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The Method of Assigning Incidences

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Abstract

Incidence calculus is a probabilistic logic in which incidences, standing for the situations in which formulae may be true, are assigned to some formulae, and probabilities are assigned to incidences. However, numerical values may be assigned to formulae directly without specifying the incidences. In this paper, we propose a method of discovering incidences under these circumstances which produces a unique output comparing with the large number of outputs from other approaches. Some theoretical aspects of this method are thoroughly studied and the completeness of the result generated from it is proved. The result can be used to calculate mass functions from belief functions in the Dempster-Shafer theory of evidence (DS theory) and define probability spaces from inner measures (or lower bounds) of probabilities on the relevant propositional language set.

1 Introduction

Incidence calculus ([2], [4]) is a probabilistic logic for dealing with uncertainty in intelligent systems. In incidence calculus, incidences are assigned to some formulae, and probabilities are assigned to incidences. An incidence is assigned to a formula if this formula is true when the incidence occurs. Incidences can be explained as the possible worlds relevant to a problem in logics, or all the possible outcomes of an event in probability theory. The probability of a formula is calculated through the set of incidences assigned to it.

Calculating probabilities of formulae through incidences results in the indirect assignment of numerical values on formulae. The initial assignment of incidences on some formulae (called axioms) plays a vital role in further propagating incidences to other formulae and obtaining probabilities of them. It is proved in [13] that the indirect encoding in incidence calculus provides the possibility of combining dependent evidence, a problem which is difficult to solve in pure numerical approaches. However in practice, numerical values may be assigned to some formulae directly without giving incidences, like the methods used in pure numeric reasoning theories, e.g., DS theory, probabilistic logic, and probability theory. It is, therefore, necessary to recover the incidence assignments (if they exist) from the given numerical assignment in these circumstances ([11]).

In [17], the task of assigning incidences was viewed as a tree searching problem and two techniques of performing this search were discussed. One of them was a depth first search while the other extended the Monte Carlo method, introduced initially in [3]. Both methods generate a number of consistent assignments of incidences, given a numerical assignment. In fact, these methods try to find all possible consistent assignments of incidences. Therefore, both of the programs are slow in terms of finding answers. Besides, they cannot be used in more general situations, represented using generalized incidence calculus theories, to which the original incidence calculus

was extended [14]. In this paper, we discuss the incidence assignment problem in the situation where incidence calculus has been extended [14]. An alternative approach to assigning incidences from numerical assignments is explored. This approach is based on a new concept: *basic incidence assignment*. The significance of the new incidence assignment method is that it can determine whether a consistent assignment is available efficiently and then construct only one assignment, which is a basic incidence assignment. Through this core assignment, a family of consistent assignments can be generated. As there is only one possible output comparing with a large amount of output from other approaches, the new method is considerablely faster then those in [17]. This result also gives a method to check whether a numerical assignment on a set is a belief function in DS theory ([8], [19]) and then to calculate its mass function when it is. The result can further be used to construct probability spaces from inner measures (or lower bounds) of probabilities on the relevant propositional language sets [9].

Although our approach only produces a single output, a basic incidence assignment, it does not mean that we may have ignored any possible outcomes discovered by other methods. In [15], the theoretical features of basic incidence assignment based approach were explored intensively. It is proved that incidence assignments discovered in our approach are more fundamental then incidence assignments obtained from other methods, and any output from other methods is subsumed by a unique output from our method.

Overall, this paper makes the following contributions regarding developing the original incidence calculus.

- The concept of basic incidence assignment is proposed which underlies the foundation for the new approach.
- The relationship between a basic incidence assignment and an incidence assignment is examined which reveals the nature of similarities among apparently different incidence functions.
- An algorithm on incidence assignments from numerical assignments is designed and implemented.
- The theoretical foundation of this approach is examined in which the completeness of the result generated from the algorithm is proved.
- An algorithm is designed to check if a numerical assignment on a set is a belief function, and obtain the corresponding mass function when it is.
- An algorithm is designed to recover the probability space from an inner measure on a propositional language set.

The rest of the paper is organized as follows. Section 2 introduces some basic definitions used throughout the paper and the basics of the original incidence calculus. Section 3 describes the generalized incidence calculus theories, and proposes basic incidence assignments. Section 4 concentrates on the relationships between basic incidence assignments and incidence functions, to see how to find one from the other. In Section 5, an algorithm for obtaining the incidence assignment from a numerical assignment is discussed. Section 6 concentrates on exploring the

theoretical nature of incidence assignments and discusses why a single output is enough to represent a number of incidence assignment, given a numerical assignment. The application of the result to DS theory and probability spaces is presented in Section 7. Finally, in Section 8 we summarize the paper.

2 The Original Incidence Calculus

We first give some basic definitions which will be used in the later sections, then introduce the original incidence calculus.

2.1 Basic definitions

Definition 2.1: *The Probability Space*

A probability space (X, χ, μ) *has:*

X*: a* sample space *usually containing all the possible worlds;*

χ*: a* σ*-algebra containing some subsets of* X*, which is defined as containing* X *and closed under complementation and countable union.*

μ*: a probability measure* $\mu : \chi \to [0, 1]$ *with the following features:*

P1. $\mu(X_i) \geq 0$ *for all* $X_i \in \chi$*;*

P2. $\mu(X) = 1$*;*

P3. $\mu(\cup_{j=1}^{\infty} X_j) = \Sigma_{j=1}^{\infty} \mu(X_j)$*, if the* X_j*'s are pairwise disjoint members of* χ*.*

A subset χ' of χ is called a *basis* of χ if it contains non-empty and disjoint elements, and if χ consists precisely of countable unions of members of χ'. For any finite χ there is a unique basis χ' of χ and it follows that

$$\Sigma_{X_i \in \chi'} \mu(X_i) = 1.$$

Given a set X, when a probability distribution μ assigns a probability on every singleton $x \in X$, σ-algebra χ and the basis χ' of χ are the same as X. The corresponding probability space is (X, X, μ).

Definition 2.2: *The Propositional Language*

- P *is a finite set of atomic propositions.*
- $\mathcal{L}(P)$ *is the propositional language formed from* P*.*

  $true, false \in \mathcal{L}(P)$*,*

  if $q \in P$*, then* $q \in \mathcal{L}(P)$*, and*

  if $\phi, \psi \in \mathcal{L}(P)$ *then* $\neg\phi \in \mathcal{L}(P)$*,* $\phi \wedge \psi \in \mathcal{L}(P)$*,* $\phi \vee \psi \in \mathcal{L}(P)$*, and* $\phi \to \psi \in \mathcal{L}(P)$*.*

  That is, $\mathcal{L}(P)$ *is closed under the operations negation* ($\neg$)*, disjunction* ($\vee$)*, conjunction* ($\wedge$) *and implication* ($\to$)*.*

Definition 2.3: *The Basic Element Set*

Assume that P is a finite set of propositions and $P = \{q_1, q_2, ..., q_n\}$. An item δ, defined as $\delta = q'_1 \wedge ... \wedge q'_n$ where q'_j is either q_j or $\neg q_j$, is called a basic element. The collection of all the basic elements, denoted as $\mathcal{A}t$, is called the basic element set from P. Any formula ψ in the language set $\mathcal{L}(P)$ can be represented as

$$\psi = \delta_1 \vee ... \vee \delta_k,$$

where $\delta_j \in \mathcal{A}t$.

2.2 The original incidence calculus

Incidence calculus ([2]) is a logic for probabilistic reasoning. In incidence calculus, probabilities are not directly associated with formulae, rather sets of possible worlds are directly associated with formulae and probabilities (or lower and upper bounds of probabilities) of formulae are calculated from these sets.

Definition 2.4: *The Original Incidence Calculus Theories*

An incidence calculus theory is a quintuple

$$< \mathcal{W}, \mu, P, \mathcal{A}, i >,$$

where

- *$\mathcal{W}$ is a finite set of possible worlds.*
- *For all $w \in \mathcal{W}$, $\mu(w)$ is the probability of w and $\mu(\mathcal{W}) = 1$, where $\mu(I) = \Sigma_{w \in I} \mu(w)$.*
- *P is a finite set of propositions. $\mathcal{A}t$ is the basic element set of P. $\mathcal{L}(P)$ is the language set of P.*
- *$\mathcal{A}$ is a distinguished set of formulae in $\mathcal{L}(P)$ called the axioms of the theory.*
- *i is a function from the axioms in $\mathcal{A}$ to $2^{\mathcal{W}}$, the set of subsets of $\mathcal{W}$. $i(\phi)$ is to be thought of as the set of possible worlds in $\mathcal{W}$ in which ϕ is true, i.e., $i(\phi) = \{w \in \mathcal{W} | \ w \models \phi\}$. $i(\phi)$ is called the incidence set of ϕ.*

i is extended to be a function from $\mathcal{L}(\mathcal{A})$ to $2^{\mathcal{W}}$ by the following defining equations of incidences.

$$\begin{aligned}
&i(true) = \mathcal{W}, \\
&i(false) = \{\}, \\
&i(\neg\phi) = \mathcal{W} \setminus i(\phi), \\
&i(\phi \wedge \psi) = i(\phi) \cap i(\psi), \\
&i(\phi \vee \psi) = i(\phi) \cup i(\psi), \\
&i(\phi \rightarrow \psi) = \mathcal{W} \setminus i(\phi) \cup i(\psi).
\end{aligned}$$

Such an incidence calculus theory is *truth functional*, *i.e.*, the incidence set of a formula can be calculated purely from its parts.

For a formula in $\mathcal{L}(P) \setminus \mathcal{L}(A)$, the lower and upper bounds of its incidence set are defined as:

$$i_*(\phi) = \bigcup_{\psi \in \mathcal{L}(A)} \{i(\psi) \mid i(\psi \rightarrow \phi) = \mathcal{W}\} \tag{1}$$

$$i^*(\phi) = \bigcap_{\psi \in \mathcal{L}(A)} \{i(\psi) \mid i(\phi \rightarrow \psi) = \mathcal{W}\} \tag{2}$$

The corresponding lower and upper bounds of probabilities of this formula are $p_*(\phi) = \mu(i_*(\phi))$ and $p^*(\phi) = \mu(i^*(\phi))$. For a formula, if $i_*(\phi) = i^*(\phi) = i(\phi)$, then $p(\phi)$ is defined as $p_*(\phi)$. We say $p(\phi)$ is the probability of formula ϕ.

Example 2.1

Suppose there are three propositions, $P = \{sunny, rainy, windy\}$, and seven possible worlds, $\mathcal{W} = \{sun, mon, tues, wed, thus, fri, sat\}$. Assume that each possible world is equally probable, *i.e.* occurs 1/7 of the time. Through a piece of evidence, we learn that four possible worlds *fri, sat, sun, mon* support *rainy*, and three possible worlds *mon, wed, fri* make *windy* true. Then the incidence sets of these two propositions are:

$$\begin{aligned} &i(rainy) = \{fri, sat, sun, mon\}, \\ &i(windy) = \{mon, wed, fri\}. \end{aligned}$$

The set of axioms $\mathcal{A}$ is $\{rainy, windy\}$. For every formula in $\mathcal{L}(A)$, it is possible to get its incidence set. For instance, for formula $\neg windy \wedge rainy$, we have

$$\begin{aligned} i(\neg windy \wedge rainy) &= i(\neg windy) \cap i(rainy) \\ &= (\mathcal{W} \setminus i(windy)) \cap i(rainy) \\ &= \{tue, thru, sat, sun\} \cap \{fri, sat, sun, mon\} \\ &= \{sat, sun\}. \end{aligned}$$

For other formulae in $\mathcal{L}(P) \setminus \mathcal{L}(A)$, such as, $(sunny \vee windy) \wedge \neg rainy = \phi$, we can only use equations (1) and (2) to obtain its bounds. In this special case, $i_*(\phi) = i(windy \wedge \neg rainy) = \{wed\}$ (as only $i((windy \wedge \neg rainy) \rightarrow \phi) = \mathcal{W}$) and $i^*(\phi) = \{tues, wed, thus\}$ (as $i(\phi \rightarrow \neg rainy) = \mathcal{W}$).

The result says that at least one day, but at most three days in that week that it is either sunny or windy but not rainy.

□

3 Generalized Incidence Calculus

In [13] and [14], we argued that incidence calculus theories introduced in Section 2 can only be used to represent a special group of information which should specify incidence functions with

truth functionality. Then we suggested to weaken the conditions on incidence functions in order to let incidence calculus represent a wider range of information. As a result, generalized incidence calculus theories (GICTs) are introduced to replace the original incidence calculus theories.

Definition 3.1: *Generalized Incidence Calculus Theories (GICTs)*

A generalized incidence calculus theory (GICT) is in the form of $< \mathcal{W}, \mu, P, \mathcal{A}, i >$ *where* $\mathcal{W}, \mu$ *and* P *are the same as in Definition 2.4.*

- $\mathcal{A}$ *is a distinguished set of formulae in* $\mathcal{L}(P)$ *called the axioms of the theory on which incidences are assigned initially.*
- i *is a function from the axioms in* $\mathcal{A}$ *to* $2^{\mathcal{W}}$*, the set of subsets of* $\mathcal{W}$*.* $i(\phi)$ *is to be thought of as the set of possible worlds in* $\mathcal{W}$ *in which* ϕ *is true.* $i(\phi)$ *is called the incidence set of* ϕ*. Function* i *satisfies the conditions*

$$\begin{aligned} &i(true) = \mathcal{W}, \\ &i(false) = \{\}, \\ &i(\phi \wedge \psi) = i(\phi) \wedge i(\psi), \phi, \psi \in \mathcal{A}. \end{aligned}$$

The condition $i(\phi \wedge \psi) = i(\phi) \wedge i(\psi)$ requires that $\mathcal{A}$ is closed under the operation $\wedge$. This condition is true whenever for any two formulae ϕ, ψ in $\mathcal{A}$, $i(\phi), i(\psi)$ and $i(\phi \wedge \psi)$ are all defined. Because, if a possible world w supports $\phi \wedge \psi$, w must also support both ϕ and ψ, so that $i(\phi \wedge \psi) \subseteq i(\phi) \cap i(\psi)$. On the other hand, if w supports both ϕ and ψ, then w supports $\phi \wedge \psi$ as well, so that $i(\phi) \cap i(\psi) \subseteq i(\phi \wedge \psi)$.

When $\mathcal{A}$ is not closed under $\wedge$ initially, we can always extend it to be closed by defining $i(\phi \wedge \psi) = i(\phi) \cap i(\psi)$. Therefore, in the rest of the paper, we take $\mathcal{A}$ as a set closed under $\wedge$. We also assume in the rest of the paper that $true$ and $false$ are always included in a set of axioms.

The function i in a GICT is also called an incidence function in the rest of the paper. Only when confusions may occur, will we make it clear whether i is a original incidence function, or a function in a GICT.

Similar to the situation in the original incidence calculus, it is not usually possible to infer the incidences of all the formulae in $\mathcal{L}(P)$, given a GICT. We can only define both the upper and lower bounds of the incidence set using the following equations.

$$i_*(\phi) = \bigcup_{\psi \in \mathcal{A}, \psi \models \phi} i(\psi) \tag{3}$$

$$i^*(\phi) = \mathcal{W} \setminus i_*(\neg\phi) \tag{4}$$

where $\psi \models \phi$ iff $\psi \rightarrow \phi = true$. That is, formula $\psi \rightarrow \phi$ is a tautology.

In particular, for any $\phi \in \mathcal{A}$, we have $i_*(\phi) = i(\phi)$.

The equations for calculating bounds here are slightly different from those defined in (1) and (2). (3) and (4) are more close to the similar concepts in other theories, such as the bounds of

beliefs in the DS theory, and the inner and outer measures of probabilities on logical sentences in [9], [10].

The lower bound of an incidence set (equation (3)) represents the set of possible worlds in which ϕ is proved to be true and the upper bound (equation (4)) represents the set of possible worlds in which $\neg\phi$ fails to be proved. Function $Prob_*(\phi) = \mu(i_*(\phi))$, the lower bound of a probability measure, gives the degree of our belief in ϕ and function $Prob^*(\phi) = \mu(i^*(\phi))$, the upper bound of a probability measure, represents the degree we fail to believe in $\neg\phi$. If for every formula ϕ in $\mathcal{A}$, $Prob_*(\phi) = Prob^*(\phi)$ holds, then $Prob(\phi)$ is defined as $Prob_*(\phi)$ and is called the probability of this formula. In this case, a GICT is shrunk into an original incidence calculus theory. Otherwise, the probability of ϕ does not exist.

In the following, when we mention a lower bound of a probability distribution on $\mathcal{A}$, we always mean the function $Prob_*(*)$ obtained through the lower bounds of incidence sets.

Now we use an example to see how to apply generalized incidence calculus theories.

Example 3.1 (Originally from [9] and used in [7])

> A person has four coats: two are blue with single-breasted, one is grey and double-breasted and one is grey and single-breasted. To choose which colour of coat to wear, this person tosses a (fair) coin. Once the colour is chosen, which specific coat is worn is determined by a mysterious procedure. What is the probability of the person wearing a single-breasted coat?

To solve this problem in generalized incidence calculus, we need to construct a GICT first. We let a set of propositions P be $P = \{grey, double\}$ where $grey$ stands for 'The coat is grey' and $double$ stands for 'The coat is double-breasted' and let $\mathcal{W} = \{w_1, w_2\}$ where w_1 is for blue coats and w_2 for grey coats. Then we have

$$\mathcal{A}t = \{grey \wedge double, \neg grey \wedge double, grey \wedge \neg double, \neg grey \wedge \neg double\}.$$

Among these basic elements, $\neg grey \wedge double$ is $false$, as there is no such coat among the four choices. It is possible to derive that w_1 supports formula $\neg grey \wedge \neg double$ and w_2 supports formula $(grey \wedge \neg double) \vee (grey \wedge double)$. Therefore, we get a GICT, $< \mathcal{W}, \mu, P, \mathcal{A}, i >$, where

$$\begin{aligned}
&\mu(w_1) = \mu(w_2) = 0.5,\\
&\mathcal{A} = \{\neg grey \wedge \neg double, (grey \wedge \neg double) \vee (grey \wedge double), true, false\},\\
&i(\neg grey \wedge \neg double) = \{w_1\},\\
&i((grey \wedge \neg double) \vee (grey \wedge double)) = \{w_2\},\\
&i(true) = \{w_1, w_2\}, i(false) = \{\}.
\end{aligned}$$

So

$$i_*(\neg double) = i(\neg grey \wedge \neg double),$$

$$i^*(\neg double) = \mathcal{W} \setminus i_*(double) = \mathcal{W},$$

and

$$p_*(\neg double) = 0.5, \qquad p^*(\neg double) = 1.$$

The answer to the question is that the probability of the person wearing a single-breasted coat lies between 0.5 and 1.

□

3.1 Basic Incidence Assignment

For each axiom $\phi \in \mathcal{A}$ of a GICT $< \mathcal{W}, \mu, P, \mathcal{A}, i >$, the incidence set $i(\phi)$ contains every possible world which makes ϕ true. If $w \in i(\psi)$ and $\psi \models \phi$ $(\psi \neq \phi)$, then w makes ϕ true, and $w \in i(\phi)$. Assume that $\psi_1, ..., \psi_j \in \mathcal{A}$ are all the axioms satisfying the condition $\psi_j \models \phi$ $(\psi_j \neq \phi)$, then $i(\phi) \setminus \cup_j i(\psi_j)$ may or may not be empty, depending on the specification of i. We are particularly interested in ϕ and those possible worlds in $i(\phi) \setminus \cup_j i(\psi_j)$, when it is not empty. What properties do these possible worlds possess? Clearly, the possible worlds in $i(\phi) \setminus \cup_j i(\psi_j)$ make only ϕ true without making any of ψ $(\psi \models \phi, \psi \neq \phi)$ true. We denote this set as $ii(\phi)$. If we let $\mathcal{A}_0$ contain all those axioms ϕ for which $ii(\phi)$ is not empty, then ii defines a function on axioms in $\mathcal{A}_0$. For other axioms, ψ, in $\mathcal{A} \setminus \mathcal{A}_0$, $ii(\psi) = \{\}$. This implies that $i(\psi)$ is purely the union of the incidence sets of some other axioms ψ_j, where $\psi_j \models \psi$. That is, the set $i(\psi)$ does not carry any extra information than that union.

Definition 3.2: *Defining a function ii from i*

Assume that i is the incidence assignment from a GICT, $< \mathcal{W}, \mu, P, \mathcal{A}, i >$, we define a new function ii on a subset of $\mathcal{A}$, $\mathcal{A}_0$, from the following equation:

$$\mathcal{A}_0 = \{\phi \mid ii(\phi) = i(\phi) \setminus \cup_{\psi_j \models \phi, \psi_j \neq \phi} i(\psi_j),\ ii(\phi) \neq \{\},\ \ \phi, \psi_j \in \mathcal{A}\}.$$

Definition 3.3: *Basic incidence assignment*

Given a set of axioms $\mathcal{A}_0$ and a set of possible worlds $\mathcal{W}$, a function $ii : \mathcal{A}_0 \to 2^{\mathcal{W}}$ is called a basic incidence assignment if ii satisfies the following conditions:

$$\begin{array}{ll} ii(\phi) \neq \{\}, & \phi \in \mathcal{A}_0; \\ ii(\phi) \cap ii(\psi) = \{\}, & \phi \neq \psi\ (\phi \not\models \psi \ \textit{or}\ \psi \not\models \phi); \\ ii(true) = \mathcal{W} \setminus \bigcup_{\phi_j \neq true} ii(\phi_j), & \phi_j \in \mathcal{A}_0, true \in \mathcal{A}_0,\ \textit{when}\ \mathcal{W} \setminus \bigcup_{\phi_j \neq true} ii(\phi_j) \neq \{\}. \end{array}$$

Clearly, in a basic incidence assignment, each possible world w is only assigned to a specific axiom. $ii(\phi_j)$ $(\phi_j \in \mathcal{A}_0)$ partitions the set $\mathcal{W}$, as $\cup_{\phi_j} ii(\phi_j) = \mathcal{W}$ as shown in the definition. If $w \in ii(\phi_j)$, we say that ϕ_j is the smallest formula which w supports (by smallest, we mean that if w also supports ϕ_l, then $\phi_j \models \phi_l$ $(\phi_j \neq \phi_l)$). Therefore, a basic incidence assignment gives a unique mapping relation between each possible world and the smallest formula of all the formulae it supports, for instance, if w supports ϕ, then w supports $\phi \vee \psi_1$, $\phi \vee \psi_2$,... etc. In a basic incidence assignment, if $w \in ii(\phi)$, then $w \notin ii(\phi \vee \psi_j)$. However, an incidence function i maps a possible world w to every formula it supports, *i.e.*, if $w \in i(\phi)$, then $w \in i(\phi \vee \psi_j)$.

In order to see the difference between i and ii, we look at an example.

Example 3.2

Following the incidence assignment in Example 2.1, the generalized incidence calculus theory is

$$< \mathcal{W}, \mu, P, \mathcal{A}, i >$$

where $\mathcal{W}$ contains seven possible worlds, and $\mu(w) = 1/7$. $P = \{sunny, windy, rainy\}$ and $\mathcal{A} = \{rainy, windy, rainy \wedge windy, true, false\}$[1]. The incidence function gives the following incidence sets to the axioms in $\mathcal{A}$.

$$\begin{aligned}
&i(rainy) = \{fri, sat, sun, mon\},\\
&i(windy) = \{mon, wed, fri\},\\
&i(rainy \wedge windy) = \{mon, fri\},\\
&i(true) = \mathcal{W} \text{ and } i(false) = \{\}.
\end{aligned}$$

An assignment ii on set $\mathcal{A} \setminus \{false\}$ can then be defined using Definition 3.2.

$$\begin{aligned}
&ii(rainy \wedge windy) = \{fri, mon\},\\
&ii(rainy) = \{sat, sun\},\\
&ii(windy) = \{wed\},\\
&ii(true) = \{tues, thur\}.
\end{aligned}$$

From this ii, the incidence function can be recovered as:

$$\begin{aligned}
&i(rainy \wedge windy) = ii(rainy \wedge windy) = \{mon, fri\},\\
&i(rainy) = ii(rainy) \cup ii(rainy \wedge windy) = \{fri, sat, sun, mon\},\\
&i(windy) = ii(windy) \cup ii(rainy \wedge windy) = \{mon, wed, fri\},\\
&i(true) = ii(true) \cup ii(rainy) \cup ii(windy) \cup ii(rainy \wedge windy) = \mathcal{W}.
\end{aligned}$$

□

sat in $ii(rainy)$ means that the smallest axiom this possible world supports is $rainy$. While the smallest axiom that fri supports is $rainy \wedge windy$ although $fri \in i(rainy)$ as well.

Function ii defined in this example is a basic incidence function of i based on Definition 3.3, and it is easy to see that the original incidence function i is recoverable from it. In general, is every function ii defined through Definition 3.2 a basic incidence assignment of that incidence function i? If yes, what is the procedure to recover its incidence function? Is the basic incidence assignment of a particular i unique? In the next section, we will answer these questions.

[1] It is worth noted that the set of axioms $\mathcal{A}$ in this generalized incidence calculus theory is closed under $\wedge$ and is different from that in the original incidence calculus theory in Example 2.1 (the original incidence calculus does not require this condition).

4 Relationships of Basic Incidence Assignments and Incidence Functions

Basic incidence assignments and incidence functions are all about mapping relations between a set of axioms and a set of possible worlds. A basic incidence assignment maps a possible world uniquely to an axiom while an incidence assignment maps a possible world to all the axioms to which it supports. So what are the general relationships between these two types of assignment? Are the relationships one to one, or one to many, or even many to many? This section goes deeper along this line to see what the answer is.

4.1 An incidence function has a unique basic incidence assignment

Algorithm A below constructs a function ii from an incidence function i in a GICT. Theorem 1 confirms that the function ii obtained from Algorithm A is the same as the function ii defined in Definition 3.2 and that this function is unique. Theorem 2 proves that function ii obtained from Algorithm A is a basic incidence assignment and the corresponding incidence function can be recoved from it. Therefore, we conclude that given an incidence function i, there is always a unique basic incidence assignment from which it can be recovered.

Algorithm A: Extraction of a function ii

Given a GICT $< \mathcal{W}, \mu, P, \mathcal{A}, i >$, function ii on a domain $\mathcal{A}_0$ ($\mathcal{A}_0 \subseteq \mathcal{A}$) can be obtained by the following procedure.

Define a subset $\mathcal{A}_0$ of $\mathcal{A} \setminus \{false\}$, as $\mathcal{A}_0 = \{\psi_1, ..., \psi_n\}$ where $\mathcal{A}_0$ satisfies the condition

$$\forall \psi_i \in \mathcal{A}_0, \forall \phi \in \mathcal{A}, \quad \phi \not\models \psi_i \ \ if \ \ \phi \neq \psi_i. \tag{5}$$

Therefore, $\mathcal{A}_0$ contains the "smallest" formulae in $\mathcal{A}$ and $\mathcal{A}_0$ is not empty[2]*. In fact, we can find at least one element belonging to $\mathcal{A}_0$ using the following procedure. For a formula $\phi \in \mathcal{A} \setminus \{false\}$, if $\exists \psi_i \in \mathcal{A}, \psi_i \neq \phi$ and $\psi_i \models \phi$, then we use ψ_i to replace ϕ and repeat the same procedure*[3]*until we obtain a formula ψ_j and we cannot find any formula which makes ψ_j true, then ψ_j will be in $\mathcal{A}_0$.*

Step 1: *for every formula $\psi \in \mathcal{A}_0$, define $ii(\psi) = i(\psi)$.*

Step 2: *define $\mathcal{A}' = \mathcal{A} \setminus (\mathcal{A}_0 \cup \{false\})$.*

Step 3: *if $\mathcal{A}'$ is left with only tautologies, go to Step 6.*

Step 4: *choose a formula ϕ_l in $\mathcal{A}'$ which satisfies the requirement that for any $\phi_j \in \mathcal{A}'$, if $\phi_j \neq \phi_l$, then $\phi_j \not\models \phi_l$.*

Define $ii(\phi_l) = i(\phi_l) \setminus \bigcup_{\psi_{lj} \in \mathcal{A}_0, \psi_{lj} \models \phi_l} ii(\psi_{lj})$.

Step 5: *delete ϕ_l from $\mathcal{A}'$ and update $\mathcal{A}_0$ to be $\mathcal{A}_0 \cup \{\phi_l\}$ when $ii(\phi_l) \neq \{\}$. Go to step 3.*

[2]We assume that $\mathcal{A}$ contains at least one axiom in addition to the tautologies and $false$. Otherwise, this GICT tells nothing but $true$ is supported by $\mathcal{W}$ and $false$ by empty.

[3]This procedure will terminate because P is finite, so are $\mathcal{L}(P)$ and the set of axioms, $\mathcal{A}$.

Step 6: *further defining* $ii(true) = \mathcal{W} \setminus \cup_{\psi_j \in \mathcal{A}_0} ii(\psi_j)$. *If* $ii(true) \neq \{\}$ *then* $ii(true)$ *represents those possible worlds which only support formula true, and add true to* $\mathcal{A}_0$. *So function* $ii : \mathcal{A}_0 \to 2^{\mathcal{W}}$ *is constructed.*

From this algorithm, we come to Theorem 1 below which proves that function ii obtained in the algorithm is the same as function ii defined in Definition 3.2.

Theorem 1 *Given a GICT,* $< \mathcal{W}, \mu, P, \mathcal{A}, i >$, *the function* ii *constructed from Algorithm A is the unique function satisfying the conditions in Definitions 3.2.*

Proof

To make the statement clear, we assume that the output of Algorithm A is $(\mathcal{A}_1, ii_1)$ where $\mathcal{A}_1$ contains a set of axioms, on which ii_1 is defined, and an output of Definition 3.2 is $(\mathcal{A}_0, ii)$.

Part I: $(\mathcal{A}_1, ii_1)$ satisfies the conditions in Definition 3.2.

(i) $ii_1(\phi) \neq \{\}$ for every $\phi \in \mathcal{A}_1$.

(ii) for $\phi \in \mathcal{A}_1$,

$$\begin{aligned} ii_1(\phi) &= i(\phi) \setminus \textstyle\bigcup_{\psi_j \in \mathcal{A}_1, \psi_j \models \phi} ii(\psi_j) \text{ (equation in Step 4)} \\ &= i(\phi) \setminus \textstyle\bigcup_{\phi_j \in \mathcal{A}_1, \phi_j \models \phi, \phi_j \neq \phi} (\bigcup_{\psi_{jl} \in \mathcal{A}_1, \psi_{jl} \models \psi_j} ii(\psi_{jl}) \cup ii(\psi_j)) \\ &\qquad \text{(based on } S \cup S = S \text{, we add some extra } ii(\psi_{jl}) \text{ sets)} \\ &= i(\phi) \setminus \textstyle\bigcup_{\phi_j \in \mathcal{A}_1, \phi_j \models \phi, \phi_j \neq \phi} i(\psi_j) \text{ (through the reverse of the equation in Step 4)} \\ &= ii(\phi) \text{ (from Definition 3.2).} \end{aligned}$$

Therefore $(\mathcal{A}_1, ii_1)$ satisfies the conditions in Definition 3.2.

Part II: Proof that ii_1 is identical with ii.

From the proof of Part I, we can see that $(\mathcal{A}_1, ii_1) \subseteq (\mathcal{A}_0, ii)$ where $\forall \phi \in \mathcal{A}_0, ii_1(\phi) = ii(\phi)$.

We only need to prove that $(\mathcal{A}_0, ii) \subseteq (\mathcal{A}_1, ii_1)$.

At the beginning of Algorithm A, we let the smallest axioms from formula (5) be in $\mathcal{A}'_1$. Then $\mathcal{A}'_1 \subseteq \mathcal{A}_0$, and $ii_1(\phi) = ii(\phi)$ for $\phi \in \mathcal{A}'_1$.

We choose an axiom $\psi \in \mathcal{A}_0 \setminus \mathcal{A}'_1$ where for any other $\psi_j \in \mathcal{A}_0 \setminus \mathcal{A}'_1$, if $\psi_j \neq \psi$, then $\psi_j \not\models \psi$. Then there is a list of ψ_l where $\psi_l \models \psi$, otherwise, ψ in $\mathcal{A}'_1$. Based on Step 4 in Algorithm A, we have

$$\begin{aligned} ii_1(\psi) &= i(\psi) \setminus \textstyle\bigcup_{\psi_l \in \mathcal{A}'_1, \psi_l \models \psi} ii_1(\psi_l) \\ &= i(\psi) \setminus \textstyle\bigcup_{\psi_l \in \mathcal{A}'_1, \psi_l \models \psi} ii(\psi_l) \text{ (as } ii_1(\psi_l) = ii(\psi_l)) \\ &\supseteq i(\psi) \setminus \textstyle\bigcup_{\psi_l \in \mathcal{A}'_1, \psi_l \models \psi} i(\psi_l). \end{aligned}$$

As $i(\psi) \setminus \bigcup_{\psi_l \in \mathcal{A}'_1, \psi_l \models \psi} i(\psi_l)$ is not empty, so is $ii_1(\psi)$. Therefore, ψ is in $\mathcal{A}_1$.

Redefine $\mathcal{A}'_1 = \mathcal{A}'_1 \cup \{\psi\}$. Repeat the above procedure for another axiom ψ_l in $\mathcal{A}_0 \setminus \mathcal{A}'_1$, we will prove that ψ_l is in $\mathcal{A}_1$. So eventually, every formula which is in $\mathcal{A}_0$ is also in $\mathcal{A}_1$. That is $(\mathcal{A}_0, ii) \subseteq (\mathcal{A}_1, ii_1)$. So $\mathcal{A}_1$ and $\mathcal{A}_0$ are the same set, and ii_1 and ii are identical on this set.

Part III: Proof that ii is the unique function of this kind.

Given an incidence function i on axiom set $\mathcal{A}$, there is only one function ii from i through Definition 3.2. Because an output from Algorithm A is equivalent to a function ii defined from Definition 3.2, and there is only one such ii given an i, there must be only one output from Algorithm A, given a i. Therefore, for a given i, the function ii, either from Algorithm A, or from Definition 3.2, is unique.

QED

Theorem 2 *Given a GICT $< \mathcal{W}, \mu, P, \mathcal{A}, i >$, the function ii defined on set $\mathcal{A}_0$ obtained from Algorithm A is a basic incidence assignment and function i in the theory can be recovered from it using equation (6).*

$$i(\phi) = \bigcup_{\phi_j \in \mathcal{A}_0, \phi_j \models \phi} ii(\phi_j). \tag{6}$$

Proof

Part I: ii is a basic incidence assignment

Suppose that $ii(\phi_i) \cap ii(\phi_j) = W' \neq \{\}$, we have the following inference procedure:

$$\begin{aligned}
&\exists w, w \in ii(\phi_i) \cap ii(\phi_j) \\
&\Longrightarrow w \in i(\phi_i) \quad \text{and} \quad w \in i(\phi_j) \\
&\Longrightarrow w \in i(\phi_i) \cap i(\phi_j) \\
&\Longrightarrow w \in i(\phi_i \wedge \phi_j) \\
&\Longrightarrow \exists \phi \neq false \text{ and } w \in i(\phi) \quad \text{and} \quad \phi = \phi_i \wedge \phi_j \text{ (as } \phi_i \neq \phi_j) \\
&\Longrightarrow \exists \phi' \neq false, \phi' \models \phi, w \in ii(\phi'), \text{ and } \phi' \models \phi_i, \phi' \models \phi_j \\
&\Longrightarrow \exists \phi' \neq false, w \notin i(\phi_i) \setminus ii(\phi') \text{ and } w \notin i(\phi_j) \setminus ii(\phi') \\
&\Longrightarrow w \notin i(\phi_i) \setminus \cup_{\phi_{il} \models \phi_i} ii(\phi_{il}) \text{ and } w \notin i(\phi_j) \setminus \cup_{\phi_{jl} \models \phi_j} ii(\phi_{jl}) \\
&\Longrightarrow w \notin ii(\phi_i) \text{ and } w \notin ii(\phi_j) \\
&\Longrightarrow w \notin ii(\phi_i) \cap ii(\phi_j).
\end{aligned}$$

Contradiction.

So the equation $ii(\phi_i) \cap ii(\phi_j) = \{\}$ holds for any two distinct elements ϕ_i and ϕ_j in $\mathcal{A}_0$. As we also have $ii(true) = \mathcal{W} \setminus \cup_j ii(\phi_j)$, when $ii(true) \neq \{\}$, so ii is a basic incidence assignment.

Part II: i can be obtained from ii

Now we prove that the incidence function i in $< \mathcal{W}, \mu, P, \mathcal{A}, i >$ can be derived from ii using equation (6).

For a formula $\phi \in \mathcal{A}_0$, there are two situations when ϕ is added into $\mathcal{A}_0$.

1) ϕ is added into $\mathcal{A}_0$ in Step 1, then

$$i(\phi) = ii(\phi) = \bigcup_{\phi_j \models \phi} ii(\phi_j).$$

2) ϕ is added into $\mathcal{A}_0$ in Step 4 and 5 when $ii(\phi) \neq \{\}$, then

$$i(\phi) = ii(\phi) \cup (\bigcup_{\phi_j \in \mathcal{A}_0, \phi_j \models \phi} ii(\phi_j)) = \bigcup_{\phi_j \in \mathcal{A}_0, \phi_j \models \phi} ii(\phi_j).$$

For a formula $\phi \in \mathcal{A} \setminus \mathcal{A}_0$, we have $ii(\phi) = \{\}$ at Step 5, equation

$$i(\phi) = \bigcup_{\phi_j \in \mathcal{A}_0, \phi_j \models \phi} ii(\phi_j)$$

holds.

Therefore, no matter in which case, the incidence set $i(\phi)$ of ϕ can always be derived from $(\mathcal{A}_0, ii)$ using equation (6).

QED

Example 4.1 (Simplified from [5])

Suppose that there are four urns of balls. The balls in the first urn are all *Blue*, the balls in the second urn are either *Blue* or *Green*, the balls in the third urn are all *Red* and the balls in the fourth urn are either *Green* or *Red*. Suppose one ball is drawn from an urn and we are interested in the colour of the drawing ball. The following propositions can be included in set P.

q_1: The ball is *Blue*;

q_2: The ball is $Blue \vee Green$;

q_3: The ball is *Red*;

q_4: The ball is $Green \vee Red$.

It is possible to establish the supporting relations between the event "drawing a ball from an urn" and the propositions in P using an incidence function i as:

$$\begin{array}{ll} i(q_1) = \{1\}, & i(q_2) = \{1, 2\}, \\ i(q_3) = \{3\}, & i(q_4) = \{3, 4\}, \\ i(true) = \{1, 2, 3, 4\}, & i(false) = \{\}. \end{array}$$

where $1, 2, 3, 4$ stand for drawing a ball from urns 1, 2, 3 and 4 respectively.

As $i(q_2 \wedge q_4) = i(q_2) \cap i(q_4) = \{\}$, the set of axioms $\mathcal{A}$ is $\mathcal{A} = \{q_1, q_2, q_3, q_4, true, false\}$ which is closed under $\wedge$ and the corresponding generalized incidence calculus theory is:

$$< \mathcal{W}, \mu, P, \mathcal{A}, i >,$$

where $\mathcal{W} = \{1, 2, 3, 4\}$ and $\mu(w) = 1/4$.

A function ii on set $\mathcal{A} \setminus \{false\}$ can be defined through Definition 3.2 and it is a basic incidence assignment.

$$\begin{array}{ll} ii(q_1) = \{1\}, & ii(q_2) = \{2\}, \\ ii(q_3) = \{3\}, & ii(q_4) = \{4\}, \\ ii(true) = \{\}. & \end{array}$$

$ii(q_4) = \{4\}$ means that the balls in the fourth urn only support proposition q_4. The incidence function i on $\mathcal{A}$ can be recovered as:

$$
\begin{aligned}
&i(q_1) = ii(q_1), && i(q_2) = ii(q_2) \cup ii(q_1),\\
&i(q_3) = ii(q_3), && i(q_4) = ii(q_3) \cup ii(q_4),\\
&i(true) = \textstyle\bigcup_j ii(q_j) = \mathcal{W} && i(false) = \{\}.
\end{aligned}
$$

□

Constructing a incidence function i from ii is not difficult if we know the incidence function i first and ii is derived from it. What is the prospect of defining an incidence function i given a basic incidence assignment ii on domain $\mathcal{A}_0$? What is the method of finding a domain $\mathcal{A}$, which should be closed under $\wedge$, on which i is defined? We will solve this problem in the next subsection.

4.2 An basic incidence assignment maps to a family of incidence assignments

Example 4.2

Given the following incidence function on a set of axioms $\mathcal{A} = \{windy, rainy, rainy \wedge windy, windy \vee sunny, rainy \wedge (windy \vee sunny), true, false\}$ as:

$$
\begin{aligned}
&i(rainy) = \{fri, sat, sun, mon\},\\
&i(windy) = \{mon, wed, fri\},\\
&i(rainy \wedge windy) = \{mon, fri\},\\
&i(windy \vee sunny) = \{mon, wed, fri\},\\
&i(rainy \wedge (windy \vee sunny)) = \{mon, fri\},\\
&i(true) = \mathcal{W} \text{ and } i(false) = \{\}.
\end{aligned}
$$

A function ii can be defined on domain $\mathcal{A}_0$=$\{windy \wedge rainy, windy, rainy, true\}$ using Definition 3.2 as

$$
\begin{aligned}
&ii(rainy \wedge windy) = \{fri, mon\},\\
&ii(rainy) = \{sat, sun\},\\
&ii(windy) = \{wed\},\\
&ii(true) = \{tues, thur\}.
\end{aligned}
$$

□

This incidence function shares the same basic incidence assignment with the incidence assignment in Example 3.2, but the original sets of axioms are different. The set of axioms in Example 3.2 is a proper subset of the set of axioms in Example 4.2. This suggests that a family of incidence functions may map to a specific basic incidence assignment. So, how to find the smallest set of axioms in a family given $(ii, \mathcal{A}_0)$ and how to generate other members of this family are the questions we need to answer.

The first theorem below proves the existence of the smallest set of axioms on which an incidence function i is derivable, given a basic incidence assignment, and the second proves that a family of sets of axioms can be generated based on the smallest set of axioms.

Theorem 3 *Given a set of axioms $\mathcal{A}_0$ with a basic incidence assignment ii, let*

$$\mathcal{A} = \mathcal{A}_0 \cup \{false\} \cup \{\phi \mid \phi = \psi_1 \wedge ... \wedge \psi_n, \psi_1, ..., \psi_n \in \mathcal{A}_0, n > 1\}.$$

Function i defined on $\mathcal{A}$ by equation (6) is an incidence function.

Proof

Part I: i is an incidence assignment on $\mathcal{A}$.

First of all, because $ii(true) = \mathcal{W} \setminus \cup_j ii(\phi_j)$, we have $i(true) = ii(true) \cup (\cup_j ii(\phi_j)) = \mathcal{W}$. We also define $i(false) = \{\}$.

As $\mathcal{A}$ is defined in the way that it is closed under $\wedge$, we only need to prove that $i(\phi \wedge \psi) = i(\phi) \cap i(\psi)$ when ϕ, ψ and $\phi \wedge \psi$ are all in $\mathcal{A}$.

Suppose that $i(\phi) \cap i(\psi) = \mathcal{W}' \neq \{\}$,

$$\begin{aligned}
&\forall w \in \mathcal{W}',\ w \in i(\phi) \cap i(\psi) \\
\Longleftrightarrow\ &\exists \phi_0,\ w \in ii(\phi_0) \quad (\phi_0 \models \phi,\ \phi_0 \models \psi) \\
\Longleftrightarrow\ &\exists \phi_0,\ w \in ii(\phi_0), \quad \phi_0 \models \phi \wedge \psi \\
\Longleftrightarrow\ &\exists \phi_0,\ w \in ii(\phi_0) \wedge ii(\phi_0) \subseteq i(\phi \wedge \psi) \text{ (from equation (6))} \\
\Longleftrightarrow\ &\qquad w \in i(\phi \wedge \psi)
\end{aligned}$$

So we have $i(\phi) \wedge i(\psi) = i(\phi \wedge \psi)$. When $i(\phi) \wedge i(\psi) = \{\}$, it is also easy to prove that $i(\phi) \wedge i(\psi) = i(\phi \wedge \psi)$. Therefore the function i defined by (6) is an incidence function.

Part II: $\mathcal{A}$ is the smallest among all possible sets of axioms which are closed under $\wedge$.

It is easy to see that $\mathcal{A}$ is smallest set which includes $\mathcal{A}_0$ and closed under $\wedge$.

QED

Theorem 4 *Given a basic incidence assignment ii on a set of axioms $\mathcal{A}_0$, let $\mathcal{A}$ be the smallest set of axioms defined in Theorem 1, then a set $\mathcal{B}$ defined using equation (7) below is a member of a family of sets, on which there is an incidence function $i_{\mathcal{B}}$ leading to ii.*

$$\mathcal{B} = \mathcal{A} \cup \{\phi \mid \phi = \psi \vee \phi', \phi' \in \mathcal{A}\} \tag{7}$$

where $\psi \in \mathcal{L}(P)$ but $\psi \notin \mathcal{A}$.

Proof

We need to prove three separate results for this theorem. Firstly, $\mathcal{B}$ is closed under $\wedge$. Secondly, there is an incidence function $i_{\mathcal{B}}$ on domain $\mathcal{B}$. Thirdly, the basic incidence assignment of $i_{\mathcal{B}}$ is the same as ii.

Part I: Proof of $\mathcal{B}$ being closed under $\wedge$.

Assume that ϕ_1 and ϕ_2 are two distinct formulae in $\mathcal{B}$.
(i) if both of these formulae are in $\mathcal{A}$, then $\phi_1 \wedge \phi_2 \in \mathcal{A} \subseteq \mathcal{B}$. $\mathcal{B}$ is closed.

(ii) if both of these formulae are in $\mathcal{B} \setminus \mathcal{A}$, we have

$$
\begin{aligned}
\phi_1, \phi_2 \in \mathcal{B} &\Longleftrightarrow \exists \phi_1' \in \mathcal{A}, \exists \phi_2' \in \mathcal{A}, \phi_1 = \phi_1' \vee \psi, \phi_2 = \phi_2' \vee \psi \\
&\Longleftrightarrow \exists \phi_1' \in \mathcal{A}, \exists \phi_2' \in \mathcal{A}, \phi_1 \wedge \phi_2 = (\phi_1' \vee \psi) \wedge (\phi_2' \vee \psi) \\
&\Longleftrightarrow \exists \phi_1' \in \mathcal{A}, \exists \phi_2' \in \mathcal{A}, \phi_1 \wedge \phi_2 = (\phi_1' \wedge \phi_2') \vee \psi \\
&\Longleftrightarrow \exists \phi'' \in \mathcal{A}, \phi'' = \phi_1' \wedge \phi_2', \phi_1 \wedge \phi_2 = \phi'' \vee \psi \\
&\Longleftrightarrow \phi_1 \wedge \phi_2 \in \mathcal{B} \text{ (because } \phi'' \vee \psi \in \mathcal{B} \text{ according to equation(7))}
\end{aligned}
$$

So $\mathcal{B}$ is closed under $\wedge$.

(iii) if one of these formulae is in $\mathcal{B} \setminus \mathcal{A}$, we can use the similar approach as in (ii) to prove that $\mathcal{B}$ is closed under $\wedge$.

Therefore, the set constructed using (7) is closed under $\wedge$.

Part II: Proof that there is an incidence function, $i_{\mathcal{B}}$, which is closed under $\wedge$ on domain $\mathcal{B}$.

Now we define a function $i_{\mathcal{B}}$ on $\mathcal{B}$ as

$$
i_{\mathcal{B}}(\phi) = \begin{cases} i_{\mathcal{A}}(\phi) & \text{if } \phi \in \mathcal{A} \\ \bigcup_{\phi_j \in \mathcal{A}, \phi_j \models \phi} i_{\mathcal{A}}(\phi_j) & \text{otherwise} \end{cases} \tag{8}
$$

where $i_{\mathcal{A}}$ is the incidence function on $\mathcal{A}$ obtained from Theorem 3. Assume that ϕ_1 and ϕ_2 are two distinct formulae in $\mathcal{B}$, we need to prove that $i_{\mathcal{B}}(\phi_1) \wedge i_{\mathcal{B}}(\phi_2) = i_{\mathcal{B}}(\phi_1 \wedge \phi_2)$.

(i) if both of these formulae are in $\mathcal{A}$, then $\phi_1 \wedge \phi_2 \in \mathcal{A}$.
$i_{\mathcal{B}}(\phi_1 \wedge \phi_2) = i_{\mathcal{A}}(\phi_1 \wedge \phi_2) = i_{\mathcal{A}}(\phi_1) \cap i_{\mathcal{A}}(\phi_2) = i_{\mathcal{B}}(\phi_1) \cap i_{\mathcal{B}}(\phi_2)$
So $i_{\mathcal{B}}$ is closed under $\wedge$.

(ii) if both of these formulae, ϕ_1, ϕ_2 are in $\mathcal{B} \setminus \mathcal{A}$, we assume that $i_{\mathcal{B}}(\phi_1) \cap i_{\mathcal{B}}(\phi_2) = \mathcal{W}' \neq \{\}$. Then we have

$$
\begin{aligned}
\forall w \in \mathcal{W}', &\Longleftrightarrow w \in i_{\mathcal{B}}(\phi_1) \cap i_{\mathcal{B}}(\phi_2) \\
&\Longleftrightarrow \exists \phi_1', \phi_2' \in \mathcal{A}, w \in i_{\mathcal{B}}(\phi_1) \cap i_{\mathcal{B}}(\phi_2), \phi_1 = \phi_1' \vee \psi, \phi_2 = \phi_2' \vee \psi \\
&\Longleftrightarrow \exists \phi_0,\ w \in ii_{\mathcal{A}}(\phi_0) \quad \phi_0 \models \phi_1',\ \phi_0 \models \phi_2', \phi_0 \in \mathcal{A}, \phi_1 = \phi_1' \vee \psi, \phi_2 = \phi_2' \vee \psi \\
&\Longleftrightarrow \exists \phi_0,\ w \in ii_{\mathcal{A}}(\phi_0), \quad \phi_0 \models \phi_1' \wedge \phi_2', \phi_1 = \phi_1' \vee \psi, \phi_2 = \phi_2' \vee \psi \\
&\Longleftrightarrow \exists \phi_0,\ w \in i_{\mathcal{A}}(\phi_1' \wedge \phi_2'), \phi_1 \wedge \phi_2 = (\phi_1' \wedge \phi_2') \vee \psi \\
&\Longleftrightarrow w \in i_{\mathcal{B}}(\phi_1 \wedge \phi_2), \text{ as } i_{\mathcal{A}}(\phi_1' \wedge \phi_2') \subseteq i_{\mathcal{B}}((\phi_1' \wedge \phi_2') \vee \psi) = i_{\mathcal{B}}(\phi_1 \wedge \phi_2).
\end{aligned}
$$

$i_{\mathcal{B}}$ is closed under $\wedge$.

(iii) Similarly, when only one of these formulae is in $\mathcal{B}$, we can still prove that $i_{\mathcal{B}}$ is closed under $\wedge$.

Because $true, false \in \mathcal{A}$, we have $i_{\mathcal{B}}(true) = i_{\mathcal{A}}(true) = \mathcal{W}$ and $i_{\mathcal{B}}(false) = i_{\mathcal{A}}(false) = \{\}$. Therefore, $i_{\mathcal{B}}$ is an incidence function on $\mathcal{B}$.

Part III: Proof of the basic incidence assignment of $i_{\mathcal{B}}$ is ii on domain $\mathcal{A}_0$.

To prove that ii is a basic incidence assignment of $i_{\mathcal{B}}$, we only need to show that $ii(\phi) = \{\}$ when $\phi \notin \mathcal{A}$.

For any $\phi \in \mathcal{B} \setminus \mathcal{A}$, we have

$$
\begin{aligned}
ii_{\mathcal{B}}(\phi) &= i_{\mathcal{B}}(\phi) \setminus \bigcup_{\phi_j \in \mathcal{B}, \phi_j \models \phi} i_{\mathcal{B}}(\phi_j) \text{ (from Definition 3.2)} \\
&= \bigcup_{\phi_j \in \mathcal{A}, \phi_j \models \phi} i_{\mathcal{A}}(\phi_j) \setminus \bigcup_{\phi_j \in \mathcal{B}, \phi_j \models \phi} i_{\mathcal{B}}(\phi_j) \text{ (replace } i_{\mathcal{B}}(\phi) \text{ using equation (8))} \\
&= \bigcup_{\phi_j \in \mathcal{A}, \phi_j \models \phi} i_{\mathcal{A}}(\phi_j) \setminus \bigcup_{\phi_j \in \mathcal{B}, \phi_j \models \phi} \left(\bigcup_{\phi_j \in \mathcal{A}, \phi_l \models \phi_j} i_{\mathcal{A}}(\phi_l)\right) \\
&\qquad \text{(replace each } i_{\mathcal{B}}(\phi_j) \text{ using equation (8))} \\
&= \bigcup_{\phi_j \in \mathcal{A}, \phi_j \models \phi} i_{\mathcal{A}}(\phi_j) \setminus \bigcup_{\phi_l \in \mathcal{A}, \phi_l \models \phi} i_{\mathcal{A}}(\phi_l) \text{ (as } \phi_j \models \phi, \phi_l \models \phi_j \text{, so } \phi_l \models \phi) \\
&= \{\} \text{ (as } \{\phi_j \mid \phi_j \in \mathcal{A}, \phi_j \models \phi\} = \{\phi_l \mid \phi_l \in \mathcal{A}, \phi_l \models \phi\}).
\end{aligned}
$$

As $(\mathcal{A}, i_{\mathcal{B}}) = (\mathcal{A}, i_{\mathcal{A}})$, we have $(\mathcal{A}, ii_{\mathcal{B}}) = (\mathcal{A}, ii_{\mathcal{A}}) = (\mathcal{A}_0, ii)$ (from Theorem 3). Therefore, the basic incidence assignment derived from $i_{\mathcal{B}}$ is the same as ii.

QED

The conclusion we derived from the above two theorems is that a single incidence assignment is mapped to a unique basic incidence assignment and a single basic incidence assignment can be mapped to a family of incidence assignments.

5 The Method of Recovering an Incidence Function from Numerical Assignments

Given a GICT, the lower bounds of probabilities on formulae can be inferred through equation (3), when i_* is obtained. However sometimes numerical assignments are given on some formulae directly without defining any incidence functions. We are interested in how to find consistent GICTs in these cases. The central part for achieving this is to define incidence functions which may be derived from a basic incidence assignment. So discovering a basic incidence assignment would be the first step in the whole recovery procedure. In this section, we discuss how to extend Algorithm A to find a basic incidence assignment and its incidence function in these circumstances.

5.1 An algorithm for assigning incidences

Algorithm B: From Numerical Assignments to Incidence Functions

Given a set of axioms $\mathcal{A}$ which is closed under $\wedge$ and a lower bound of probability distribution $Prob_*$ on $\mathcal{A}$, construct a function ii on $\mathcal{A}_0$ ($\mathcal{A}_0 \subseteq \mathcal{A}$) and a function i on $\mathcal{A}$.

Step 1: *Let $\mathcal{A}_0$ be the subset of $\mathcal{A} \setminus \{false\}$ as defined in equation (5). If there are l elements in $\mathcal{A}_0$ each of which satisfies $Prob_*(\phi_j) > 0$ (if $Prob_*(\phi_j) = 0$, it is not necessary to introduce a possible world w) then l possible worlds will be defined and $\mathcal{W} = \{w_1, w_2, ..., w_l\}$.*

For $j = 1, ..., l, \phi_j \in \mathcal{A}_0$, define

$$
\begin{aligned}
&Prob'_*(w_j) = Prob_*(\phi_j), \\
&\mu(w_j) = Prob'_*(w_j), \\
&ii(\phi_j) = \{w_j\}, \\
&\mathcal{A}' = \mathcal{A} \setminus (\mathcal{A}_0 \cup \{false\}).
\end{aligned}
$$

Step 2: *Choose a formula ψ from $\mathcal{A}'$ which satisfies the condition that $\forall \psi' \in \mathcal{A}'$, $\psi' \not\models \psi$ if $\psi' \neq \psi$.*

Then define $Prob'_*(\psi) = Prob_*(\psi) - \Sigma_{\phi_j \in \mathcal{A}_0, \phi_j \models \psi} Prob'_*(\phi_j)$.

If $Prob'_*(\psi) > 0$, then add a new possible world w_{l+1} to $\mathcal{W}$ and define

$$\begin{aligned}
&\mu(w_{l+1}) = Prob'_*(\psi),\\
&ii(\psi) = \{w_{l+1}\},\\
&\mathcal{A}_0 := \mathcal{A}_0 \cup \{\psi\},\\
&\mathcal{A}' := \mathcal{A}' \setminus \{\psi\},\\
&l := l + 1.
\end{aligned}$$

If $Prob'_*(\psi) = 0$, let $\mathcal{A}' = \mathcal{A}' \setminus \{\psi\}$. No new possible world will be created to match ψ.

If $Prob'_*(\psi) < 0$, this assignment is not consistent, stop the procedure.

Repeat this step until $\mathcal{A}'$ is left with only tautologies.

Step 3: *If $\Sigma_j(Prob'_*(\phi_j)) < 1$, then add a possible world w_{l+1} to $\mathcal{W}$. Define*

$$\begin{aligned}
&\mu(w_{l+1}) = 1 - \Sigma_j Prob'_*(\phi_j),\\
&\mathcal{A}_0 = \mathcal{A}_0 \cup \{true\},\\
&ii(true) = \{w_{l+1}\}.
\end{aligned}$$

Step 4: *The final set of possible worlds is $\mathcal{W} = \{w_1, w_2, ..., w_{l+1}\}$ and the probability distribution is $\mu(w_i)$, and $\Sigma_i \mu(w_i) = 1$. ii is a function on $\mathcal{A}_0$. Finally, we define a function i as:*

$$i(\phi) = \bigcup_{\phi_j \in \mathcal{A}_0, \phi_j \models \phi} ii(\phi_j), \;\; \phi \in \mathcal{A}$$

$$i(false) = \{\}.$$

If there are n elements in $\mathcal{A}$ then there are at most $n+1$ elements in $\mathcal{W}$.

Theorem 5 *Given $(\mathcal{A}, Prob_*)$ where $\mathcal{A}$ is a set of axioms closed under $\wedge$ and $Prob_*$ is an assignment of lower bounds of probabilities on $\mathcal{A}$, functions i and ii obtained from Algorithm B are an incidence function and a basic incidence assignment respectively. The corresponding generalized incidence calculus theory $< \mathcal{W}, \mu, P, \mathcal{A}, i >$ can produce $Prob_*$ on $\mathcal{A}$.*

Proof

Part I: ii and i are a basic incidence assignment and an incidence function respectively.

For any two formulae ϕ and ψ in $\mathcal{A}_0$, we have

$$\begin{aligned}
&ii(\phi) \cap ii(\psi) = \{\} \;\; (when \;\; \phi \neq \psi)\\
&ii(true) = \{w_{n+1}\} = \mathcal{W} \setminus \cup_{\phi_j \in \mathcal{A}} ii(\phi_j).
\end{aligned}$$

So ii is a basic incidence assignment. Therefore $i(\psi) = \cup_{\phi_j \models \psi} ii(\phi_j)$ is an incidence function on $\mathcal{A}$ based on Theorem 3.

The corresponding generalized incidence calculus theory is

$$< \mathcal{W}, \mu, P, \mathcal{A}, i > .$$

Part II: The lower bounds of probabilities of formulae $\mu(i_*(\phi))$ is the same as the original numerical assignment $Prob_*(\phi)$.

For any $\psi \in \mathcal{A}$, we can calculate the lower bound of its probability, denoted by p_{i*} (in order to distinguish it from $Prob_*$), as follows.

$$\begin{aligned}
p_{i*}(\psi) &= \mu(i_*(\psi)) \\
&= \mu(\cup_{\phi_j \in \mathcal{A}, \phi_j \models \psi} i(\phi_j)) \\
&= \mu(\cup_{\phi_j \in \mathcal{A}, \phi_j \models \psi} \cup_{\phi_{jl} \in \mathcal{A}, \phi_{jl} \models \phi_j} ii(\phi_{jl})) \\
&= \mu(\cup_{\phi_{jl} \in \mathcal{A}, \phi_{jl} \models \psi} ii(\phi_{jl})) \\
&= \mu(\cup_{\phi_{jl} \in \mathcal{A}, \phi_{jl} \models \psi, \phi_{jl} \neq \psi} ii(\phi_{jl})) + \mu(ii(\psi)) \\
&= \Sigma_{\phi_{jl} \in \mathcal{A}, \phi_{jl} \models \psi} \mu(ii(\phi_{jl})) + \mu(ii(\psi)) \\
&= \Sigma_{\phi_{jl} \in \mathcal{A}, \phi_{jl} \models \psi} Prob'_*(\phi_{jl}) + Prob'_*(\psi) \\
&= Prob_*(\psi).
\end{aligned}$$

So this theory produces the same lower bounds of probabilities for those formulae in $\mathcal{A}$ as $Prob_*$.

QED

This algorithm is entirely based on the result that $ii(\phi) \cap ii(\psi) = \{\}$. In Algorithm B, for a formula ϕ, we keep deleting those portions in $Prob_*(\phi)$ which can be carried by formulae ϕ_j (where $\phi_j \models \phi$) until we obtain the last part which must be carried by ϕ itself. This last portion will only be contributed by its basic incidence set.

5.2 Unique output of the algorithm

In this section we first give an example to demonstrate the use of Algorithm B. The example is reconstructed from [12]. Then we prove that the output of the algorithm is unique regardless of the order of selecting axioms.

Example 5.1

Suppose that we have P, $\mathcal{L}(P)$ and a set of axioms $\mathcal{A} = \{a, b, c, a \wedge b, a \wedge c, b \wedge c, a \wedge b \wedge c, false, true\}$ with the lower bound of a probability distribution $Prob_*$ as

$$\begin{aligned}
&Prob_*(a) = 0.760, && Prob_*(b) = 0.640, \\
&Prob_*(c) = 0.480, && Prob_*(a \wedge b) = 0.525, \\
&Prob_*(a \wedge c) = 0.350, && Prob_*(b \wedge c) = 0.225, \\
&Prob_*(a \wedge b \wedge c) = 0.165, && Prob_*(true) = 1, \\
&Prob_*(false) = 0.
\end{aligned}$$

The set $\mathcal{A}$ is closed under the operator $\wedge$. Using Algorithm B, an incidence function is defined from the following steps.

Step 1. The set $\mathcal{A}_0$ is $\{a \wedge b \wedge c\}$ which contains the smallest formula in $\mathcal{A}$. This means that there is one possible world, $w_1 \in \mathcal{W}$, supporting formula $a \wedge b \wedge c$ and $\mu(w_1) = 0.165$. We also have

$$\begin{aligned}
&Prob'_*(a \wedge b \wedge c) = Prob_*(a \wedge b \wedge c) = 0.165,\\
&\mu(w_1) = 0.165,\\
&ii(a \wedge b \wedge c) = \{w_1\},\\
&\mathcal{A}' = \mathcal{A} \setminus (\mathcal{A}_0 \cup \{false\}).
\end{aligned}$$

Step 2. Choose formula $a \wedge b$ from $\mathcal{A}'$. Because $a \wedge b \wedge c$ is the only formula in $\mathcal{A}_0$ and it has the property that $a \wedge b \wedge c \models a \wedge b$, we have

$$Prob'_*(a \wedge b) = Prob_*(a \wedge b) - Prob'_*(a \wedge b \wedge c) = 0.525 - 0.165 = 0.36.$$

Since $Prob'_*(a \wedge b) > 0$, we have $\mathcal{W} = \mathcal{W} \cup \{w_2\}$ and define

$$\begin{aligned}
&ii(a \wedge b) = \{w_2\},\\
&\mu(w_2) = Prob'_*(a \wedge b),\\
&\mathcal{A}_0 = \mathcal{A}_0 \cup \{a \wedge b\},\\
&\mathcal{A}' = \mathcal{A}' \setminus \{a \wedge b\},\\
&l = 2.
\end{aligned}$$

Repeating this step for each of the remaining elements in $\mathcal{A}'$, we get

$$\begin{array}{llll}
\mathcal{W} = \mathcal{W} \cup \{w_3\}, & ii(a \wedge c) = \{w_3\}, & \mu(w_3) = 0.185, & \mathcal{A}_0 = \mathcal{A}_0 \cup \{a \wedge c\};\\
\mathcal{W} = \mathcal{W} \cup \{w_4\}, & ii(b \wedge c) = \{w_4\}, & \mu(w_4) = 0.06, & \mathcal{A}_0 = \mathcal{A}_0 \cup \{b \wedge c\};\\
\mathcal{W} = \mathcal{W} \cup \{w_5\}, & ii(a) = \{w_5\}, & \mu(w_5) = 0.05, & \mathcal{A}_0 = \mathcal{A}_0 \cup \{a\};\\
\mathcal{W} = \mathcal{W} \cup \{w_6\}, & ii(b) = \{w_6\}, & \mu(w_6) = 0.055, & \mathcal{A}_0 = \mathcal{A}_0 \cup \{b\};\\
\mathcal{W} = \mathcal{W} \cup \{w_7\}, & ii(c) = \{w_7\}, & \mu(w_7) = 0.07, & \mathcal{A}_0 = \mathcal{A}_0 \cup \{c\}.
\end{array}$$

Also, $l = 7$.

Step 3. As

$$\Sigma_j Prob'_*(\phi_j) = 0.165 + 0.36 + 0.185 + 0.06 + 0.05 + 0.055 + 0.07 = 0.945,$$

we have $\mathcal{W} = \mathcal{W} \cup \{w_8\}$ and define

$$\begin{aligned}
&\mu(w_8) = 1 - \Sigma_j \mu(ii(\phi_j)) = 1 - \mu(\{w_1, ..., w_7\}) = 0.055,\\
&\mathcal{A}_0 = \mathcal{A}_0 \cup \{true\},\\
&ii(true) = \{w_8\}.
\end{aligned}$$

Step 4. We obtain $\mathcal{W} = \{w_1, ..., w_8\}$ with probability distribution μ on it. ii is a basic incidence assignment on $\mathcal{A}_0$. The incidence function derived from $i(\phi) = \cup_{\phi_j \models \phi} ii(\phi_j)$ on axioms set $\mathcal{A}$ is as shown below.

$$
\begin{array}{ll}
i(a \wedge b \wedge c) = \{w_1\}, & i(a \wedge b) = \{w_1, w_2\}, \\
i(a \wedge c) = \{w_1, w_3\}, & i(b \wedge c) = \{w_1, w_4\}, \\
i(a) = \{w_1, w_2, w_3, w_4\}, & i(b) = \{w_1, w_2, w_4, w_6\}, \\
i(c) = \{w_1, w_3, w_4, w_7\}, & i(true) = \mathcal{W}, \\
i(false) = \{\}. &
\end{array}
$$

□

When we apply Algorithm B, there may be more than one formula satisfying the conditions in Step 2, but the order of choosing these formulae has no effect on the final result. In this example, after we choose $a \wedge b \wedge c$ in Step 1 and come to Step 2, it does not matter whether we choose $a \wedge b$ or $a \wedge c$ first. The final result remains the same.

Algorithms A and B were both implemented on a Sun Sparc 4 Workstation in Sicstus Prolog 2.1. The execution time for this example is 0.759 (milliseconds) in the order $\{a \wedge b \wedge c, a \wedge c, b \wedge c, a, b, a \wedge b, c, false, true\}$. If the axioms are reordered as $\{a, b, c, b \wedge c, a \wedge c, a \wedge b, a \wedge b \wedge c, false, true\}$, there is little difference. The runtime for the latter case is 1.189 (milliseconds). The algorithm creates a set of possible worlds with 8 elements.

Theorem 6 *Applying Algorithm B to $(\mathcal{A}, Prob_*)$ produces the same result regardless of the order of selecting formulae in Step 2.*

Proof

Assume that $\mathcal{A}_0 = \{\phi_1, ..., \phi_k\}$ after Step 1 and there are two formulae ψ_1, ψ_2 satisfying the condition specified in Step 2.

In Step 1, for every $\phi_j \in \mathcal{A}_0$, we have

$$Prob'_*(\phi_j) = Prob_*(\phi_j),$$

$$\mu(w_j) = Prob'_*(\phi_j),$$

$$ii(\phi_j) = \{w_j\}.$$

Assume that we choose ψ_1 first in Step 2, then we have

$$Prob'_*(\psi_1) = Prob_*(\psi_1) - \Sigma_{\phi_j \in \mathcal{A}_0, \psi_j \models \psi_1} Prob'_*(\phi_j).$$

Now we choose ψ_2 and obtain

$$
\begin{aligned}
Prob'_*(\psi_2) &= Prob_*(\psi_2) - \Sigma_{\phi_j \in \mathcal{A}_0 \cup \psi_1, \phi_j \models \psi_2} Prob'_*(\phi_j) \text{ (because } \psi_1 \text{ is a smallest formula now)} \\
&= Prob_*(\psi_2) - \Sigma_{\phi_j \in \mathcal{A}_0, \phi_j \models \psi_2} Prob'_*(\phi_j) \text{ (because } \psi_1 \not\models \psi_2),
\end{aligned}
$$

which indicates that adding ψ_1 into set $\mathcal{A}_0$ has no effect on the outcome of $Prob'_*(\psi_2)$.

In the same way we can prove that choosing ψ_2 first and then ψ_1 gives exactly the same $Prob'_*(\psi_1)$ and $Prob'_*(\psi_2)$ as above. That is, $Prob'_*$ on set $\mathcal{A}_0 \cup \{\psi_1, \psi_2\}$ remains the same no matter which formula in $\{\psi_1, \psi_2\}$ is chosen first. Similarly, we can prove the theorem for any set of formulae $\{\psi_1, ..., \psi_n\}$ satisfying the condition in Step 2.

QED

Applying Algorithm B to $(\mathcal{A}, Prob_*)$ only produces one basic incidence assignment, as the order of choosing axioms has no effect on the final output. Through this basic incidence assignment, not only an incidence function can be defined on $\mathcal{A}$, but also a family of incidence assignments can be obtained through Theorem 5, which would share the same set of possible worlds, but with a different set of axioms.

The size of a set of possible worlds created from Algorithm B is related only to the size of the initial set of axioms. What would happen if we have some ideas about the size of a set of possible worlds already (say, a set of possible worlds with 100 elements) and that is larger than the set obtained from Algorithm B? In other words, there may be many pairs, $(\mathcal{W}_j, \mu_j)$, which would generate $Prob_*$ on $\mathcal{A}$, as long as an appropriate i is available, as the situations in [17].

In the next section, the nature of the basic incidence assignment created from Algorithm B will be examined, and this assignment will be compared with any potential consistent assignment, given a numerical assignment on a closed set. The result shows that the pair $(\mathcal{W}, \mu)$ obtained from Algorithm B subsumes all the possible pairs $(\mathcal{W}_j, \mu_j)$ from other approaches. In another word, given a pair $(\mathcal{W}_j, \mu_j)$, $\mathcal{W}$ partitions $\mathcal{W}_j$, and for each element A in the partition, there is an element $w \in \mathcal{W}$ with $\mu(w) = \sum_{w_j \in A} \mu_j(w_j)$.

5.3 Comparison with related approaches

There are two algorithms in [16] , [17] for assigning incidences on axioms based on a probability assignment, one of which is the extension of a method in [3] in the original incidence calculus. The common feature of the two algorithms is that a set of possible worlds has to be fixed first, such as a set of 100 elements, each of which with probability 1/100. Given a set of axioms, both algorithms try to divide possible worlds into groups and assign each group to an axiom. Therefore, not only the number of axioms but also the interrelationship of these axioms affect the division procedure. For instance, assume that there are only two axioms a, b in $\mathcal{A}$, then the assignment procedure could be simply done by choosing two subsets of the set of possible worlds which can produce $Prob_*(a)$ and $Prob_*(b)$ respectively. However, if $Prob_*(a \wedge b)$ is known as well in addition to $Prob_*(a)$ and $Prob_*(b)$, then we need not only two subsets of possible worlds W_1, W_2 to match $Prob_*(a)$ and $Prob_*(b)$, but also another subset W_3 which matches $Prob_*(a \wedge b)$ with the condition that $W_3 = W_1 \cap W_2$. That is, the complexity of these two algorithms increases along with the interrelationship of axioms considerably. Because of this, the order of axioms also affects the efficiency of the algorithms [16], [17]

In summary, there are three factors associated with the complexity of each algorithm in [16], [17]: the number of axioms, the relations among axioms, and the order of axioms in addition to the requirement of a fixed number of possible worlds. However, in our algorithm, only one factor affects the complexity, that is, the number of axioms. Besides, the new algorithm does not require a set of possible worlds to be predefined.

Example 5.1 has also been tested [16], [17] using the two algorithms, with a predefined set of possible worlds containing 100 elements, each of which with 1/100 probability. Although there are only 8 axioms in $\mathcal{A}$, both algorithms take a long time to find a consistent assignment of incidences (with runtime 374.520 and 355.060 seconds respectively). Our algorithm only needs

1.189 (milliseconds) runtime to find a consistent assignment (all these experiments were carried out on Sun Sparc 4 stations in Sicstus Prolog). This example illustrates that the relations among axioms could slow the algorithms down enormously, much worse than the size of set of axioms. In real world cases, axioms always have some interrelations.

6 Properties of Basic Incidence Assignments

In [15], the theoretical background of the basic incidence assignment was investigated to explain why Algorithms B produces only one basic incidence assignment comparing with a large number of outputs from other approaches, and to examine whether Algorithm B has omitted any alternative incidence assignments implied by the numerical assignment.

In this section, we will summarize the research carried out in [15]. We argue that one basic incidence assignment is the only possible outcome of Algorithm B. All the consistent incidence assignments from the numerical assignment can be generated from this unique basic incidence assignment.

6.1 Similarity of separate incidence assignments

Definition 6.1 *A regular GICT*

A GICT $< \mathcal{W}, \mu, P, \mathcal{A}, i >$ *is said to be regular iff*

$$\forall w \in \mathcal{W}, \mu(w) \neq 0.$$

That is, there are no possible worlds with zero probability. Any GICT can be converted to an equivalent regular theory by simply removing the excess possible worlds with zero probability. For each w with $\mu(w) = 0$, redefine

$$\begin{aligned}
&\mathcal{W}' = \mathcal{W} \setminus \{w\}, \\
&\forall w \in \mathcal{W}', \mu'(w) = \mu(w), \\
&\forall \phi \in \mathcal{A}, i'(\phi) = i(\phi) \setminus \{w\}.
\end{aligned}$$

In the rest of this section, we assume that all GICTs have already been transformed in this way, if necessary, and when we refer to two GICTs as being equal, we mean subject to removal of these possible worlds.

Definition 6.2 *Similarity of possible worlds*

Given an generalized incidence calculus theory (GICT), $< \mathcal{W}, \mu, P, \mathcal{A}, i >$*, two possible worlds* $w_1, w_2 \in \mathcal{W}$ *are similar* ($w_1 \sim w_2$) *iff*

$$\forall \phi \in \mathcal{A}, w_1 \in i(\phi) \Longleftrightarrow w_2 \in i(\phi).$$

Clearly, similarity is an equivalence relation. It is reflexive ($w \sim w$), symmetric ($w_1 \sim w_2 \Rightarrow w_2 \sim w_1$) and transitive ($w_1 \sim w_2 \wedge w_2 \sim w_3 \Rightarrow w_1 \sim w_3$).

Therefore, we can talk about equivalence classes of similar possible worlds, and indeed in any GICT $\mathcal{W}$ will be partitioned by the set of all such classes.

Definition 6.3 *Fundamental generalized incidence calculus theory (GICT)*

A GICT, $< \mathcal{W}, \mu, P, \mathcal{A}, i >$, is fundamental iff it has no distinct similar possible worlds. That is, iff

$$\forall w_1, w_2 \in \mathcal{W}, w_1 \sim w_2 \Rightarrow w_1 = w_2.$$

Definition 6.4 *Direct subsumption of GICT*

A GICT, $< \mathcal{W}', \mu', P, \mathcal{A}, i' >$, is directly subsumed by $< \mathcal{W}, \mu, P, \mathcal{A}, i >$ iff

- *For some $w_1', w_2' \in \mathcal{W}'$ with $w_1' \sim w_2'$ and some $w_3 \notin \mathcal{W}'$, $\mathcal{W} = (\mathcal{W}' \setminus \{w_1', w_2'\}) \cup \{w_3\}$*
- $$\mu(w) = \begin{cases} \mu'(w) & : w \neq w_3 \\ \mu'(w_1') + \mu'(w_2') & : w = w_3 \end{cases}$$
- $$i(\phi) = \begin{cases} i'(\phi) & : w_1' \notin i'(\phi) \\ (i'(\phi) \setminus \{w_1', w_2'\}) \cup \{w_3\} & : w_1' \in i'(\phi) \end{cases}$$

In this definition, the fact that $w_1' \in i'(\phi)$ will imply that $w_2' \in i'(\phi)$, since $w_1' \sim w_2'$. So a subsumption replaces two possible worlds which occur in the same set of incidences with a single possible world whose probability is the sum of the probabilities of the previous two.

Definition 6.5 *Subsumption of GICTs*

A GICT $< \mathcal{W}_0, \mu_0, P, \mathcal{A}, i_0 >$ is subsumed by $< \mathcal{W}_n, \mu_n, P, \mathcal{A}, i_n >$ iff there is a list of GICTs

$$[< \mathcal{W}_0, \mu_0, P, \mathcal{A}, i_0 >, ..., < \mathcal{W}_n, \mu_n, P, \mathcal{A}, i_n >]$$

such that $\forall j = 1, ..., n$, $< \mathcal{W}_{j-1}, \mu_{j-1}, P, \mathcal{A}, i_{j-1} >$ is directly subsumed by $< \mathcal{W}_j, \mu_j, P, \mathcal{A}, i_j >$.

It is worth pointing out that the list can be singleton. That is any GICT subsumes itself.

Theorem 7 *A fundamental GICT can only be subsumed by itself.*

Proof

Assume that a GICT subsumes a fundamental one, then there will be a chain of direct subsumption between the two. A direct subsumption acts on a pair of similar worlds in a GICT, but there can be no such pair in a fundamental theory and so the chain must have zero length. Therefore the fundamental GICT is subsumed by itself.

QED

Since probabilities of formulae (or lower bounds of probabilities, to be precise) are calculated through the incidence sets of these formulae, we can have the following statement.

If $< \mathcal{W}_1, \mu_1, P, \mathcal{A}, i_1 >$ subsumes $< \mathcal{W}_2, \mu_2, P, \mathcal{A}, i_2 >$, then

$$\forall \phi \in P, \mu_1(i_1(\phi)) = \mu_2(i_2(\phi))$$

where μ applied to a set is the sum of the results of applying μ to the individual members of that set.

Theorem 8 *Subsumption preserves lower bounds of probability.*

Proof

As a subsumption is made up purely of a series of direct subsumptions, it is sufficient to show that direct subsumption preserves lower bounds of probability. This, however, is not difficult, as $Prob_*(i(\phi)) = \mu(i(\phi))$. Let $< \mathcal{W}, \mu, P, \mathcal{A}, i >$directly subsume $< \mathcal{W}', \mu', P, \mathcal{A}, i' >$. Let w be the new possible world which is introduced in Definition 6.4. Now, if $w \notin i'(\phi)$ then

$$Prob'_*(\phi) = \mu'(i'(\phi)) = \mu'(i(\phi)) = Prob_*(\phi).$$

If, on the other hand, $w \in i'(\phi)$, then

$$\begin{aligned}
Prob'_*(\phi) &= \mu'(i'(\phi)) \\
&= \mu'((i(\phi) \setminus \{w_1, w_2\}) \cup \{w\}) \\
&= \mu'(i(\phi) \setminus \{w_1, w_2\}) + \mu'(w) \\
&= \mu(i(\phi) \setminus \{w_1, w_2\}) + \mu(w_1) + \mu(w_2) \\
&= \mu((i(\phi) \setminus \{w_1, w_2\}) \cup \{w_1\} \cup \{w_2\}) \\
&= \mu(i(\phi)) \\
&= Prob_*(\phi).
\end{aligned}$$

QED

Note that because subsumption preserves lower bounds of probabilities in the above way, is also preserves similarity of possible worlds. To illustrate this, we will define the function of occurrence.

Definition 6.6: *Occurrence function*

In a GICT, $< \mathcal{W}, \mu, P, \mathcal{A}, i >$, the occurrence (o) is a function mapping the possible worlds (in $\mathcal{W}$) to the subsets of the axioms (in $\wedge(\mathcal{A})$ which means that $\mathcal{A}$ is closed under $\wedge$) defined by

$$o(w) = \{\phi \in \mathcal{A} \mid w \in i(\phi)\}.$$

So the occurrence of a possible world is the set of axioms in whose incidence sets it occurs. It is then trivial to show from Definition 6.2, that

$$w_1 \sim w_2 \Longleftrightarrow o(w_1) = o(w_2).$$

Subsumption preserves occurrence (and hence similarity) since a possible world in a subsumption theory will either have existed in the original GICT with the same occurrence, or it will have replaced two similar possible worlds. In this later case, the occurrence of the new possible world will (by Definitions 6.4 and 6.5) be the same as that of the first two, and it will therefore be similar to any other possible world which was similar to the first two.

Definition 6.7: *Fundamental Subsumption*

A GICT has a fundamental subsumption iff it is subsumed by a fundamental GICT.

Theorem 9 *Fundamental subsumption is unique up to renaming.*

Proof

Let us assume that a GICT, $< \mathcal{W}, \mu, P, \mathcal{A}, i >$, is subsumed by two fundamental GICTs, $< \mathcal{W}_1, \mu_1, P, \mathcal{A}, i_1 >$ and $< \mathcal{W}_2, \mu_2, P, \mathcal{A}, i_2 >$. We must provide an isomorphism between these to show that they are unique up to renaming. Clearly the sets of atomic propositions and axioms are the same in both cases.

Each direct subsumption acts on a pair of similar possible worlds, and replaces that pair with a single possible world. As stated earlier, subsumption preserves similarity of possible worlds. That is for any equivalence class of $\sim$ and a direct subsumption acting on two possible worlds in it, the equivalence structure will be retained, with the substitution of the new possible world for the old two. A fundamental subsumption continues this process until each equivalence class is reduced to a single element; if any has more than one, the GICT is not fundamental as similarities exist, and none can be reduced to zero. If the number of equivalence classes in $\mathcal{W}$ is n, then the number of elements in both $\mathcal{W}_1$ and $\mathcal{W}_2$ will therefore also be n. We form an isomorphism between the two sets of possible worlds based on the original equivalence classes; an element $w_1 \in \mathcal{W}_1$ is mapped to $w_2 \in \mathcal{W}_2$ when w_1 and w_2 are subsumed from the same equivalence class in $\mathcal{W}$.

By Definition 6.4, μ_1 and μ_2 of each element will be the same of μ applied to all members of the corresponding equivalence set. As this original set is the same for both cases, μ_1 and μ_2 are isomorphic.

i_1 and i_2 are also isomorphic, this is clear as subsumption is incidence preserving.

QED

Theorem 10 *Every GICT is subsumed by a fundamental GICT.*

Proof

Let $< \mathcal{W}, \mu, P, \mathcal{A}, i >$ be a GICT. If it is fundamental, then it subsumes itself and we are done. If it is not fundamental, then there are two possible worlds, $w_1, w_2 \in \mathcal{W}$ such that $w_1 \sim w_2$. We can therefore define a new GICT $< \mathcal{W}', \mu', P, \mathcal{A}, i' >$ which directly subsumes our original theory (by Definition 6.4). Clearly, if $< \mathcal{W}', \mu', P, \mathcal{A}, i' >$ directly subsumes $< \mathcal{W}, \mu, P, \mathcal{A}, i >$ then $\mathcal{W}'$ has one less element than $\mathcal{W}$. We can continue this process of direct subsumption, reducing the size of $\mathcal{W}$ at every step. As $\mathcal{W}$ is finite, and GICTs with less than two possible worlds are necessarily fundamental, we will eventually reach a fundamental GICT, which will subsume the original GICT by the definition of subsumption as a chain of direct subsumptions.

QED

So we have now shown that every GICT is subsumed by one and only one fundamental GICT. We are now ready to define our equivalence relation on GICTs.

Definition 6.8: Similarity of GICTs

Two GICTs $< \mathcal{W}_1, \mu_1, P, \mathcal{A}, i_1 >$ *and* $< \mathcal{W}_2, \mu_2, P, \mathcal{A}, i_2 >$ *are similar* ($< \mathcal{W}_1, \mu_1, P, \mathcal{A}, i_1 > \sim < \mathcal{W}_2, \mu_2, P, \mathcal{A}, i_2 >$) *iff they are subsumed by the same fundamental GICT.*

It is clear that this is an equivalence relation, since every GICT is subsumed by itself, it is reflexive; the symmetricity is trivial; and the transitivity again follows trivially from the above results.

6.2 Fundamental nature of basic incidence assignment

Based on previous discussion, it is therefore necessary to prove that a GICT derived from applying the basic incidence assignment algorithm is fundamental. We must, before attempting to prove this, address the problem of possible worlds with zero probability. Algorithm B has ensured that no such possible worlds would have been created. So the derived GICT is a regular one.

Theorem 11 *Any GICT derived from applying Algorithm B to* $(\mathcal{A}, Prob_*)$ *(*$\mathcal{A}$ *is closed under* $\wedge$*) is fundamental, for that family of GICTs which have the same set of axioms.*

Proof

Given a numerical assignment on a set of axioms $\mathcal{A}$, which is closed under $\wedge$, Algorithm B will produce a basic incidence assignment ii on $\mathcal{A}_0$ ($\mathcal{A}_0 \subseteq \mathcal{A}$). A number of incidence assignments can be derived from this basic incidence assignment on different sets of axioms (Theorems 3 and 4) where $\mathcal{A}$ is the smallest set among all of them. We need to prove that every such derived GICT is fundamental to that family of GICTs which have the same set of axioms. To do this, we only need to take the GICT with $\mathcal{A}$ as the set of axioms as an example, as others can be proved similarly.

So we need to prove that GICT, $< \mathcal{W}, \mu, P, \mathcal{A}, i >$, derived from the basic incidence assignment ii on $\mathcal{A}_0$ ($\mathcal{A}_0 \subseteq \mathcal{A}$) is fundamental to all other GICTs in the form $< \mathcal{W}_j, \mu_j, P, \mathcal{A}, i_j >$ which share $\mathcal{A}$ but with different sets of possible worlds and probability distributions.

To prove this, we must show that no two distinct possible worlds in this GICT are similar - that is that for every pair of possible worlds w_1, w_2, there must be at least one $\phi \in \mathcal{A}$ such that either $w_1 \in i(\phi) \wedge w_2 \notin i(\phi)$ or $w_1 \notin i(\phi) \wedge w_2 \in i(\phi)$.

For simplicity, we will take w_1, w_2 to be ordered so that w_1 was created before w_2 in the algorithm. From the procedure of the algorithm, we know that there is a formula $\phi_1 \in \mathcal{A}$, that $ii(\phi_1) = \{w_1\}$, and likewise for w_2. Clearly $w_1 \in i(\phi_1)$. We will show that $w_2 \notin i(\phi_1)$, which is sufficient, as w_1, w_2 are arbitrary.

In Algorithm B, we have

$$i(\phi_1) = \cup_{\phi_j \models \phi_1} ii(\phi_j).$$

So $w_2 \in i(\phi_1)$ means that $ii(\phi_l) = \{w_2\}$ for a specific ϕ_l, because of the property of basic incidence assignment $ii(\phi_i) \cap ii(\phi_j) = \{\}$. Therefore, $w_2 \in i(\phi_1)$ implies that $\phi_l \models \phi_1$. However, if $\phi_l \models \phi_1$, then ϕ_1 should not have been selected before ϕ_l, which contradicts the choice of w_2. Therefore this GICT is fundamental.

Similarly, as every other GICT derived from this basic incidence assignment has the same set of possible worlds, every such GICT must be fundamental.

QED

So, we have shown that every GICT is subsumed by a fundamental one, and that GICTs derived from Algorithm B are fundamental. What we have to show next is that any GICT derived from any other incidence assignment approach is subsumed by a GICT from Algorithm B, given $(\mathcal{A}, Prob_*)$, where $\mathcal{A}$ is closed as usual.

Definition 6.9 *Strict Incidence Set*

In a GICT, the strict incidence set of a subset of axioms is the intersection of the incidence sets of all the axioms in the subset with the complement of the incidence sets of all the axioms not in the subset. In other words, for a subset, $\mathcal{A}'$, of set $\mathcal{A}$,

$$si(\mathcal{A}') = (\bigcap_{\phi \in \mathcal{A}'} i(\phi)) \cap (\bigcap_{\psi \in \mathcal{A} \setminus \mathcal{A}'} \mathcal{W} \setminus i(\phi)).$$

Intuitively, the strict incidence set of a set of axioms is the set of possible worlds in which those axioms, and only those axioms, are true.

Definition 6.10 *Basic Probability Portion*

Given a GICT $< \mathcal{W}, \mu, P, A, i >$, assume that ii is the basic incidence assignment derived from i on domain $\mathcal{A}_0$, then probability $\mu(ii(\phi))$ for $\phi \in \mathcal{A}_0$ is called the basic probability portion carried by ϕ, denoted as $bp(\phi)$. For any $\psi \in \mathcal{A} \setminus \mathcal{A}_0$, we define $bp(\psi) = 0$.

Clearly, following Definitions 3.3 and 6.10, we have

$$\sum_{\phi \in \mathcal{A}_0} bp(\phi) = 1.$$

This means that the basic probability portion of an axiom is the portion in its lower bound of probability that carried by that axiom, that cannot be obtained from other axioms.

Corollary 1 *Given a GICT, assume that the basic incidence assignment from it is ii on domain $\mathcal{A}_0$, then w_1 and w_2 are similar (Definition 6.2) if $w_1, w_2 \in ii(\phi)$.*

Proof

It is straightforward based on Definitions 3.3 and 6.2 and Theorem 2.

QED

Corollary 2 *In a fundamental GICT, the basic probability partion of an axiom ϕ is either 0, when $ii(\phi) = \{\}$, or $\mu(w)$ when $ii(\phi) = \{w\}$.*

Proof

This follows directly from Definitions 3.3 and 6.10, and Definition 6.3.

QED

Theorem 12 *A consistent lower bound of a probability distribution $Prob_*$ on a set of axioms $\mathcal{A}$, which is closed under conjunction, has one and only one fundamental consistent GICT, with $\mathcal{A}$ as the set of axioms.*

Proof

From Theorems 3 and 4 we know that GICT, $< \mathcal{W}, \mu, P, \mathcal{A}, i >$, derived from Algorithm B is named as the smallest theory because the set of axioms , $\mathcal{A}$, in this theory contains fewer elements than that in other derived theories, with sets of axioms as $\mathcal{B}_j$.

Assume that the corresponding GICTs of all consistent incidence assignments discovered from many other approaches are in set Q. Further assume that a subset of Q is Q_1 containing those GICTs which all have a comment set of axioms, $\mathcal{A}$. Then we need to prove that this smallest theory is the only fundamental GICT subsumes all GICTs in Q_1. Similarly, we can prove that each GICT in $Q \setminus Q_1$ is subsumed by a GICT obtained through Theorem 4.

Part I: Now, we will prove that the smallest GICT is the only fundamental GICT among all GICTs in Q_1.

Assume that we have a non-contradictory lower bound $Prob_*$ of a probability distribution on a finite set of n axioms $\mathcal{A}$, which is closed under conjunction. The consistency of the lower bound ensures that a consistent incidence assignment is derivable from Algorithm B, which is fundamental. We must now show that it is unique.

If there are n axioms in $\mathcal{A}$, then there are at most $n - 1$ possible worlds being created in Algorithm B. Each possible world w makes $ii(\phi) = \{w\}$ true (for a ϕ) and none for $false$. If the basic incidence assignment ii is defined on domain $\mathcal{A}_0$, then $bp(\phi) = \mu(w)$ for $\phi \in \mathcal{A}_0$ and $ii(\phi) = \{w\}$.

We now choose a GICT from Q_1 arbitrarily, say $< \mathcal{W}_1, \mu_1, P, \mathcal{A}, i_1 >$. There is a unique basic incidence assignment ii_1 derivable from i_1. We assume ii_1 is defined on domain $\mathcal{A}_1$. Then based on Definition 6.10, we have $bp_1(\psi) = \mu_1(W'), W' \subseteq \mathcal{W}_1$, for $\psi \in \mathcal{A}_1$ and $bp_1(\psi) = 0$ if $\psi \notin \mathcal{A}_1$.

However, as bp and bp_1 both define basic probability portions on axioms from the same consistent probability distribution, $bp(\phi)$ must be the same as $bp_1(\phi)$. Therefore, $\mathcal{A}_0$ and $\mathcal{A}_1$ specify the same set of axioms, for each of which the basic probability portion is not zero. That is:

$$\mu(ii(\phi)) = \mu_1(ii_1(\phi)), \phi \in \mathcal{A}_0, \phi \in \mathcal{A}_1.$$

In other words, we have

$$ii(\phi) = \{w\}, \mu(w) = \mu_1(ii_1(\phi)).$$

(i) if for every $\phi \in \mathcal{A}_1$, $ii_1(\phi)$ has only one element w', then we can rename this possible world as w, where $ii(\phi) = \{w\}$. So after renaming all the elements in $\mathcal{W}_1$, we have $\mathcal{W} = \mathcal{W}_1$. The smallest GICT is identical with $< \mathcal{W}_1, \mu_1, P, \mathcal{A}, i_1 >$.

(ii) if there exists an axiom ψ where $ii_1(\psi)$ has more than one element, say $ii_1(\psi) = \{w_1, w_2, ..., w_n\}$, then we have $w_1 \sim w_2 \sim ... \sim w_n$ (from Corollary 1). Therefore, according to Definitions 6.4 and 6.5, the smallest GICT $< \mathcal{W}, \mu, P, \mathcal{A}, i >$ subsumes $< \mathcal{W}_1, \mu_1, P, \mathcal{A}, i_1 >$.

In summary, any GICT in Q_1, $< \mathcal{W}_1, \mu_1, P, \mathcal{A}, i_1 >$, is either identical with the smallest GICT, when the former is also a fundamental one, or subsumed by the smallest GICT. So the smallest GICT is the unique fundamental GICT subsumes all the GICTs in Q_1.

Part II:

Similarly, we can prove that any GICT from any incidence assignment approach is either identical with a GICT derived from Theorem 4 or subsumed by it.

Therefore, there is only one fundamental GICT for that family of GICTs which all have the same set of axioms, and the fundamental one is the one derived from the basic incidence assignment approach.

QED

So, we have proved that for any lower bound of a consistent probability distribution on a set of axioms closed under conjunction, there is a unique fundamental incidence assignment and that this assignment is the one found by Algorithm B. Under this condition, any consistent incidence assignment produced by [17] is equivalent to the one produced in this paper, further more, any consistent assignment produced by *any* method will be subsumed by this fundamental one. As there are an infinite number of such assignment (take any possible world and divide it into two, each with half the probability and repeat the procedure), Algorithm B is simplier than other approaches.

7 Extending the Result to DS Theory and Probability Spaces

One of the meaningful extensions of Algorithm B is to determine whether a numerical distribution is a belief function (in DS theory) when $\mathcal{A}$ is the whole language set $\mathcal{L}(P)$, and to determine the mass function when it is. The algorithm can also be used to recover the corresponding probability space when $Prob_*$ is thought of as an inner measure (or a lower bound) on $\mathcal{A}$ in probability structures [9], [10].

7.1 Deriving mass functions in DS theory

7.1.1 DS theory

In DS theory, a piece of evidence is always described on a special set, called *a frame of discerment* Θ, which contains mutually exclusive and exhaustive answers for a question. This piece of evidence can either be in the form of a *mass function*, denoted as m, or in the form of a *belief function*, denoted as *bel*. The conditions for these two functions are

$$\Sigma_{A \subseteq \Theta} m(A) = 1; \;\; m(\{\}) = 0.$$
$$bel(\Theta) = 1,$$

$$bel(\cup_1^n A_i) \geq \Sigma_i bel(A_i) - \Sigma_{i>j} bel(A_i \cap A_j) + ... + (-1)^{-n} bel(\cap_i A_i),$$

where $n = \mid \Theta \mid$.

The relationship between a belief function and its mass function is unique. They can be recovered from each other as follows.

$$bel(A) = \Sigma_{B \subseteq A} m(B),$$
$$m(A) = \Sigma_{B \subseteq A, B \neq \emptyset} (-1)^{a-b} bel(B),$$

where $a - b = \mid (A \wedge \neg B) \mid$ and $A, B \in L(P)$ [21]. $\mid A \mid$ stands for the total number of elements in A.

If $m(A) > 0$, then A is called a *focal element* of this belief function.

7.1.2 Relationships between mass functions and basic incidence assignments

Incidence calculus is about calculating incidences and bounds of probabilities on formulae while DS theory is about calculating beliefs on subsets of a set. To establish some relationships between the two theories, we need to find a common domain to which both theories can talk.

In Section 2, the basic element set, $\mathcal{A}t$, of a given set of atomic propositions P is defined. $\mathcal{A}t$ satisfies the definition of a frame of discernment, so it is a frame of discernment and both belief functions and mass functions can be defined on it. Following the one-to-one relationship between $2^{\mathcal{A}t}$ and $\mathcal{L}(P)$ in Definition 2.3, we can map each formula ϕ to a subset of $\mathcal{A}t$ and denote this subset as A_ϕ.

Therefore, given a belief function *bel* on $\mathcal{A}t$, we can define a belief function on $\mathcal{L}(P)$ as $bel'(\phi) = bel(A_\phi)$ where $A_\phi \subseteq \mathcal{A}t$ and $\phi = \vee\delta_i$, $\delta_i \in A_\phi$. Therefore we can also talk about a belief function on a language set $\mathcal{L}(P)$.

Let $< \mathcal{W}, \mu, P, \mathcal{A}, i >$ be a GICT, and ii be the corresponding basic incidence assignment on $\mathcal{A}_0$. Then

$$\sum_{\phi \in \mathcal{A}_0} \mu(ii(\phi)) = 1.$$

If we define $m(\phi) = \mu(ii(\phi))$, m is a mass function and the elements in $\mathcal{A}_0$ are in fact the corresponding focal elements. So if we can discover a basic incidence assignment, we can also discover a mass function. This result also supports what we have proved in [13].

7.2 Deriving mass functions

Based on the discussion in the previous subsection, in the following we show an alternative way to derive a mass function from a numerical assignment by means of incidence calculus, when this assignment is a belief function. This is described in Algorithm C below.

Algorithm C: Deriving Mass Functions:

Given a numerical assignment $Prob_*$ on the set $\mathcal{A} = \mathcal{L}(P)$, determine whether $Prob_*$ is a belief function on this language set [4] *and obtain its mass function if it is.*

[4] In fact, this language set can be any frame of discernment.

Step 1: *Delete all those elements in $\mathcal{A}$ in which $Prob_*(*) = 0$. Similar to the action in Algorithm B, define a subset $\mathcal{A}_0$ out of $\mathcal{A}$ using formula (5).*

For every $\phi \in \mathcal{A}_0$, define $m(\phi) = Prob'_*(\phi) = Prob_*(\phi)$.

Define $\mathcal{A}' = \mathcal{A} \setminus \mathcal{A}_0$.

Step 2: *Choose a formula ψ from $\mathcal{A}'$ which satisfies the condition that $\forall \psi' \in \mathcal{A}'$, $\psi' \not\models \psi$ if $\psi' \neq \psi$.*

Define $Prob'_*(\psi) = Prob_*(\psi) - \Sigma_{\phi_j \in \mathcal{A}_0, \phi_j \models \psi} Prob'_*(\phi_j)$.

If $Prob'_*(\psi) > 0$, define

$$\begin{aligned} &\mathcal{A}_0 = \mathcal{A}_0 \cup \{\psi\}, \\ &\mathcal{A}' = \mathcal{A}' \setminus \{\psi\}, \\ &m(\psi) = Prob'_*(\psi). \end{aligned}$$

If $Prob'_*(\psi) = 0$ then ψ is not a focal element of this belief function.

If $Prob'_*(\phi) < 0$ then this assignment is not a belief function, stop the procedure.

Repeat this step until $\mathcal{A}'$ is empty.

Step 3: *All the elements in $\mathcal{A}_0$ will be the focal elements of this belief function and the function m defined in Step 2 is the corresponding mass function. It is easy to prove that $\Sigma_A m(A) = 1$.*

The algorithm tries to find the focal elements of a belief function one by one. Once all the focal elements are found and the uncertain values of these elements are defined, the corresponding mass function is known.

Example 7.1:

Assume that there are four elements in $\mathcal{A}t = \{a, b, c, d\}$. $\mathcal{A} = \mathcal{L}(P)$ is $\mathcal{A} = \{a, b, c, d, a \vee b, a \vee c, a \vee d, b \vee c, b \vee d, c \vee d, a \vee b \vee c, a \vee c \vee d, a \vee b \vee d, b \vee c \vee d, a \vee b \vee c \vee d = true, false\}$ and the corresponding numerical assignment on elements of $\mathcal{A}$ are

$$\begin{aligned} &Prob_*(\{a\}) = 0.5, && Prob_*(\{d\}) = 0.3, \\ &Prob_*(\{a \vee b\}) = 0.7, && Prob_*(\{a \vee c\}) = 0.5, \\ &Prob_*(\{a \vee d\}) = 0.8, && Prob_*(\{b \vee d\}) = 0.3, \\ &Prob_*(\{c \vee d\}) = 0.3, && Prob_*(\{a \vee b \vee c\}) = 0.7, \\ &Prob_*(\{a \vee c \vee d\}) = 0.8, && Prob_*(\{a \vee b \vee d\}) = 1, \\ &Prob_*(\{b \vee c \vee d\}) = 0.3, && Prob_*(\{a \vee b \vee c \vee d\} = true) = 1. \end{aligned}$$

All the rest formulae have zero value of lower bounds.

Applying Algorithm C on $(\mathcal{A}, Prob_*)$, the calculating procedure for a mass function is as follows.

Step 1. After deleting those elements with 0 degrees of belief, we have $\mathcal{A} = \{a, d, a \vee c, a \vee b, a \vee d, b \vee d, c \vee d, a \vee b \vee c, a \vee c \vee d, a \vee b \vee d, b \vee c \vee d, a \vee b \vee c \vee d = true\}$

$\mathcal{A}_0 = \{a, d\}$.

Define $m(a) = Prob'_*(a) = Prob_*(a) = 0.5$, $m(d) = Prob'_*(d) = Prob_*(d) = 0.3$. $\mathcal{A}' = \mathcal{A} \backslash \mathcal{A}_0$.

Step 2. Get $a \vee c$ from $\mathcal{A}'$. Because $a \models a \vee c$, we have $Prob'_*(a \vee c) = Prob_*(a \vee c) - Prob'_*(a) = 0.5 - 0.5 = 0$. So $a \vee c$ is not a focal element. Define $\mathcal{A}' = \mathcal{A}' \setminus \{a \vee c\}$.

Repeat this procedure until we get $a \vee b$ and we have $Prob'_*(a \vee b) = 0.7 - 0.5 = 0.2$. Define

$$\begin{aligned} &m(a \vee b) = Prob'_*(a \vee b) = 0.2, \\ &\mathcal{A}_0 = \mathcal{A}_0 \cup \{a \vee b\}, \\ &\mathcal{A}' = \mathcal{A}' \setminus \{a \vee b\}. \end{aligned}$$

Repeat this procedure until $\mathcal{A}'$ is empty.

Step 3. We get $\mathcal{A}_0 = \{a, d, a \vee b\}$ and the mass function m gives $m(a) = 0.5, m(d) = 0.3, m(a \vee c) = 0.2$.

□

7.3 Recovering probability spaces

In [9], [10], Fagin and Halper suggested a method to assign probability measures on formulae instead of on sets. In this method, given a probability space $(\mathcal{W}, \chi, \mu)$[5], an inner measure on a propositional language set $\mathcal{L}(P)$ can be defined through a truth assignment $\pi(w) : \mathcal{L}(P) \rightarrow \{\mathbf{true}, \mathbf{false}\}$ as follows. If $\pi(w)(\phi) = \mathbf{true}$, ϕ is said to be true at w; otherwise we say that ϕ is false at w. ϕ^π is defined to contain all those elements in $\mathcal{W}$ in which ϕ is true. If we define $Prob_*(\phi) = \mu_*(\phi^\pi)$ where μ_* is the inner measure of μ, then $Prob_*$ is called an inner measure of a probability distribution on $\mathcal{L}(P)$. It is proved in [10] that a belief function on such a language set is also an inner measure which is generated from a probability space. Therefore it is also interesting to apply the above technique to recover a probability space when we know an inner measure $Prob_*$ of probabilities on $\mathcal{L}(P)$.

Algorithm D: Recovering Probability Spaces:

Given an inner measure $Prob_*$ on the set $\mathcal{A} = \mathcal{L}(P)$, recover the initial probability space from which $Prob_*$ is derived.

Step 1: *Delete all those elements in $\mathcal{A}$ in which $Prob_*(*) = 0$. Similar to the initial step in Algorithm B, define a subset $\mathcal{A}_0$ out of $\mathcal{A}$ using formula (5).*

For every $\phi_j \in \mathcal{A}_0$, define $\pi(w_j)(\phi_j) = \mathbf{true}$ and $\mu(w_j) = Prob'_*(\phi_j) = Prob_*(\phi_j)$.

Define $\mathcal{A}' = \mathcal{A} \setminus \mathcal{A}_0$. Assume that there are l elements in $\mathcal{A}_0$. So there are l possible worlds created.

Step 2: *Choose a formula ψ from $\mathcal{A}'$ which satisfies the condition that $\forall \psi' \in \mathcal{A}'$, $\psi' \not\models \psi$ if $\psi' \neq \psi$.*

Define $Prob'_*(\psi) = Prob_*(\psi) - \Sigma_{\phi_j \in \mathcal{A}_0, \phi_j \models \psi} Prob'_*(\phi_j)$.

If $Prob'_*(\psi) > 0$, define

[5] In order to be consistent with incidence calculus, we take a set of possible worlds as a set of sample space.

$$\mathcal{A}_0 = \mathcal{A}_0 \cup \{\psi\},$$
$$\mathcal{A}' = \mathcal{A}' \setminus \{\psi\},$$
$$\mu(w_{l+1}) = Prob'_*(\psi),$$
$$\pi(w_{l+1})(\psi) = \mathbf{true},$$
$$l = l + 1.$$

If $Prob'_*(\psi) = 0$ then there is no need to create an extra possible world to match ψ alone.

If $Prob'_*(\phi) < 0$ then this assignment is not a correct inner measure, stop the procedure.

Repeat this step until $\mathcal{A}'$ is empty.

Step 3: *Set $\chi' = \{\{w_1\}, ..., \{w_{l+1}\}\}$ is the basis of an unspecified probability space. It is easy to prove that $\Sigma_{w_j}\mu(w_j) = 1$.*

The corresponding probability space will be $(\mathcal{W}, \chi, \mu)$ where χ is the $\sigma-$algebra generated by the basis χ'. In the simplest case, the probability space can just be (χ', χ', μ).

Through Theorem 3, we can prove that from probability space (χ', χ, μ) and mapping π, the lower bound $Porb_*$ on $\mathcal{L}(P)$ can be re-calculated.

More details about probability space, probability structure and its relation with DS theory can be found in [9], [10].

Example 7.2

Continuing Example 7.1, if we take $Prob_*$ as an inner measure of a probability measure on $\mathcal{A}$ from an unknown probability space, this space can be recovered as $(\mathcal{W}, \chi', \mu)$ where the basis for χ is $\chi' = \{\{w_1\}, \{w_2\}, \{w_3\}\}$, and $\mu(w_1) = 0.5, \mu(w_2) = 0.3, \mu(w_3) = 0.2$.

□

8 Summary

Dealing with uncertainty is an important task in many automated reasoning systems. Quite a few numerical and symbolic approaches have been proposed and discussed ([1], [2], [6], [18], [19], [20], [22] etc). Incidence calculus is one among these. The main difference between incidence calculus and pure numerical approaches is the indirect assignment of numerical values, that is, the assignment of numerical values on statements through possible worlds. This enables incidence calculus to deal with dependencies among evidence.

However, when numerical values are assigned on statements directly rather than on possible worlds, incidence calculus cannot be applied directly. A set of possible worlds needs to be constructed, so do the assignments between possible worlds and statements, and between numerical values and possible worlds, before applying incidence calculus.

In this paper, we discussed how to construct a set of possible world from an assignment of probabilities in generalized incidence calculus. An important concept, basic incidence assignment, is proposed which possesses some significant features that incidence functions do not. Each incidence function has a unique basic incidence assignment and many different incidence functions may

have the same basic incidence assignment. So it is more meaningful to recover a basic incidence assignment than an incidence function from a numerical assignment. This is the main achievement of our algorithm.

Comparing to the methods discussed in [17], our algorithm is superior to them in terms of lower computational complexity. Only one output is essential and this output is easy to define, while the other methods try to find all consistent assignments. In our algorithm, the size of the set of possible worlds entirely depends on the size of $\mathcal{A}$. For example, if there are only n elements in $\mathcal{A}$, then we can define a set of possible worlds containing at most $n-1$ elements. We also proved that any incidence assignment generated from a consistent numerical assignment is subsumed by a fundamental incidence assignment derived from the same numerical assignment using our approach.

When we extend the result to DS theory and the probability space, we follow the known result that a lower bound in incidence calculus is equivalent to a belief function ([13]) and a belief function is, in turn, equivalent to an inner measure in probability structures ([10]) when these three theories concern the same problem space. Therefore the incidence assignment method can not only be used to define an incidence assignment but also be used to construct an undefined probability space. In the latter case, a basis for an $\sigma-$algebra of a probability space is constructed.

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