $\mathcal{L}(Q)$ is also a lower semilattice, it is enough to observe that it is closed under $\cap$, as if $X, Y \in P_{<\infty}(Q)$ then $(X] \cap (Y] = (\{x \wedge y : x \in X, y \in Y\}]$. A straightforward calculation shows that the lattice is distributive.

Next we show that $\mathcal{L}(Q)$ enjoys the universal property of free objects, making it the free distributive lattice on the lower semilattice Q. Let $i : Q \to (P_{<\infty}(Q)]$ be given by $i(x) = (\{x\}]$. Then i preserves meets. If now $L = \langle L, \vee, \wedge, 0 \rangle$ is a distributive lattice, and $f : Q \to L$ preserves meets, then the mapping $h : (P_{<\infty}(Q)] \to L$, given by $h((X]) = \bigvee_{x \in X} f(x)$ is easily seen to be the unique lattice-theoretic homomorphism so that $f = h \circ i$. To see that h preserves meets, notice that by distributivity and properties of f,

$$\begin{aligned} h((X]) \wedge h((Y]) &= (\bigvee_{x \in X} f(x)) \wedge (\bigvee_{y \in Y} f(y)) \\ &= \bigvee_{x \in X, y \in Y} (f(x) \wedge f(y)) \\ &= \bigvee_{x \in X, y \in Y} f(x \wedge y) = h((X] \cap (Y]). \end{aligned}$$

If $Q = \langle Q, \wedge, 0 \rangle$ is a lower semilattice with least element 0, then the least element of $\mathcal{L}(Q)$ is $(\{0\}]$. □

Remark 32. Notice that if $Q = \langle Q, \wedge, 0 \rangle$ is a lower semilattice with least element then $\mathcal{L}(Q)$ can be alternatively viewed as the free distributive lattice with least element generated by the lower semilattice Q with least element. The definition of the free distributive lattice with least element is defined as in Definition 25, but requiring all maps to preserve least element. That is, the free distributive lattice with least element generated by the lower semilattice with least element $\langle Q, \wedge, 0 \rangle$ is a distributive lattice D with least element which has a function $i : Q \to D$ which preserves meets and least element, so that D, i satisfy the universal property: If L is any distributive lattice with least element and $f : Q \to L$ preserves meets and least element, then there is a unique lattice-homomorphism preserving least element $h : D \to L$ so that $f = h \circ i$.

Corollary 33. *The free distributive lattice $\mathcal{L}(\omega^{<\omega})$ on the lower semilattice $\omega^{<\omega} = \langle \omega^{<\omega}, \wedge \rangle$ embeds as an initial segment of the degrees $\boldsymbol{Ceers} \setminus \boldsymbol{Fin}$.*

Proof. If $Q = \langle Q, \leqslant \rangle$ is a poset, and $X \in P_{<\infty}(Q)$ then there exists a finite $\leqslant$-antichain X' such that $(X] = (X']$: This X' is unique, and we denote it by X^M, since it is the set of the $\leqslant$-maximal elements in X. It is not difficult to see that if $X, Y \in P_{<\infty}(Q)$ then

$$(X] \subseteq (Y] \Leftrightarrow (\forall x \in X^M)(\exists y \in Y^M)[x \leqslant y].$$

Therefore we can use items (1) and (3) of Theorem 29 to show that the mapping

$$(X] \mapsto B_{\sigma_0, \ldots, \sigma_{n-1}}$$

(where X is a nonempty finite subset of $\omega^{<\omega}$ and $X^M = \{\sigma_0, \ldots, \sigma_{n-1}\}$) order-theoretically embeds $\mathcal{L}(\langle \omega^{<\omega}, \subseteq \rangle)$ to the degrees of ceers. Item (2) of Theorem 29 shows that, up to

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equivalence, the range of this embedding is an initial segment of **Ceers** $\setminus$ **Fin**. The least element in the embedding is the degree of Id. □

Notice that the embedding of the previous corollary maps meets of $\mathcal{L}(\omega^{<\omega})$ to meets of **Ceers** (since the image is an initial segment), but maps joins to joins of the image, which need not be joins of **Ceers**. We now show that in fact the embedding distinctly does not map joins to joins. In fact, even in the simplest case where $\sigma \wedge \tau \neq \lambda$, A_σ and A_τ do not have a join. In the following, if u, v are numbers, then $\langle u, v\rangle$ denotes the string of length 2 comprised of u, v.

Observation 34. *For any $i, j, k \in \omega$ with $j \neq k$, $A_{\langle i,j\rangle}$ and $A_{\langle i,k\rangle}$ do not have a join in the ceers. In particular, $B_{\langle i,j\rangle,\langle i,k\rangle}$ is not a join since $B_{\langle i,j\rangle,\langle i,k\rangle} \not\leqslant A_{\langle i,j\rangle} \oplus A_{\langle i,k\rangle}$*

Proof. From Theorem 29, $B_{\langle i,j\rangle,\langle i,k\rangle}$ bounds no other degree which is above both $A_{\langle i,j\rangle}$ and $A_{\langle i,k\rangle}$, thus if there is a join of $A_{\langle i,j\rangle}$ with $A_{\langle i,k\rangle}$, the join must be $B_{\langle i,j\rangle,\langle i,k\rangle}$. So, it suffices to show $B_{\langle i,j\rangle,\langle i,k\rangle} \not\leqslant A_{\langle i,j\rangle} \oplus A_{\langle i,k\rangle}$.

We will show this by taking a supposed reduction g of $B_{\langle i,j\rangle,\langle i,k\rangle}$ to $A_{\langle i,j\rangle} \oplus A_{\langle i,k\rangle}$ and showing that in this case some S_α for α of the form $\alpha = (\alpha^-)^\frown\infty$ on the true path in the construction of $A_{\langle i,j\rangle}$ must be sent entirely to the evens or entirely to the odds, but it can't be the odds as $A_{\langle i,j\rangle} \leqslant A_{\langle i,j\rangle}{\restriction}S_\alpha$ and $A_{\langle i,j\rangle} \not\leqslant A_{\langle i,k\rangle}$. The symmetric argument shows that some $S_{\alpha'}$ for α' on the true path in the construction of $A_{\langle i,k\rangle}$ must be sent entirely to the odds. But then the two copies of A_i which are equivalent in $B_{\langle i,j\rangle,\langle i,k\rangle}$ via $\sim$ are not equivalent in their g-images, yielding a contradiction.

Thus suppose that $B_{\langle i,j\rangle,\langle i,k\rangle}$ reduces to $A_{\langle i,j\rangle} \oplus A_{\langle i,k\rangle}$ via the function g. Let X be the set of x so that $g(2x)$ is odd. That is, X is the pre-image of $A_{\langle i,k\rangle}$ on the copy of $A_{\langle i,j\rangle}$ inside $B_{\langle i,j\rangle,\langle i,k\rangle}$. Similarly, let Y be the set of x so that $g(2x)$ is even. That is, Y is the preimage of $A_{\langle i,j\rangle}$ on the copy of $A_{\langle i,j\rangle}$ inside $B_{\langle i,j\rangle,\langle i,k\rangle}$. Since $X = W_a$ and $Y = W_b$ for some $a, b \in \omega$ give a partition of ω, either the true path in the construction of $A_{\langle i,j\rangle}$ has ∞ as the ath bit or the bth bit. To see this, suppose both the ath bit and the bth bit of the true path is an f. Then for any β along the true path of length greater than a or b, $S_\beta \setminus K$ (referencing the sets with these names in the construction of $A_{\langle i,j\rangle}$) would be disjoint from both X and Y. But X and Y partition ω, so this is impossible.

If the former is the case (i.e., the ath bit is ∞), then by Lemma 20 we have $A_{\langle i,j\rangle} \leqslant A_{\langle i,j\rangle}{\restriction}S_\alpha \leqslant A_{\langle i,j\rangle}{\restriction}X \leqslant A_{\langle i,k\rangle}$ which contradicts Corollary 23, thus the latter must be the case. But then an entire S_α is contained in Y for some $\alpha \subset \mathrm{Tp}$. Thus every class in K intersects Y. So, the entire copy of A_i inside $A_{\langle i,j\rangle}$ must be sent to even numbers. The symmetric argument (taking X to be the set of x so that $g(2x+1)$ is even, and Y to be the set of x so that $g(2x+1)$ is odd) shows that the entire copy of A_i inside $A_{\langle i,k\rangle}$ must be sent to odd numbers. But then g is not a reduction after all, because the two copies of A_i (the one inside $A_{\langle i,j\rangle}$ and the one inside $A_{\langle i,k\rangle}$) are equivalent in $B_{\langle i,j\rangle,\langle i,k\rangle}$, but their images are not equivalent in $A_{\langle i,j\rangle} \oplus A_{\langle i,k\rangle}$. □

We now characterize the initial segments of $\mathcal{L}(\omega^{<\omega})$, giving a characterization of the countable distributive lattices that we know how to embed as an initial segment of **Ceers** $\setminus$ **Fin**.

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Corollary 35. *A countable distributive lattice L is isomorphic to an initial segment of $\mathcal{L}(\omega^{<\omega})$ if and only if*

(1) L satisfies the descending chain condition (i.e. there is no infinite descending chain);
(2) the poset of its join-irreducible elements is order-theoretic isomorphic to a subtree of $\omega^{<\omega}$.

Therefore, any countable distributive lattice satisfying these conditions can be embedded as an initial segment of $\boldsymbol{Ceers} \setminus \boldsymbol{Fin}$.

Proof. Let L satisfy the descending chain condition, and let J be the partially ordered set of its join-irreducible elements. Then every element a of L can be identified with the finite antichain of J comprised of the maximal elements of J which are below a, [6, § III.2]. Therefore, (looking at the proof of Lemma 31) it is easy to see that if J is a lower semilattice then L is isomorphic with $\mathcal{L}(J)$. Suppose in addition that J is (up to isomorphism) a subtree of $\omega^{<\omega}$. Then clearly this isomorphism extends to an isomorphism of $\mathcal{L}(J)$ with an initial segment of $\mathcal{L}(\omega^{<\omega})$. It follows that if L satisfies the two conditions of the corollary then L is isomorphic to an initial segment of $\mathbf{Ceers} \setminus \mathbf{Fin}$.

In the other direction, suppose that L is isomorphic to an initial segment of $\mathcal{L}(\omega^{<\omega})$. Then trivially L satisfies the descending chain condition. Moreover the isomorphism must send the join-irreducible elements of L to an initial segment of join-irreducible elements of $\mathcal{L}(\omega^{<\omega})$ which are ceers of the form A_σ, so after all J is isomorphic to a subtree of $\omega^{<\omega}$. □

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