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Abstract We define cut-free display calculi for knowledge logics where an indiscernibility relation is associated to each set of agents, and where agents decide the membership of objects using this indiscernibility relation. To do so, we first translate the knowledge logics into polymodal logics axiomatised by primitive axioms and then use Kracht's results on properly displayable logics to define the display calculi. Apart from these technical results, we argue that Display Logic is a natural framework to define cut-free calculi for many other logics with relative accessibility relations.

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1 Introduction

Background. Formal logic has been used by various authors to analyse and reason about knowledge. The possible-worlds semantics for knowledge logics initiated by Hintikka [Hin62] has been very fruitful for modelling reasoning about knowledge (see e.g. the reference book [FHMV95]). This approach has also been successfully used for reasoning about protocols in distributed systems (see e.g. [Hal87]). Starting from different postulates about knowledge, Orłowska [Orło89] has introduced knowledge logics based on the assumption that the knowledge of agents depends on the degree of certainty with which they perceive objects from a given domain. The semantical structures are of the form $(OB, AGT, (R_Q)_{Q \subseteq AGT})$ where the elements of OB are interpreted as objects for which data is stored in some information system in the sense of [Paw81], AGT is a set of agents and $(R_Q)_{Q \subseteq AGT}$ is a family of binary relations over OB such that $(o, o') \in R_Q$ iff o and o' cannot be "distinguished" by the set Q of agents. Thus $R_Q(o) \stackrel{\text{def}}{=} \{o' \in OB : (o, o') \in R_Q\}$ is the set of objects which members of Q cannot "distinguish" from o.

As is usual for modal logics, different notions of "distinguishability" can be captured by imposing different conditions on each R_Q. Moreover, the relations in $(R_Q)_{Q \subseteq AGT}$ are not independent since it is required that (1) $R_{Q \cup Q'} = R_Q \cap R_{Q'}$ for any $Q, Q' \subseteq AGT$ and (2) $R_\emptyset = OB \times OB$. Condition (1) means that two sets of agents can distinguish more objects than each one can individually, and (2) means that the empty set of agents cannot distinguish any two objects.

In that setting, based on an alternative ontology, a knowledge operator $\triangle(Q) : \mathcal{P}(OB) \rightarrow \mathcal{P}(OB)$ is definable where $\triangle(Q)(X) \subseteq OB$ consists of those objects which the set Q of agents can distinguish as belonging, or not belonging, to some given $X \subseteq OB$:

$$\triangle(Q)(X) \stackrel{\text{def}}{=} \{o \in OB : R_Q(o) \subseteq X\} \cup \{o \in OB : R_Q(o) \subseteq (OB \setminus X)\}$$

Intuitively, an object $o \in OB$ is in $\triangle(Q)(X)$ if the objects that are Q-indistinguishable from o are either all inside X, or all outside X. Actually, $\triangle(Q)$ is similar to Aumann's knowledge operator in the event-based approach of knowledge [Aum76], and also corresponds to the modal operator $\triangle$ in logics of non-contingency (see e.g. [Hin62, MR66, Hum95, Kuh95]).

There are many existing logics based on Orlowska's approach (see e.g. [FdCO85, Orło89, Vak91, Bal97, Kon97a, Dem98a]). For instance, some of these logics, based on rough set theory [Paw82], provide a fruitful framework to model reasoning in the presence of incomplete information (see e.g. [Vak91, Kon97a]).

Our objectives. Formalising proof systems for such knowledge logics within the display logic framework (see e.g. [Bel82]) is the main objective of the paper. Existing proof systems for these logics are either Hilbert-style systems [Bal97, BO98, Dem98b] which are not amenable to mechanisation, or are Rasiowa-Sikorski-style and Gentzen-style proof systems [Kon97a] which require the use of nominals. Since adding nominals greatly increases the expressive power of the logics[3], we wish to define calculi for the logics without nominals. Such a motivation can be put in parallel with the definition of a cut-free calculus for relation algebras that does not involve point variables [Gor97].

To understand why we highly value the calculi in the display logic framework (abbreviated by **DL**), it might be worth recalling that **DL** is a proof-theoretical framework introduced in [Bel82] that admits a very general cut-elimination theorem. Moreover, **DL** generalises the structural language of Gentzen's sequents in a rather abstract way by using multiple complex structural connectives instead of Gentzen's comma. The term "display" comes from the nice property that any occurrence of a structure in a sequent can be displayed either as the entire antecedent or as the entire succedent of some sequent which is *structurally equivalent* to the initial sequent.

Our contribution. The main contribution of the paper is the definition of cut-free display calculi $\delta\mathcal{L}$ for the knowledge logics described above. Since these logics do not easily fit into the classical polymodal display framework defined in [Kra96, Wan97], we first define faithful translations of these knowledge logics into polymodal logics $\mathcal{L}_\cap$ in which the modal operators are the universal modality $[\mathtt{U}]$ [GP92] or are of the form $[\mathtt{c}_1 \cap \ldots \cap \mathtt{c}_n]$ where each $\mathtt{c}_i$ is a constant interpreted as a binary relation. The operator $\cap$ is interpreted as the intersection operation.

The crucial observation is that the polymodal logics $\mathcal{L}_\cap$ are *properly displayable* in the sense of Kracht [Kra96]. That is, by applying the general

[3] See e.g. [PT91] where the intersection modal operator is modally defined *only* when nominals are allowed.

results from [Kra96], the polymodal logics $\mathcal{L}_{\cap}$ admit cut-free display calculi $\delta\mathcal{L}_{\cap}$. The rules of the display calculi $\delta\mathcal{L}$ for the knowledge logics mimic those of the display calculi $\delta\mathcal{L}_{\cap}$ for the polymodal logics. Completeness is then proved syntactically as is standard with **DL** (see e.g. [Gor96, Kra96]). Alternatively, completeness can be shown starting from the Hilbert-style system for the knowledge logics, but this way does not provide any insight into the syntactic relationships between $\delta\mathcal{L}$ and $\delta\mathcal{L}_{\cap}$. More importantly, the techniques used here are applicable to other logics with relative accessibility relations, which is why we emphasise the translations between the logics and between the calculi.

Along the way, we also characterise the computational complexity of various basic manipulations found in display calculi based on Boolean negation. For instance, checking that two sequents are structurally equivalent can be done in *quadratic-time* in the size of the input sequents. Additionally, several quantitative analyses about the size of the proofs obtained in the calculi for the knowledge logics are given in the paper.

A very important feature of **DL** is the ability to combine several families of structural operations. In the paper, we also show how to separate the display logic inferences which reason about the algebra of modal terms (typically Boolean terms) from those that reason about the deducibility relation of the logics. Hence, our contribution is not merely technical: we believe that **DL** is a fairly natural framework for defining many other calculi for logics with relative accessibility relations (see e.g. [BO98]). In that sense, the present paper can be viewed as a case study that opens the door for further investigation.

Related work. On one hand, Hilbert-style calculi for several logics with relative accessibility relations, including some of the logics treated here, have been defined in [Bal97, Dem98b]. But such systems are known to be particularly inefficient for mechanisation. On the other hand, Rasiowa-Sikorski proof systems and sequent proof systems have been defined in [Kon97b] for logics with relative accessibility relations, but these proof systems require the use of nominals. Although some of the logics in [Kon97b] contain some of the logics of the present paper, the extension with names is powerful since it increases the expressive power of the logics. Indeed, even decidability has not been shown for some of these logics (see partial positive decidability results in [DK98]). Consequently, there is a need for calculi exclusively dedicated to the language without names.[4]

Here we define such calculi in the general **DL** framework, thereby showing that **DL** is powerful enough to deal with logics with relative accessibility relations. Unlike in [Kon97a], where the use of nominals is essential, neither nominals nor prefixed formulae are needed in the **DL** framework.

As is well-known, other general proof-theoretical frameworks exist for non-classical logics: Labelled Deductive Systems [Gab96] and Relational Proof Systems [Orło88, Orło91, Orło92] to name two. But the use of labels in the former, and the use of explicit point variables in the latter is somewhat akin to the use of names.

In any case, **DL** has already shown its generality since cut-free display calculi have been defined for substructural logics [Res98, Gor98], for modal and polymodal logics [Wan94, Kra96, Wan97, DG98b], for intuitionistic logics [Gor95] and for relation algebras [Gor97]. Numerous enriched sequent-style calculi such as prefixed sequent calculi and hypersequent calculi can be naturally encoded in **DL** (see e.g. [Min97, Wan98]). Finally, two particular display calculi have been mechanised using the proof assistant `Isabelle` by Dawson [DG98a].

Plan of the paper. The rest of the paper is structured as follows. In Section 2, we recall the definition of the knowledge logics under study [Orło89]. In Section 3, we introduce the required polymodal logics axiomatised by primitive axioms and we recall their cut-free display calculi following [Kra96]. In Section 4.1, we show how to translate knowledge logics into these polymodal logics. Finally, Section 4.2 mainly contains the definition of the cut-free display calculi whereas Section 4.3 presents the cut-elimination theorem and the completeness proof.

2 Logics with relative accessibility relations

Given a set $\mathtt{For}_0 = \{\mathtt{p}_1, \mathtt{p}_2, \ldots\}$ of *propositional variables* and a set $\mathtt{A}_0 = \{\delta_1, \delta_2, \ldots\}$ of *agent constants*, the agent expressions $\alpha \in \mathtt{A}$ and the formulae

[4] A similar need exists for the Data Analysis Logic DAL, for which no Hilbert-style axiomatisation without nominals is known, although its extension with nominals is axiomatised in [Gar86].

$\phi \in \mathtt{For}$ are inductively defined as follows for $\delta_i \in \mathtt{A}_0$ and $\mathtt{p}_i \in \mathtt{For}_0$:

$$\alpha \ ::= \ \delta_i \ | \ -\alpha \ | \ \alpha_1 \cup \alpha_2 \ | \ \alpha_1 \cap \alpha_2$$

$$\phi \ ::= \bot \ | \ \top \ | \ \mathtt{p}_i \ | \ \phi_1 \wedge \phi_2 \ | \ \phi_1 \vee \phi_2 \ | \ \neg\phi \ | \ \triangle(\alpha)\phi$$

Standard abbreviations include $\Rightarrow$ and $\Leftrightarrow$. As usual in complexity theory, for any syntactic object $\mathtt{O}$, we write $|\mathtt{O}|$ to denote its *length* (or *size*) for some reasonably (unspecified) succinct encoding.

An $\mathtt{A}$-*interpretation* m is a map $m : \mathtt{A} \rightarrow \mathcal{P}(Ag)$ such that Ag is a non-empty set, $\mathcal{P}(Ag)$ is the set of all subsets of Ag, and for any $\alpha_1, \alpha_2 \in \mathtt{A}$:

1. $m(\alpha_1 \cap \alpha_2) = m(\alpha_1) \cap m(\alpha_2)$;
2. $m(\alpha_1 \cup \alpha_2) = m(\alpha_1) \cup m(\alpha_2)$;
3. $m(-\alpha_1) = Ag \setminus m(\alpha_1)$.

For any $\alpha, \beta \in \mathtt{A}$ we write $\alpha \equiv \bot$ [resp. $\alpha \equiv \beta$, $\alpha \sqsubseteq \beta$] when for any $\mathtt{A}$-interpretation m, $m(\alpha) = \emptyset$ [resp. $m(\alpha) = m(\beta)$, $m(\alpha) \subseteq m(\beta)$]. The relations $\equiv$ and $\sqsubseteq$ are known to be decidable (by decidability of classical propositional logic).

Definition 2.1. A *frame* $\mathcal{F}$ is a structure $\mathcal{F} = (OB, Ag, (R_P)_{P \subseteq Ag})$ such that OB is a non-empty set of *objects*, Ag is a non-empty set of *agents* and $(R_P)_{P \subseteq Ag}$ is a family of binary relations over OB such that
(a) for all $Q, Q' \subseteq Ag$, $R_{Q \cup Q'} = R_Q \cap R_{Q'}$ (b) $R_\emptyset = OB \times OB$.

A *model* $\mathcal{M}$ is a structure $\mathcal{M} = (OB, Ag, (R_P)_{P \subseteq Ag}, m)$ such that $\mathcal{F} = (OB, Ag, (R_P)_{P \subseteq Ag})$ is a frame and $m : \mathtt{For}_0 \cup \mathtt{A} \rightarrow \mathcal{P}(OB) \cup \mathcal{P}(Ag)$ is a mapping such that $m(\mathtt{p}) \subseteq OB$ for all $\mathtt{p} \in \mathtt{For}_0$, and the restriction of m to $\mathtt{A}$ is an $\mathtt{A}$-interpretation. As usual, we say that the model $\mathcal{M}$ is *based on* $\mathcal{F}$. ∇

Let $\mathcal{M} = (OB, Ag, (R_P)_{P \subseteq Ag}, m)$ be a model and $o \in OB$. The formula ϕ is *satisfied by the object* $o \in OB$ *in* $\mathcal{M}$ $\stackrel{\text{def}}{\Leftrightarrow}$ $\mathcal{M}, o \models \phi$ where the satisfaction relation $\models$ is inductively defined as follows:

$$\mathcal{M}, o \models \mathtt{p} \ \stackrel{\text{def}}{\Leftrightarrow} \ o \in m(\mathtt{p}), \text{ for any } \mathtt{p} \in \mathtt{For}_0$$

$$\mathcal{M}, o \models \triangle(\alpha)\phi \ \stackrel{\text{def}}{\Leftrightarrow} \ \begin{cases} \text{either } \mathcal{M}, o' \models \phi \text{ for all } o' \in R_{m(\alpha)}(o) \\ \text{or } \mathcal{M}, o' \models \neg\phi \text{ for all } o' \in R_{m(\alpha)}(o). \end{cases}$$

We omit the standard conditions for the propositional connectives and the logical constants. A formula ϕ is *true* in a model $\mathcal{M}$ (written $\mathcal{M} \models \phi$) $\stackrel{\text{def}}{\Leftrightarrow}$ for all $o \in OB$, $\mathcal{M}, o \models \phi$. A formula ϕ is *true* in a frame $\mathcal{F}$ (written $\mathcal{F} \models \phi$) $\stackrel{\text{def}}{\Leftrightarrow}$ ϕ is true in every model based on $\mathcal{F}$. In the sequel, by a *B-logic*[5] $\mathcal{L}$ we understand a pair $\langle \mathtt{L}, X^{\mathcal{M}} \rangle$ such that $\mathtt{L} \subseteq \mathtt{For}$ and $X^{\mathcal{M}}$ is a class of models based on a given class of frames. A formula $\phi \in \mathtt{L}$ is said to be *$\mathcal{L}$-valid* $\stackrel{\text{def}}{\Leftrightarrow}$ ϕ is true in all the models in $X^{\mathcal{M}}$. A formula $\phi \in \mathtt{L}$ is said to be *$\mathcal{L}$-satisfiable* $\stackrel{\text{def}}{\Leftrightarrow}$ $\neg\phi$ is not $\mathcal{L}$-valid.

We write $T_{\triangle} \stackrel{\text{def}}{=} \langle \mathtt{For}, X_T^{\mathcal{M}} \rangle$ [resp. $B_{\triangle} \stackrel{\text{def}}{=} \langle \mathtt{For}, X_B^{\mathcal{M}} \rangle$, $S5_{\triangle} \stackrel{\text{def}}{=} \langle \mathtt{For}, X_{S5}^{\mathcal{M}} \rangle$] to denote the B-logic such that a model $(OB, Ag, (R_P)_{P \subseteq Ag}, m) \in X_T^{\mathcal{M}}$ [resp. $\in X_B^{\mathcal{M}}$, $\in X_{S5}^{\mathcal{M}}$] $\stackrel{\text{def}}{\Leftrightarrow}$ for all $\emptyset \neq Q \subseteq Ag$, R_Q is reflexive [resp. R_Q is reflexive and symmetric, R_Q is an equivalence relation].

Both $B_{\triangle}$ and $S5_{\triangle}$ were introduced in [Orło89] and they are the very object of study in the present paper. By taking one's favourite (mono)modal logic $\mathbf{L}$, one can easily define the corresponding B-logic $\mathcal{L}$ when $\mathbf{L}$ is characterised by a class of standard Kripke frames closed under intersection. However, in $B_{\triangle}$ and $S5_{\triangle}$ the standard necessity operator $[\alpha]$ can be defined by $[\alpha]\phi \stackrel{\text{def}}{=} \phi \wedge \triangle(\alpha)\phi$, and this might not always be the case for a B-logic derived from an arbitrary (mono)modal logic $\mathbf{L}$. Furthermore, Vakarelov's copying construction (see e.g. [Vak91]), used in the proof of the forthcoming Theorem 3.1 is known to behave badly with transitive relations. Hence, displaying $\mathrm{S4}_{\triangle}$ will still remain an open question at the end of the present paper. Decidability of $\mathrm{S4}_{\triangle}$-satisfiability is also open.

In what follows, a B-logic $\mathcal{L} = \langle \mathtt{For}, \mathcal{X} \rangle$ is said to be *closed under intersection* $\stackrel{\text{def}}{\Leftrightarrow}$ there exists a class $\mathcal{C}$ of standard Kripke frames of the form (W, R) such that

1. $\mathcal{C}$ is closed under intersection: that is, for all $(W_1, R_1), (W_2, R_2) \in \mathcal{C}$ if $W_1 \cap W_2 \neq \emptyset$ then $(W_1 \cap W_2, R_1 \cap R_2) \in \mathcal{C}$; and
2. $\mathcal{X}$ contains exactly the class of models $(OB, Ag, (R_P)_{P \subseteq Ag}, m)$ such that for all $\emptyset \neq P \subseteq Ag$, $(OB, R_P) \in \mathcal{C}$.

[5] "logic with Boolean terms"

3 Polymodal logics axiomatised by primitive axioms

In this section, we investigate polymodal logics $\mathcal{L}_\cap$ that are not our main object of study (after all we want to display the B-logics from [Orło89]). However, as stated in Section 1, these polymodal logics share with the B-logics various properties that help to understand the B-logics themselves.

3.1 Hilbert-style axiomatisation

Given a set $\mathtt{For}_0 = \{\mathtt{p}_1, \mathtt{p}_2, \ldots\}$ of atomic propositions and a set $\mathtt{M}_0 = \{\mathtt{c}_0, \mathtt{c}_1, \ldots\}$ of modal constants, the modal terms $\mathtt{a} \in \mathtt{M}$ (which are different from $\mathtt{U}$) and the formulas $\phi \in \mathtt{For}_\cap$ for the $\cap$-logics are inductively defined as follows for $\mathtt{c}_i \in \mathtt{M}_0$ and $\mathtt{p}_i \in \mathtt{For}_0$:

$$\mathtt{a} ::= \mathtt{c}_i \mid \mathtt{a}_1 \cap \mathtt{a}_2$$

$$\phi ::= \bot \mid \top \mid \mathtt{p}_i \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \neg\phi \mid [\mathtt{a}]\phi \mid [\mathtt{U}]\phi$$

Standard abbreviations include $\langle\mathtt{a}\rangle$, $\langle\mathtt{U}\rangle$, $\Rightarrow$ and $\Leftrightarrow$.

Definition 3.1. An $\cap$-*frame* is a structure $(W, (R_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}\cup\{\mathtt{U}\}})$ where W is a non-empty set of *states* and $(R_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}\cup\{\mathtt{U}\}}$ is a family of binary relations over W such that $R_{\mathtt{U}} = W \times W$ and $R_{\mathtt{a}\cap\mathtt{b}} = R_{\mathtt{a}} \cap R_{\mathtt{b}}$ for all $\mathtt{a}, \mathtt{b} \in \mathtt{M}$.

An $\cap$-*model* $\mathcal{M} = (W, (R_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}\cup\{\mathtt{U}\}}, m)$ is a structure such that $\mathcal{F} = (W, (R_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}})$ is an $\cap$-frame and m is a mapping $m : \mathtt{For}_0 \to \mathcal{P}(W)$. $\mathcal{M}$ is said to be *based on* $\mathcal{F}$. ∇

Observe that the modal operator $[\mathtt{U}]$ is the standard universal modal operator [GP92, Hem96]. The satisfiability relation $\models$, and the associated notions of $\mathcal{L}_\cap$-satisfiability and $\mathcal{L}_\cap$-validity are defined as usual for polymodal logics. In what follows, by an $\cap$-*logic* $\mathcal{L}_\cap$ we understand a pair $\langle \mathtt{For}_\cap, X^{\mathcal{M}} \rangle$ such that $X^{\mathcal{M}}$ is the class of all $\cap$-models based on a given class of $\cap$-frames.

For any B-logic $\mathcal{L}$ closed under intersection, we write $\mathcal{L}_\cap$ to denote the corresponding $\cap$-logic, i.e. with the same conditions on the binary relations of the $\cap$-models. Unless otherwise stated, in the rest of the paper, $\mathcal{L}$ denotes a B-logic in $\{T_\triangle, B_\triangle, S5_\triangle\}$.

Let $\mathtt{M}_0(\mathtt{a})$ denote the set of modal constants (from $\mathtt{M}_0$) that occur in $\mathtt{a}$, let $\mathtt{Th}(\mathcal{L}_\cap)$ be the smallest set of $\cap$-formulae such that $\mathtt{Th}(\mathcal{L}_\cap)$ is closed under *modus ponens*, *uniform substitution* and *necessitation* for $[\mathtt{U}]$ and each $[\mathtt{a}]$, and let $\mathtt{Th}(\mathcal{L}_\cap)$ also contain every tautology of classical propositional logic together with the formulae

(K) $[\mathtt{a}](\mathtt{p} \Rightarrow \mathtt{q}) \wedge [\mathtt{a}]\mathtt{p} \Rightarrow [\mathtt{a}]\mathtt{q}$ for any $\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}$

(T) $[\mathtt{a}]\mathtt{p} \Rightarrow \mathtt{p}$ for any $\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}$

(B) $\mathtt{p} \Rightarrow [\mathtt{a}]\langle\mathtt{a}\rangle\mathtt{p}$ when $\mathcal{L} = B_\triangle$

(5) $\langle\mathtt{a}\rangle\mathtt{p} \Rightarrow [\mathtt{a}]\langle\mathtt{a}\rangle\mathtt{p}$ for $\mathtt{a} \in \mathtt{M}$ when $\mathcal{L} = S5_\triangle$

(5U) $\langle\mathtt{U}\rangle\mathtt{p} \Rightarrow [\mathtt{U}]\langle\mathtt{U}\rangle\mathtt{p}$

(U) $[\mathtt{U}]\mathtt{p} \Rightarrow [\mathtt{a}]\mathtt{p}$ for $\mathtt{a} \in \mathtt{M}$

($\cap$) $[\mathtt{a}]\mathtt{p} \vee [\mathtt{b}]\mathtt{p} \Rightarrow [\mathtt{a} \cap \mathtt{b}]\mathtt{p}$ for $\mathtt{a}, \mathtt{b} \in \mathtt{M}$

(AC) $[\mathtt{a}]\mathtt{p} \Leftrightarrow [\mathtt{b}]\mathtt{p}$ for $\mathtt{a}, \mathtt{b} \in \mathtt{M}$ such that $\mathtt{M}_0(\mathtt{a}) = \mathtt{M}_0(\mathtt{b})$.

Alternatively, the schemata ($\cap$) and (AC) can be replaced by:

$$[\mathtt{a}]\mathtt{p} \Rightarrow [\mathtt{b}]\mathtt{p} \text{ when } \mathtt{M}_0(\mathtt{a}) \subseteq \mathtt{M}_0(\mathtt{b}).$$

Thus, each $\mathtt{Th}(\mathcal{L}_\cap)$ can be seen as a traditional axiomatic Hilbert system allowing us to write $\phi \in \mathtt{Th}(\mathcal{L}_\cap)$ to mean that ϕ is derivable in $\mathtt{Th}(\mathcal{L}_\cap)$. By adapting constructions in [Vak91, Bal97] we can show that each such Hilbert system is sound and complete with respect to the intended semantics:

Theorem 3.1. An $\cap$-formula ϕ is $\mathcal{L}_\cap$-valid iff $\phi \in \mathtt{Th}(\mathcal{L}_\cap)$.

Since the standard Kripke frame of the canonical model for $\mathtt{Th}(\mathcal{L}_\cap)$ is not an $\cap$-frame, we make a substantial use of the copying technique from [Bal97]. The full proof is rather long and tedious and it is omitted here.

To express the next lemma, let $\mathtt{M}_0(\phi)$ denote the set of modal constants occurring in ϕ, as before, but let $\mathtt{M}(\phi)$ denote the set of modal expressions $\mathtt{a}$ such that ϕ has a subformula of the form $[\mathtt{a}]\psi$.

Lemma 3.2. If $\phi \in \mathtt{Th}(\mathcal{L}_\cap)$, then there is a derivation $\langle\phi_1, \ldots, \phi_n\rangle$ of ϕ such that for any $i \in \{1, \ldots, n\}$, $\mathtt{M}_0(\phi_i) \subseteq \mathtt{M}_0(\phi)$ and $max\{|\mathtt{a}| : \mathtt{a} \in \mathtt{M}(\phi_i)\} \leq max\{|\mathtt{a}| : \mathtt{a} \in \mathtt{M}(\phi)\}$.

At first sight, this lemma may appear analogous to cut-elimination (elimination of the introduction of irrelevant terms in a proof). However, the idea of the proof of Lemma 3.2 consists in considering a fragment $\mathtt{Th}_{\phi}(\mathcal{L}_{\cap})$ of $\mathtt{Th}(\mathcal{L}_{\cap})$ restricted to $\mathtt{M}' \subset \mathtt{M}$ where $\mathtt{M}'$ contains all possible modal expressions $\mathtt{a}$ built using the constants from $\mathtt{M}_0(\phi)$ such that $|\mathtt{a}| \leq max\{|\mathtt{b}| : \mathtt{b} \in \mathtt{M}(\phi)\}$. Completeness of $\mathtt{Th}_{\phi}(\mathcal{L}_{\cap})$ with respect to $\mathcal{L}_{\cap}$ for this fragment of the language including ϕ can then be obtained easily. The lemma follows since $\mathtt{Th}_{\phi}(\mathcal{L}_{\cap}) \subseteq \mathtt{Th}(\mathcal{L}_{\cap})$.

Lemma 3.2 is used to prove the forthcoming Theorem 3.8 that states that the application of some rules in the forthcoming display calculus $\delta\mathcal{L}$ can be constrained.

3.2 Display calculi

As stated previously, there are numerous existing display calculi. We extend Wansing's [Wan94] formulation since it is tailored for classical polymodal logics. Following Kracht's terminology, each $\mathcal{L}_{\cap}$ can be *properly displayed* ([Kra96, Theorem 21]). That is, each $\mathcal{L}_{\cap}$ admits a display calculus $\delta\mathcal{L}_{\cap}$ for which cut-elimination holds since each $\delta\mathcal{L}_{\cap}$ satisfies the conditions (C1)-(C8) from [Bel82]. Moreover, axioms from the Hilbert-style system $\mathtt{Th}(\mathcal{L}_{\cap})$ for $\mathcal{L}_{\cap}$ are encoded in $\delta\mathcal{L}_{\cap}$ by purely *structural rules*, i.e. rules that involve only *structure variables*. In this section, we shall explicitly recall the display calculus $\delta\mathcal{L}_{\cap}$ for $\mathcal{L}_{\cap}$ obtained by application of Kracht's results. This allows us to smoothly introduce various notions and to state properties that are used to define the display calculi for the knowledge logics (see Section 4.2 and Section 4.3). Hence, this section is mainly included to make the paper self-contained.

On the structural side, we have Boolean structural connectives $*$ (unary), $\circ$ (binary), I (nullary) and the family $(\bullet_{\mathtt{a}})_{\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}}$ of modal structural connectives. A *structure* $\mathtt{X} \in \mathtt{struc}(\delta\mathcal{L}_{\cap})$ is inductively defined as follows:

$$\mathtt{X} ::= \phi \mid *\mathtt{X} \mid \mathtt{X}_1 \circ \mathtt{X}_2 \mid I \mid \bullet_{\mathtt{a}}\mathtt{X}$$

for $\phi \in \mathtt{For}_{\cap}$, $\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}$. A *sequent* is defined as a pair of structures of the form $\mathtt{X} \vdash \mathtt{Y}$ with $\mathtt{X}$ the *antecedent* and $\mathtt{Y}$ the *succedent*. The rules of $\delta\mathcal{L}_{\cap}$ are presented in Figures 1-5.

$$(\mathrm{Id})\quad \mathtt{p} \vdash \mathtt{p} \qquad\qquad (\mathrm{cut})\quad \frac{\mathtt{X} \vdash \phi \quad \phi \vdash \mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}}$$

Figure 1: Fundamental logical axioms and cut rule

$$\frac{\mathtt{X} \circ \mathtt{Y} \vdash \mathtt{Z}}{\mathtt{X} \vdash \mathtt{Z} \circ *\mathtt{Y}} \qquad \frac{\mathtt{X} \circ \mathtt{Y} \vdash \mathtt{Z}}{\mathtt{Y} \vdash *\mathtt{X} \circ \mathtt{Z}} \qquad \frac{\mathtt{X} \vdash \mathtt{Y} \circ \mathtt{Z}}{\mathtt{X} \circ *\mathtt{Z} \vdash \mathtt{Y}} \qquad \frac{\mathtt{X} \vdash \mathtt{Y} \circ \mathtt{Z}}{*\mathtt{Y} \circ \mathtt{X} \vdash \mathtt{Z}}$$

$$\frac{*\mathtt{X} \vdash \mathtt{Y}}{*\mathtt{Y} \vdash \mathtt{X}} \qquad \frac{\mathtt{X} \vdash *\mathtt{Y}}{\mathtt{Y} \vdash *\mathtt{X}} \qquad \frac{**\mathtt{X} \vdash \mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}} \qquad \frac{\mathtt{X} \vdash **\mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}} \qquad \frac{\mathtt{X} \vdash \bullet_{\mathtt{a}}\mathtt{Y}}{\bullet_{\mathtt{a}}\mathtt{X} \vdash \mathtt{Y}}$$

Figure 2: Display postulates

The *display postulates* (reversible rules) in Figure 2 deal with the manipulation of structural connectives. The expression

$$\frac{\mathtt{X} \vdash \bullet_{\mathtt{a}}\mathtt{Y}}{\bullet_{\mathtt{a}}\mathtt{X} \vdash \mathtt{Y}}$$

should be understood as a rule schema. So $\mathtt{a}$ is just a *metavariable*, and Figure 2 contains a countably infinite set of rules. Such a distinction is necessary in order to be able to apply scrupulously the cut-elimination theorem from [Bel82]. All forthcoming rules obey the same reading.

In any structure $\mathtt{Z}$, the structure $\mathtt{X}$ occurs *negatively* [resp. *positively*] $\overset{\text{def}}{\Leftrightarrow}$ $\mathtt{X}$ occurs in the scope of an odd number [resp. an even number] of occurrences of $*$ [Bel82]. In a sequent $\mathtt{Z} \vdash \mathtt{Z}'$, an occurrence of $\mathtt{X}$ is an *antecedent part* [resp. *succedent part*] $\overset{\text{def}}{\Leftrightarrow}$ it occurs positively in $\mathtt{Z}$ [resp. negatively in $\mathtt{Z}'$] or it occurs negatively in $\mathtt{Z}'$ [resp. positively in $\mathtt{Z}$] [Bel82]. Two sequents $\mathtt{X} \vdash \mathtt{Y}$ and $\mathtt{X}' \vdash \mathtt{Y}'$ are said to be *structurally equivalent* $\overset{\text{def}}{\Leftrightarrow}$ there is derivation of the first sequent from the second using only the display postulates defined in Figure 2.

Theorem 3.3. [Bel82] For every sequent $\mathtt{Z} \vdash \mathtt{Z}'$ and every antecedent [resp. succedent] part $\mathtt{X}$ of $\mathtt{Z} \vdash \mathtt{Z}'$, there is a structurally equivalent sequent $\mathtt{X} \vdash \mathtt{Y}$

[resp. $\mathtt{Y} \vdash \mathtt{X}$] that has $\mathtt{X}$ (alone) as its antecedent [resp. succedent]. $\mathtt{X}$ is said to be *displayed* in $\mathtt{X} \vdash \mathtt{Y}$ [resp. $\mathtt{Y} \vdash \mathtt{X}$].

The proof of Theorem 3.3 immediately gives the following corollary.

Corollary 3.4. Displaying requires linear-time w.r.t input sequent size.

Here, the good point for mechanisation is that although the display postulates from Figure 2 can be viewed as low level manipulations on structures, displaying the occurrence of a structure is not a time-consuming task. The *problem*[6] PDS *of structural equivalence* is the set of pairs of sequents that are structurally equivalent (for some reasonably succinct encoding of the sequents). At first glance, the rules in Figure 2, which determine the notion of structurally equivalent, do not guarantee that PDS can be solved in a tractable way. Indeed,

1. All the rules in Figure 2 are reversible and therefore there is no measure that allows us to state that applying a rule to a sequent strictly decreases its size with respect to this measure.
2. There is at least one rule that can be applied to any sequent $\mathtt{X} \vdash \mathtt{Y}$.

However, a closer examination of the rules allows us to show that PDS is in the complexity class **P**.

Theorem 3.5. PDS requires quadratic-time in the size of the input sequents.

Proof: A sequent or structure is said to be *reduced* $\stackrel{\text{def}}{\Leftrightarrow}$ it contains no structures of the form $**\mathtt{X}$ and $*(\mathtt{X} \circ \mathtt{Y})$ (see also the notion of *normal form* in [Kra96]). Every sequent is structurally equivalent to a reduced sequent. Such a sequent can be computed in linear-time by:

1. replacing $**\mathtt{X}$ by $\mathtt{X}$;
2. replacing $*(\mathtt{X} \circ \mathtt{Y})$ by $(*\mathtt{Y} \circ *\mathtt{X})$.

[6] A *problem* is understood as a set of strings as is usual in complexity theory.

$$\frac{}{I \vdash \top}\ (\vdash \top) \quad \frac{I \vdash \mathtt{X}}{\top \vdash \mathtt{X}}\ (\top \vdash) \quad \frac{\mathtt{X} \vdash I}{\mathtt{X} \vdash \bot}\ (\vdash \bot) \quad \frac{}{\bot \vdash I}\ (\bot \vdash)$$

$$\frac{\mathtt{X} \vdash *\phi}{\mathtt{X} \vdash \neg\phi}\ (\vdash \neg) \quad \frac{*\phi \vdash \mathtt{X}}{\neg\phi \vdash \mathtt{X}}\ (\neg \vdash) \quad \frac{\mathtt{X} \vdash \phi \quad \mathtt{Y} \vdash \psi}{\mathtt{X} \circ \mathtt{Y} \vdash \phi \wedge \psi}\ (\vdash \wedge) \quad \frac{\phi \circ \psi \vdash \mathtt{X}}{\phi \wedge \psi \vdash \mathtt{X}}\ (\wedge \vdash)$$

$$\frac{\mathtt{X} \vdash \phi \circ \psi}{\mathtt{X} \vdash \phi \vee \psi}\ (\vdash \vee) \quad \frac{\phi \vdash \mathtt{X} \quad \psi \vdash \mathtt{Y}}{\phi \vee \psi \vdash \mathtt{X} \circ \mathtt{Y}}\ (\vee \vdash) \quad \frac{\phi \vdash \mathtt{X}}{[\mathsf{a}]\phi \vdash \bullet_{\mathsf{a}}\mathtt{X}}\ ([\mathsf{a}] \vdash) \quad \frac{\mathtt{X} \vdash \bullet_{\mathsf{a}}\phi}{\mathtt{X} \vdash [\mathsf{a}]\phi}\ (\vdash [\mathsf{a}])$$

Figure 3: Operational rules

Point 2. above may seem incongruous at first sight, but keep in mind that $\circ$ is overloaded à la Gentzen's comma so that $*(\mathtt{X}\circ\mathtt{Y})$ and $(*\mathtt{Y}\circ*\mathtt{X})$ in the same position are dual representations of the same formula. Thus, this procedure is reminiscent of the way to compute a negated normal form for a formula of classical propositional logic. There is however, one notable difference: the rules in Figure 2 do not enforce the commutativity of $\circ$.

Let $\mathtt{X} \vdash \mathtt{Y}$ and $\mathtt{X}' \vdash \mathtt{Y}'$ be sequents. Let ϕ be the leftmost formula that occurs as a structure in $\mathtt{X}$[7]. Assume this occurrence of ϕ is an antecedent part (the case when it is a succedent part is analogous). If $\langle \mathtt{X} \vdash \mathtt{Y}, \mathtt{X}' \vdash \mathtt{Y}' \rangle \in \mathtt{PDS}$, then there is an antecedent part occurrence of ϕ in $\mathtt{X}' \vdash \mathtt{Y}'$ such that $\langle \phi \vdash \mathtt{Z}, \mathtt{X}' \vdash \mathtt{Y}' \rangle \in \mathtt{PDS}$ and $\langle \phi \vdash \mathtt{Z}, \mathtt{X} \vdash \mathtt{Y} \rangle \in \mathtt{PDS}$ for some reduced structure $\mathtt{Z}$. So first display the leftmost structural occurrence of ϕ in $\mathtt{X} \vdash \mathtt{Y}$ such that the succedent $\mathtt{Z}$ is reduced (actually $\mathtt{Z}$ is unique). Then, display any antecedent part structural occurrence of ϕ in $\mathtt{X}' \vdash \mathtt{Y}'$ such that the succedent $\mathtt{Z}'$ is reduced (for each such occurrence, $\mathtt{Z}'$ is unique). If there is some $\mathtt{Z}'$ equal to $\mathtt{Z}$, then $\langle \mathtt{X} \vdash \mathtt{Y}, \mathtt{X}' \vdash \mathtt{Y}' \rangle \in \mathtt{PDS}$ otherwise $\langle \mathtt{X} \vdash \mathtt{Y}, \mathtt{X}' \vdash \mathtt{Y}' \rangle \notin \mathtt{PDS}$. **Q.E.D.**

In Figure 3 and in Figure 4, the constructs $([\mathsf{a}] \vdash)$, $(\vdash [\mathsf{a}])$, (nec^l_{a}) and (nec^r_{a}) should be understood as rule schemata for $\mathsf{a} \in \mathtt{M} \cup \{\mathtt{U}\}$. Observe that the symmetry between the rules (Figure 4) is not total. However, using the basic structural rules (Figure 2), the missing rules are derivable (to weaken a succedent for instance). The rules in Figure 4 induce that $\circ$ is a commutative, associative, idempotent, binary operation with neutral element I. This is

[7] A formula ϕ that is a subformula of another formula ψ is not a structural occurence.

$$\frac{\mathtt{X} \vdash \mathtt{Z}}{I \circ \mathtt{X} \vdash \mathtt{Z}} \ (I_l) \qquad \frac{\mathtt{X} \vdash \mathtt{Z}}{\mathtt{X} \vdash I \circ \mathtt{Z}} \ (I_r) \qquad \frac{I \vdash \mathtt{Y}}{*I \vdash \mathtt{Y}} \ (Q_l) \qquad \frac{\mathtt{X} \vdash I}{\mathtt{X} \vdash *I} \ (Q_r)$$

$$\frac{\mathtt{X} \vdash \mathtt{Z}}{\mathtt{Y} \circ \mathtt{X} \vdash \mathtt{Z}} \ (weak_l) \quad \frac{\mathtt{X} \vdash \mathtt{Z}}{\mathtt{X} \circ \mathtt{Y} \vdash \mathtt{Z}} \ (weak_r)$$

$$\frac{\mathtt{X}_1 \circ (\mathtt{X}_2 \circ \mathtt{X}_3) \vdash \mathtt{Z}}{(\mathtt{X}_1 \circ \mathtt{X}_2) \circ \mathtt{X}_3 \vdash \mathtt{Z}} \ (assoc_l) \qquad \frac{\mathtt{Z} \vdash \mathtt{X}_1 \circ (\mathtt{X}_2 \circ \mathtt{X}_3)}{\mathtt{Z} \vdash (\mathtt{X}_1 \circ \mathtt{X}_2) \circ \mathtt{X}_3} \ (assoc_r)$$

$$\frac{\mathtt{Y} \circ \mathtt{X} \vdash \mathtt{Z}}{\mathtt{X} \circ \mathtt{Y} \vdash \mathtt{Z}} \ (com_l) \quad \frac{\mathtt{Z} \vdash \mathtt{Y} \circ \mathtt{X}}{\mathtt{Z} \vdash \mathtt{X} \circ \mathtt{Y}} \ (com_r) \quad \frac{\mathtt{X} \circ \mathtt{X} \vdash \mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}} \ (contr_l) \quad \frac{\mathtt{Y} \vdash \mathtt{X} \circ \mathtt{X}}{\mathtt{Y} \vdash \mathtt{X}} \ (contr_r)$$

$$\frac{I \vdash \mathtt{X}}{\bullet_{\mathtt{a}} I \vdash \mathtt{X}} \ (nec^l_{\mathtt{a}}) \quad \frac{\mathtt{X} \vdash I}{\mathtt{X} \vdash \bullet_{\mathtt{a}} I} \ (nec^r_{\mathtt{a}})$$

Figure 4: Other basic structural rules

exactly the behaviour of the comma in Gentzen's calculus for classical logic. Such a refined decomposition of the properties of $\circ$, partly explains how a large number of substructural logics can be displayed (see e.g. [Gor98]).

The structural rules in Figure 5 are translations of the primitive axioms of $\mathcal{L}_\cap$ into *primitive display rules* [Kra96]. We invite the reader to consult [Kra96] for a precise definition of primitivity since its exact definition is not pertinent here. Anyway, a primitive axiom is always a Sahlqvist formula [Sah75]. Primitivity of the axioms guarantees a display calculus satisfying the conditions (C1)-(C8) [Bel82] and therefore enjoying cut-elimination.

All the rules in Figure 5 are indeed rule schemata for $\mathtt{a}, \mathtt{b}, \mathtt{d} \in \mathtt{M}$, except for $(refl_{\mathtt{a}})$, $(sym_{\mathtt{a}})$, $(trans_{\mathtt{a}})$ where $\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}$. Alternatively, the rules $(weak^1_{\mathtt{a},\mathtt{b}})$, $(weak^2_{\mathtt{a},\mathtt{b}})$, $(com_{\mathtt{a},\mathtt{b}})$, $(ass^1_{\mathtt{a},\mathtt{b},\mathtt{d}})$ and $(ass^2_{\mathtt{a},\mathtt{b},\mathtt{d}})$ can be replaced by

$$\frac{* \bullet_{\mathtt{a}} * \mathtt{X} \vdash \mathtt{Y}}{* \bullet_{\mathtt{b}} * \mathtt{X} \vdash \mathtt{Y}}$$

when $\mathtt{M}_0(\mathtt{a}) \subseteq \mathtt{M}_0(\mathtt{b})$. As usual in **DL**, this possible replacement highlights that it is often better to consider low level rules which may be omitted to

$$\frac{* \bullet_{\mathsf{a}} * \mathtt{X} \vdash \mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}}\ (refl_{\mathsf{a}}) \qquad \frac{* \bullet_{\mathsf{U}} * \mathtt{X} \vdash \mathtt{Y}}{* \bullet_{\mathsf{a}} * \mathtt{X} \vdash \mathtt{Y}}\ (uni_{\mathsf{a}})$$

$$\frac{* \bullet_{\mathsf{a}} * \mathtt{X} \vdash \mathtt{Y}}{* \bullet_{\mathsf{a}} \bullet_{\mathsf{a}} * \mathtt{X} \vdash \mathtt{Y}}\ (trans_{\mathsf{a}}),\ \ (\mathcal{L} = S5_{\triangle})$$

$$\frac{* \bullet_{\mathsf{a}} * (\mathtt{Z} \circ * \bullet_{\mathsf{a}} * \mathtt{X}) \vdash \mathtt{Y}}{\mathtt{X} \circ * \bullet_{\mathsf{a}} * \mathtt{Z} \vdash \mathtt{Y}}\ (sym_{\mathsf{a}}),\ \ (\mathcal{L} \neq T_{\triangle})$$

$$\frac{* \bullet_{\mathsf{a}} * \mathtt{X} \vdash \mathtt{Y}}{* \bullet_{\mathsf{a} \cap \mathsf{b}} * \mathtt{X} \vdash \mathtt{Y}}\ (weak^{1}_{\mathsf{a},\mathsf{b}}) \quad \frac{* \bullet_{\mathsf{b}} * \mathtt{X} \vdash \mathtt{Y}}{* \bullet_{\mathsf{a} \cap \mathsf{b}} * \mathtt{X} \vdash \mathtt{Y}}\ (weak^{2}_{\mathsf{a},\mathsf{b}}) \quad \frac{* \bullet_{\mathsf{a} \cap \mathsf{b}} * \mathtt{X} \vdash \mathtt{Y}}{* \bullet_{\mathsf{b} \cap \mathsf{a}} * \mathtt{X} \vdash \mathtt{Y}}\ (com_{\mathsf{a},\mathsf{b}})$$

$$\frac{* \bullet_{(\mathsf{a} \cap \mathsf{b}) \cap \mathsf{d}} * \mathtt{X} \vdash \mathtt{Y}}{* \bullet_{\mathsf{a} \cap (\mathsf{b} \cap \mathsf{d})} * \mathtt{X} \vdash \mathtt{Y}}\ (ass^{1}_{\mathsf{a},\mathsf{b},\mathsf{d}}) \quad \frac{* \bullet_{\mathsf{a} \cap (\mathsf{b} \cap \mathsf{d})} * \mathtt{X} \vdash \mathtt{Y}}{* \bullet_{(\mathsf{a} \cap \mathsf{b}) \cap \mathsf{d}} * \mathtt{X} \vdash \mathtt{Y}}\ (ass^{2}_{\mathsf{a},\mathsf{b},\mathsf{d}})$$

Figure 5: Other structural rules

obtain other similar calculi via structural variation (see [Gor98]).

Theorem 3.6. [Kra96] An $\mathcal{L}_{\cap}$-formula $\phi \in \mathtt{Th}(\mathcal{L}_{\cap})$ iff $I \vdash \phi$ is derivable in $\delta\mathcal{L}_{\cap}$.

Theorem 3.7. [Bel82] $\delta\mathcal{L}_{\cap}$ enjoys cut-elimination, i.e. if there is a proof of $\mathtt{X} \vdash \mathtt{Y}$ in $\delta\mathcal{L}_{\cap}$, then there is a cut-free proof of $\mathtt{X} \vdash \mathtt{Y}$ in $\delta\mathcal{L}_{\cap}$.

In Figure 3, each complex term $\mathsf{a} \in \mathtt{M}$ is associated with its own modal structural connective $\bullet_{\mathsf{a}}$. In [Wan97], Wansing defines a display calculus for PDL^{-} where each program constant c has a corresponding modal structural connective $\bullet_{\mathsf{c}}$. PDL^{-} is defined in [Wan97] as the Propositional Dynamic Logic containing the operators $\cup$ (non-deterministic choice), ; (composition), ? (test) with the program constants interpreted using *serial and single-alternative* binary relations. Thus, Wansing [Wan97] implicitly uses the valid equivalences:

$$[\mathsf{a} \cup \mathsf{b}]\phi \Leftrightarrow [\mathsf{a}]\phi \wedge [\mathsf{b}]\phi \qquad [\mathsf{a};\mathsf{b}]\phi \Leftrightarrow [\mathsf{a}][\mathsf{b}]\phi \qquad [\phi?]\psi \Leftrightarrow (\phi \Rightarrow \psi)$$

For $\mathcal{L}_\cap$, we cannot make use of such an economical encoding (by only considering the family $(\bullet_{\mathtt{c}})_{\mathtt{c} \in \mathtt{M}_0 \cup \{\mathtt{U}\}}$) since the intersection operator $\cap$ is not *modlly definable* [GT75], which is the source of several problems.

For each modal expression $\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}$, the $(refl_{\mathtt{a}})$-rule can be viewed as a cut-rule in disguise. Indeed, reading this rule upwards, given $\mathtt{X} \vdash \mathtt{Y}$, one has to choose which $\mathtt{a}$ to consider in order to apply some $(refl_{\mathtt{a}})$-rule for some $\mathtt{a} \in \mathtt{M} \cup \{\mathtt{U}\}$. Theorem 3.8 below states that if $I \vdash \phi$ is derivable in $\delta\mathcal{L}_\cap$, there are at most $max\{|\mathtt{b}| : \mathtt{b} \in \mathtt{M}(\phi)\}^n$ choices for $\mathtt{a}$ where $n = card(\mathtt{M}_0(\phi))$. Modulo associativity and commutativity of $\cap$, the number of choices drastically decreases.

Theorem 3.8. Let ϕ be an $\mathcal{L}_\cap$-formula such that $I \vdash \phi$ is derivable in $\delta\mathcal{L}_\cap$. Then, there is a cut-free derivation of $I \vdash \phi$ in $\delta\mathcal{L}_\cap$ such that any instance of the rule $(refl_{\mathtt{a}})$ in this derivation satisfies $\mathtt{M}_0(\mathtt{a}) \subseteq \mathtt{M}_0(\phi)$ and $|\mathtt{a}| \leq max\{|\mathtt{a}| : \mathtt{a} \in \mathtt{M}(\phi)\}$.

The proof of Theorem 3.8 follows from Lemma 3.2 when one considers the proof of Kracht's results on properly displayable logics [Kra96]. By induction on the formation of ϕ we can also prove

Proposition 3.9. $\phi \vdash \phi$ is cut-free derivable in $\delta\mathcal{L}_\cap$ for all $\mathcal{L}_\cap$-formulae ϕ.

In the rest of the paper, we therefore assume that the identity rule (Id) from Figure 1 is of the form $\psi \vdash \psi$ where ψ is a formula variable.

4 Displaying logics with relative accessibility relations

In this section, we first show how $\mathcal{L}$-satisfiability can be faithfully translated into $\mathcal{L}_\cap$-satisfiability. The translation shall then be the key point to define the display calculus $\delta\mathcal{L}$ that mimics the rules of $\delta\mathcal{L}_\cap$.

4.1 Satisfiability-preserving maps

In this section, we assume that $\mathcal{L}$ is a B-logic closed under intersection. This is the only place in the paper where $\mathcal{L}$ is not necessarily a member of

$\{T_\triangle, B_\triangle, S5_\triangle\}$. We reproduce here some arguments from [Dem98b] in order to generalise the translation from LKO (our $S5_\triangle$) into $S5_\omega^{\cap,U}$ (our $(S5_\triangle)_\cap$) defined in [Dem98b] to $\mathcal{L}$ closed under intersection.

Assume $\phi \in \mathtt{For}$ and let $\mathtt{A}_0(\phi)$ denote the set of agent constants occurring in ϕ. Without loss of generality, we can assume that if $card(\mathtt{A}_0(\phi)) = n$, then $\mathtt{A}_0(\phi) = \{\delta_1, \ldots, \delta_n\}$. Indeed, $\mathcal{L}$-satisfiability and $\mathcal{L}$-validity are not sensitive to the renaming of constants. When $\mathtt{A}_0(\phi) = \emptyset$, ϕ is $\mathcal{L}$-valid iff ϕ is $\mathcal{L}_\cap$-valid iff ϕ is valid in classical propositional logic. So assume in the sequel that $n \geq 1$, that is $\mathtt{A}_0(\phi) \neq \emptyset$. For any integer $k \in \{0, \ldots, 2^n - 1\}$ we write $bit_i(k)$ for the i^{th} bit of the natural number k in binary form and write α_k^* to denote the agent expression $\alpha_1 \cap \ldots \cap \alpha_n$ where for any $i \in \{1, \ldots, n\}$,

$$\alpha_i \stackrel{\text{def}}{=} \begin{cases} \delta_i & \text{if } bit_i(k) = 0 \\ -\delta_i & \text{otherwise.} \end{cases}$$

For any agent expression $\alpha \in \mathtt{A}$ such that $\mathtt{A}_0(\alpha) \subseteq \{\delta_1, \ldots, \delta_n\}$, either $\alpha \equiv \perp$ or there is a unique set $Y = \{\alpha_{i_1}^*, \ldots, \alpha_{i_l}^*\}$ such that $\alpha \equiv \alpha_{i_1}^* \cup \ldots \cup \alpha_{i_l}^*$. Suppose $\alpha \in \mathtt{A}$ occurs in ϕ in some subformula $\triangle(\alpha)\psi$ such that $\alpha \not\equiv \perp$ and $\alpha \equiv \alpha_{i_1}^* \cup \ldots \cup \alpha_{i_l}^*$ for some $\{i_1, \ldots, i_l\} \subseteq \{0, \ldots, 2^n - 1\}$. The normal form[8] of α, written $\mathtt{N}(\alpha)$, is merely the expression $\alpha_{i_1}^* \cup \ldots \cup \alpha_{i_l}^*$. In the case when $\alpha \equiv \perp$, $\mathtt{N}(\alpha) \stackrel{\text{def}}{=} \delta_1 \cap -\delta_1$. We write $\mathtt{N}(\phi)$ to denote the formula obtained from ϕ by replacing each occurrence of α by $\mathtt{N}(\alpha)$. $\mathtt{N}(\phi)$ is unique modulo associativity and commutativity of $\cup$ and $\cap$, and $\mathtt{N}(\phi)$ can be computed in deterministic time $2^{p(|\phi|)}$ for some polynomial $p(m)$.

Using the technique from [Dem98b], we shall define a mapping g from $\mathtt{For}$ into $\mathtt{For}_\cap$ that takes advantage of the normal forms and such that $g(\phi) \stackrel{\text{def}}{=} f_\phi(\mathtt{N}(\phi))$ where f_ϕ is homomorphic for the propositional connectives:

- for any $\mathtt{p} \in \mathtt{For}_0$, $f_\phi(\mathtt{p}) \stackrel{\text{def}}{=} \mathtt{p}$; $f_\phi(\perp) \stackrel{\text{def}}{=} \perp$; $f_\phi(\top) \stackrel{\text{def}}{=} \top$;
- $f_\phi(\triangle(\alpha_{i_1}^* \cup \ldots \cup \alpha_{i_l}^*)\psi) \stackrel{\text{def}}{=} [\mathtt{c}_{i_1} \cap \ldots \cap \mathtt{c}_{i_l}]\neg f_\phi(\psi) \vee [\mathtt{c}_{i_1} \cap \ldots \cap \mathtt{c}_{i_l}] f_\phi(\psi)$;
- $f_\phi(\triangle(\delta_1 \cap -\delta_1)\psi) \stackrel{\text{def}}{=} [\mathtt{U}]\neg f_\phi(\psi) \vee [\mathtt{U}] f_\phi(\psi)$.

Note that f_ϕ depends on ϕ only because $card(\mathtt{A}_0(\phi)) = n$. The definition of f_ϕ can be refined by the renaming technique (see e.g. [Min88]) so that

[8] The canonical disjunctive normal form for classical propositional logic [Lem65, Kon97a].

$f_\phi(\psi)$ is not copied twice in the conditions defining $f_\phi(\triangle(\alpha)\psi)$. It suffices to appropriately associate a new proposition $\mathtt{p}_{f_\phi(\psi)}$ to $f_\phi(\psi)$: the details are omitted here.

Lemma 4.1. $\phi \in \mathtt{For}$ is $\mathcal{L}$-satisfiable iff $g(\phi)$ is $\mathcal{L}_\cap$-satisfiable.

Proof: For the direction from left to right, assume there exist an $\mathcal{L}$-model $\mathcal{M} = (OB, Ag, (R_Q)_{Q\subseteq Ag}, m)$ and $o \in OB$ such that $\mathcal{M}, o \models \mathtt{N}(\phi)$. Let $\mathcal{M}'$ be the $\mathcal{L}_\cap$-model $(OB, (R'_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}\cup\{\mathtt{U}\}}, m')$ such that:

1. for all $k \in \{0, \ldots, 2^n - 1\}$, $R'_{\mathtt{c}_k} \stackrel{\text{def}}{=} R_{m(\alpha_k^*)}$
2. for all $\mathtt{c} \in \mathtt{M}_0 \setminus \{\mathtt{c}_0, \ldots, \mathtt{c}_{2^n-1}\}$ $R'_{\mathtt{c}} \stackrel{\text{def}}{=} R'_{\mathtt{c}_0}$ (arbitrary value)
3. for all $\mathtt{a} \in \mathtt{M}$, $R'_{\mathtt{a}} \stackrel{\text{def}}{=} \bigcap_{\mathtt{c}\in\mathtt{M}_0(\mathtt{a})} R'_{\mathtt{c}}$
4. $R'_{\mathtt{U}} \stackrel{\text{def}}{=} OB \times OB$
5. for all $\mathtt{p} \in \mathtt{For}_0$, $m'(\mathtt{p}) \stackrel{\text{def}}{=} m(\mathtt{p})$.

Writing $sub(\phi)$ for the set of *subformulae* of ϕ including ϕ, one can show by structural induction that for any $o' \in OB$ and any $\psi \in sub(\phi)$: $\mathcal{M}, o' \models \psi$ iff $\mathcal{M}', o' \models g(\psi)$. Therefore $\mathcal{M}', o \models g(\phi)$.

For the direction from right to left, assume there exist an $\mathcal{L}_\cap$-model $\mathcal{M}' = (W', (R'_{\mathtt{a}})_{\mathtt{a}\in\mathtt{M}\cup\{\mathtt{U}\}}, m')$ and $w \in W'$ such that $\mathcal{M}', w \models g(\phi)$. Let $\mathcal{M} \stackrel{\text{def}}{=} (W', Ag, (R_Q)_{Q\subseteq Ag}, m)$ where:

1. $Ag \stackrel{\text{def}}{=} \{0, \ldots, 2^n - 1\}$
2. the restriction of m to $\mathtt{For}_0$ is m'
3. for $i \in \{1, \ldots, n\}$, $m(\delta_i) \stackrel{\text{def}}{=} \{k \in Ag : bit_i(k) = 0\}$. The interpretation of the other constants in $\{\delta_{n+1}, \ldots\}$ is not constrained until the restriction of m to $\mathtt{A}$ is an $\mathtt{A}$-interpretation, which is always possible.
4. $R_\emptyset \stackrel{\text{def}}{=} W' \times W'$ and for any $\emptyset \neq Q \subseteq Ag$, $R_Q \stackrel{\text{def}}{=} \bigcap_{k\in Q} R'_{\mathtt{c}_k}$.

The construction of $\mathcal{M}$ is designed to guarantee that for any $k \in \{0, \ldots, 2^n - 1\}$, $m(\alpha_k^*)$ is non-empty, as otherwise $R_{m(\alpha_k^*)}$ has to be the universal relation. Indeed, $m(\alpha_k^*)$ is equal to $\{k\}$. The definition of $\mathcal{M}$ is correct and $\mathcal{M}$ is an $\mathcal{L}$-model. By structural induction one can show that for any $\psi \in sub(\phi)$ and any $w' \in W'$: $\mathcal{M}', w' \models g(\psi)$ iff $\mathcal{M}, w' \models \psi$. Hence $\mathcal{M}, w \models \phi$. **Q.E.D.**

Since $g(\neg\phi) \Leftrightarrow \neg g(\phi)$ is $\mathcal{L}_\cap$-valid, Lemma 4.1 also entails that ϕ is $\mathcal{L}$-valid iff $g(\phi)$ is $\mathcal{L}_\cap$-valid.

$$\frac{\mathtt{X} \vdash \bullet_\alpha \phi \circ \bullet_\alpha * \phi}{\mathtt{X} \vdash \triangle(\alpha)\phi}\ (\vdash \triangle(\alpha)) \qquad \frac{\phi \vdash \mathtt{X} \quad \mathtt{Y} \vdash \phi}{\triangle(\alpha)\phi \vdash \bullet_\alpha \mathtt{X} \circ \bullet_\alpha * \mathtt{Y}}\ (\triangle(\alpha) \vdash)$$

Figure 6: Operational rules

$$\frac{* \bullet_\alpha * \mathtt{X} \vdash \mathtt{Y}}{\mathtt{X} \vdash \mathtt{Y}}\ (refl_\alpha) \quad \frac{* \bullet_\alpha * \mathtt{X} \vdash \mathtt{Y}}{* \bullet_\alpha \bullet_\alpha * \mathtt{X} \vdash \mathtt{Y}}\ (trans_\alpha),\ (\mathcal{L} = S5_\triangle)$$

$$\frac{* \bullet_\alpha *(\mathtt{Z} \circ * \bullet_\alpha * \mathtt{X}) \vdash \mathtt{Y}}{\mathtt{X} \circ * \bullet_\alpha * \mathtt{Z} \vdash \mathtt{Y}}\ (sym_\alpha),\ (\mathcal{L} \neq T_\triangle) \quad \frac{* \bullet_\beta * \mathtt{X} \vdash \mathtt{Y}}{* \bullet_\alpha * \mathtt{X} \vdash \mathtt{Y}}\ (\sqsubseteq_{\beta,\alpha}),\ \text{with } \beta \sqsubseteq \alpha$$

Figure 7: Additional structural rules

4.2 The display calculi

The display calculus $\delta\mathcal{L}$ defined below is composed of axioms and inference rules that partly mimic those of $\delta\mathcal{L}_\cap$. As for $\delta\mathcal{L}_\cap$, the Boolean structural connectives are $*$ (unary), $\circ$ (binary), I (nullary). But now we introduce a new family $(\bullet_\alpha)_{\alpha \in \mathtt{A}}$ of (unary) modal structural connectives. A structure $\mathtt{X} \in \mathtt{struc}(\delta\mathcal{L})$ is inductively defined as follows for $\phi \in \mathtt{For}$ and $\alpha \in \mathtt{A}$:

$$\mathtt{X} ::= \phi \ \mid\ *\mathtt{X} \ \mid\ \mathtt{X}_1 \circ \mathtt{X}_2 \ \mid\ I \ \mid\ \bullet_\alpha$$

The fundamental logical axioms and cut rule (Figure 1), the basic structural rules (Figure 2 and Figure 4 with obvious modifications) and the operational rules (Figure 3) for $\delta\mathcal{L}$ are by definition those for $\delta\mathcal{L}_\cap$ except that the rules introducing $[\mathtt{a}]\phi$ (as antecedent and succedent) are replaced by the rules introducing $\triangle(\alpha)\phi$ (as antecedent and succedent) described in Figure 6. The additional structural rules for $\delta\mathcal{L}$ are presented in Figure 7.

In Figure 6 and Figure 7, α and β belong to $\mathtt{A}$. Since $\delta\mathcal{L}$ satisfies the conditions (C1)-(C8) [Bel82], $\delta\mathcal{L}$ enjoys cut-elimination. Section 4.3 contains a soundness and completeness proof as well as a cut-elimination theorem established by backward translation, using only the fact that $\delta\mathcal{L}_\cap$ enjoys cut-elimination, thereby highlighting the relationships between these calculi.

4.3 Soundness, completeness and cut-elimination

To prove soundness of $\delta\mathcal{L}$ with respect to $\mathcal{L}$-validity, we use the mappings $a : \mathtt{struc}(\delta\mathcal{L}) \to \mathtt{For}$ and $s : \mathtt{struc}(\delta\mathcal{L}) \to \mathtt{For}$ defined below when $\mathcal{L} \neq T_{\triangle}$:

$$a(\phi) \stackrel{\text{def}}{=} s(\phi) \stackrel{\text{def}}{=} \phi \text{ for any } \phi \in \mathtt{For}$$

$$\begin{array}{llllll}
a(I) & \stackrel{\text{def}}{=} & \top & s(I) & \stackrel{\text{def}}{=} & \bot \\
a(*\mathtt{X}) & \stackrel{\text{def}}{=} & \neg s(\mathtt{X}) & s(*\mathtt{X}) & \stackrel{\text{def}}{=} & \neg a(\mathtt{X}) \\
a(\mathtt{X} \circ \mathtt{Y}) & \stackrel{\text{def}}{=} & a(\mathtt{X}) \wedge a(\mathtt{Y}) & s(\mathtt{X} \circ \mathtt{Y}) & \stackrel{\text{def}}{=} & s(\mathtt{X}) \vee s(\mathtt{Y}) \\
a(\bullet_{\alpha}\mathtt{X}) & \stackrel{\text{def}}{=} & a(\mathtt{X}) \vee \neg\triangle(\alpha)a(\mathtt{X}) & s(\bullet_{\alpha}\mathtt{X}) & \stackrel{\text{def}}{=} & s(\mathtt{X}) \wedge \triangle(\alpha)s(\mathtt{X})
\end{array}$$

In the case when $\mathcal{L} = T_{\triangle}$, the definition of $a(\bullet_{\alpha}\mathtt{X})$ must be changed to

$$a(\bullet_{\alpha}\mathtt{X}) \stackrel{\text{def}}{=} a(\mathtt{X}) \vee \neg\triangle^{-}(\alpha)a(\mathtt{X})$$

where $\triangle^{-}(\alpha)$ is the backward modality associated with $\triangle(\alpha)$. That is, as is standard with **DL**, we extend the language by adding the family $\{\triangle^{-}(\alpha) : \alpha \in \mathtt{A}\}$ of "reverse" unary modal operators. The satisfiability relation $\models$ is modified appropriately by adding the clause:

$$\mathcal{M}, o \models \triangle^{-}(\alpha)\phi \stackrel{\text{def}}{\Leftrightarrow} \left\{ \begin{array}{l} \text{either } \mathcal{M}, o' \models \phi \text{ for all } o' \in R^{-1}_{m(\alpha)}(o) \\ \text{or } \mathcal{M}, o' \models \neg\phi \text{ for all } o' \in R^{-1}_{m(\alpha)}(o) \end{array} \right.$$

where $R^{-1}_{m(\alpha)}$ is the converse relation of $R_{m(\alpha)}$.

In the case when $\mathcal{L} \in \{B_{\triangle}, S5_{\triangle}\}$, there is no need to introduce new modalities since the relations in the corresponding models are symmetric.

Theorem 4.2. If $\mathtt{X} \vdash \mathtt{Y}$ is derivable in $\delta\mathcal{L}$, then $a(\mathtt{X}) \Rightarrow s(\mathtt{Y})$ is $\mathcal{L}$-valid.

The proof is by induction on the length of the given derivation of $\mathtt{X} \vdash \mathtt{Y}$. The maps a and s are slight variants of standard translations that can be found for instance in [Kra96]. The interest of a and s is not only in the soundness proof but also in the way the structural connectives should be interpreted depending on their occurrence, either as antecedent parts or as succedent parts.

Corollary 4.3. (soundness) If $I \vdash \phi$ is derivable in $\delta\mathcal{L}$, then ϕ is $\mathcal{L}$-valid.

In order to prove completeness, we define a reverse map g^{-1} such that any $\mathcal{L}_\cap$-formula ϕ is $\mathcal{L}_\cap$-valid iff $g^{-1}(\phi)$ is $\mathcal{L}$-valid.

Let ϕ be an $\mathcal{L}_\cap$-formula such that $\mathtt{M}_0(\phi) = \{\mathtt{c}_1, \ldots, \mathtt{c}_m\}$. Let n be the smallest natural number such that $2^n - 1 \geq m$. As usual, for any $k \in \{0, \ldots, 2^n - 1\}$, we define the agent expression $\alpha_k^* \stackrel{\text{def}}{=} \alpha_1 \cap \ldots \cap \alpha_n$ where for any $i \in \{1, \ldots, n\}$, if $bit_i(k) = 0$, then $\alpha_i \stackrel{\text{def}}{=} \delta_i$ otherwise $\alpha_i \stackrel{\text{def}}{=} -\delta_i$. We define the mapping $g_n^{-1} : \mathtt{M} \cup \{\mathtt{U}\} \cup \mathtt{struc}(\delta\mathcal{L}_\cap) \to \mathtt{A} \cup \mathtt{struc}(\delta\mathcal{L})$ as follows

- g_n^{-1} is homomorphic for the Boolean connectives and the Boolean structural connectives;
- $g_n^{-1}(I) \stackrel{\text{def}}{=} I$; $\quad g_n^{-1}(\mathtt{p}_i) = \mathtt{p}_i$ for $i \in \omega$; $\quad g_n^{-1}(\top) \stackrel{\text{def}}{=} \top$; $\quad g_n^{-1}(\bot) \stackrel{\text{def}}{=} \bot$;
- for all $k \in \{0, \ldots, 2^n - 1\}$, $g_n^{-1}(\mathtt{c}_k) \stackrel{\text{def}}{=} \alpha_k^*$;
- for all $\mathtt{c} \in \mathtt{M}_0 \setminus \{\mathtt{c}_0, \ldots, \mathtt{c}_{2^n-1}\}$, $g_n^{-1}(\mathtt{c}) = \delta_1$ (arbitrary value);
- $g_n^{-1}(\mathtt{U}) \stackrel{\text{def}}{=} \delta_1 \cap -\delta_1$; $\qquad g_n^{-1}(\mathtt{a} \cap \mathtt{b}) \stackrel{\text{def}}{=} g_n^{-1}(\mathtt{a}) \cup g_n^{-1}(\mathtt{b})$;
- $g_n^{-1}([\mathtt{a}]\psi) \stackrel{\text{def}}{=} g_n^{-1}(\psi) \wedge \triangle(g_n^{-1}(\mathtt{a}))g_n^{-1}(\psi)$; $\qquad g_n^{-1}(\bullet_{\mathtt{a}}\mathtt{X}) = \bullet_{g_n^{-1}(\mathtt{a})} g_n^{-1}(\mathtt{X})$.

One can check that $\mathtt{N}(g_n^{-1}(\phi)) = g_n^{-1}(\phi)$, and that $g(g_n^{-1}(\phi))$ is $\mathcal{L}_\cap$-valid iff ϕ is $\mathcal{L}_\cap$-valid since $\mathtt{p} \wedge ([\mathtt{a}]\mathtt{p} \vee [\mathtt{a}]\neg\mathtt{p}) \Leftrightarrow [\mathtt{a}]\mathtt{p}$ is $\mathcal{L}_\cap$-valid. Hence, by Lemma 4.1, ϕ is $\mathcal{L}_\cap$-valid iff $g_n^{-1}(\phi)$ is $\mathcal{L}$-valid providing a definition for g^{-1}.

Let $\mathtt{A}_0(\phi)$ [resp. $\mathtt{A}_0(\beta)$] denote the set of agent constants (from $\mathtt{A}_0$) occurring in the formula ϕ [resp. in the agent expression β].

Lemma 4.4. Let ϕ be an $\mathcal{L}$-formula with $\mathtt{A}_0(\phi) = \{\delta_1, \ldots, \delta_n\}$ and, in the $(\sqsubseteq_{\beta,\alpha})$-rule, let (3) be the condition $\mathtt{A}_0(\beta) \subseteq \mathtt{A}_0(\phi)$. Then, $I \vdash \phi$ has a cut-free proof satisfying (3) in $\delta\mathcal{L}$ iff $I \vdash g_n^{-1}(g(\phi))$ has a cut-free proof satisfying (3) in $\delta\mathcal{L}$.

Proof: First, one can show that $I \vdash \phi$ admits a cut-free proof in $\delta\mathcal{L}$ iff $I \vdash \mathtt{N}(\phi)$ admits a cut-free proof in $\delta\mathcal{L}$ as follows. One can easily observe that to each occurrence of $\alpha \in \mathtt{A}$ in ϕ there is a matching occurrence of $\mathtt{N}(\alpha) \in \mathtt{A}$ in $\mathtt{N}(\phi)$. Let Π be a cut-free proof of $I \vdash \phi$ satisfying the condition of the lemma. Let Π' be the *proof* obtained from Π by:

- replacing every occurrence of $\alpha \in \mathtt{A}$ in a formula by $\mathtt{N}(\alpha)$;
- replacing every instance of some $(\sqsubseteq_{\beta,\alpha})$-rule by an appropriate instance of the $(\sqsubseteq_{\mathtt{N}(\beta),\mathtt{N}(\alpha)})$-rule.

Then Π' is a cut-free proof of $I \vdash \mathtt{N}(\phi)$ satisfying the condition of the lemma.

Conversely, let Π be a cut-free proof of $I \vdash \mathtt{N}(\phi)$ satisfying the condition of the lemma. Let Π' be the *proof* obtained from Π by:

- replacing every occurrence of $\alpha \in \mathtt{A}$ in a formula by α' such that $\alpha = \mathtt{N}(\alpha')$ and the agent expressions satisfy the matching mentioned at the beginning of the proof;
- replacing every instance of any $(\sqsubseteq_{\beta,\alpha})$-rule by an appropriate instance of the $(\sqsubseteq_{\beta,\alpha'})$-rule, with α' satisfying the condition as above.

Then Π' is a cut-free proof of $I \vdash \phi$ satisfying the condition of the lemma.

Second, $g_n^{-1}(g(\phi)) = h(\mathtt{N}(\phi))$ where $h : \mathtt{For} \to \mathtt{For}$ is a formula mapping defined as follows:

- for any $i \in \omega$, $h(\mathtt{p}_i) \stackrel{\text{def}}{=} \mathtt{p}_i$;
- h is homomorphic for the Boolean connectives;
- $h(\triangle(\alpha)\psi) \stackrel{\text{def}}{=} (\neg h(\psi) \wedge \triangle(\alpha)\neg h(\psi)) \vee (h(\psi) \wedge \triangle(\alpha)h(\psi))$.

Let (4) be the condition that $I \vdash \triangle(\alpha)\psi$ admits a cut-free proof in $\delta\mathcal{L}$ iff $I \vdash (\neg\psi \wedge \triangle(\alpha)\neg\psi) \vee (\psi \wedge \triangle(\alpha)\psi)$ admits a cut-free proof in $\delta\mathcal{L}$. If (4) holds, then $I \vdash h(\mathtt{N}(\phi))$ admits a cut-free proof in $\delta\mathcal{L}$ iff $I \vdash \mathtt{N}(\phi)$ admits a cut-free proof in $\delta\mathcal{L}$. Condition (4) can be proved by checking the following points:

- for any structure $\mathtt{X}$ and any formula ψ, the sequent $\mathtt{X} \vdash \neg\psi$ admits a cut-free proof in $\delta\mathcal{L}$ iff $\mathtt{X} \vdash *\psi$ admits a cut-free proof in $\delta\mathcal{L}$;
- for any structure $\mathtt{X}$ and any formula ψ, the sequent $\mathtt{X} \vdash \triangle(\alpha)\psi$ admits a cut-free proof in $\delta\mathcal{L}$ iff $\mathtt{X} \vdash \triangle(\alpha)\neg\psi$ admits a cut-free proof in $\delta\mathcal{L}$;
- for all formulae ψ, ψ', the sequent $I \vdash \psi$ admits a cut-free proof in $\delta\mathcal{L}$ iff $I \vdash (\psi \wedge \psi') \vee (\psi \wedge \neg\psi')$ admits a cut-free proof in $\delta\mathcal{L}$.

Q.E.D.

Theorem 4.5. For any $\mathcal{L}$-formula ϕ, if $I \vdash g(\phi)$ has a cut-free proof in $\delta\mathcal{L}_{\cap}$, then $I \vdash \phi$ has a cut-free proof in $\delta\mathcal{L}$.

Proof: Consider a cut-free derivation of $I \vdash g(\phi)$ satisfying the restrictions stated in Theorem 3.8. In the case when $\mathtt{A}_0(\phi) = \emptyset$, a cut-free derivation of $I \vdash \phi$ in $\delta\mathcal{L}$ can be fairly easily obtained: after all ϕ is simply a formula of classical propositional logic.

In the rest of the proof, assume that $\mathtt{A}_0(\phi) = \{\delta_1, \ldots, \delta_n\}$, that is $\mathtt{A}_0(\phi)$ is non-empty. We show that in the given cut-free proof of $I \vdash g(\phi)$, for every sequent $\mathtt{X} \vdash \mathtt{Y}$ with cut-free proof Π, the sequent $g_n^{-1}(\mathtt{X}) \vdash g_n^{-1}(\mathtt{Y})$ admits a cut-free proof, say $g_n^{-1}(\Pi)$, in $\delta\mathcal{L}$. So, we shall conclude $g_n^{-1}(I) \vdash g_n^{-1}(g(\phi))$ admits a cut-free derivation in $\delta\mathcal{L}$. By Lemma 4.4, $I \vdash \phi$ admits a cut-free derivation in $\delta\mathcal{L}$.

As expected, the proof is by induction on the length of the derivations. When $\mathtt{X} \vdash \mathtt{Y}$ is of the form $\psi \vdash \psi$, the base case is immediate. Similarly, the proof poses no difficulty when the last rule is a basic structural rule (from Figure 2 and Figure 4) or an operational structural rule introducing a Boolean connective (Figure 3). This is due to the fact that g_n^{-1} is homomorphic for the Boolean connectives and the Boolean structural connectives.

In what follows, we write

$$\frac{s'}{s}\ (dp)$$

to denote that the sequent s is obtained from the sequent s' by an unspecified finite number (possibly zero) of applications of display postulates from Figure 2.

Now, let us treat the case when the last rule is $(\vdash [\mathsf{a}])$. The proof

$$\dfrac{\begin{array}{c}\vdots\ \Pi\\ \mathtt{X} \vdash \bullet_{\mathsf{a}}\psi\end{array}}{\mathtt{X} \vdash [\mathsf{a}]\psi}\ (\vdash [\mathsf{a}])$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ g_n^{-1}(\Pi)\\ \mathtt{X}' \vdash \bullet_\alpha\psi'\end{array}}{* \bullet_\alpha * * \psi' \vdash *\mathtt{X}'}\ (dp)}{*\psi' \vdash *\mathtt{X}'}\ (refl_\alpha)}{\mathtt{X}' \vdash \psi'}\ (dp) \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ g_n^{-1}(\Pi)\\ \mathtt{X}' \vdash \bullet_\alpha\psi'\end{array}}{\mathtt{X}' \circ * \bullet_\alpha *\psi' \vdash \bullet_\alpha\psi'}\ (weak_r)}{\mathtt{X}' \vdash \bullet_\alpha\psi' \circ \bullet_\alpha * \psi'}\ (dp)}{\mathtt{X}' \vdash \triangle(\alpha)\psi'}\ (\vdash \triangle(\alpha))}{\dfrac{\mathtt{X}' \circ \mathtt{X}' \vdash \psi' \wedge \triangle(\alpha)\psi'}{\mathtt{X}' \vdash \psi' \wedge \triangle(\alpha)\psi'}\ (contr_l)}\ (\vdash \wedge)$$

with $\alpha \stackrel{\text{def}}{=} g_n^{-1}(\mathtt{a})$, $\psi' \stackrel{\text{def}}{=} g_n^{-1}(\psi)$ and $\mathtt{X}' \stackrel{\text{def}}{=} g_n^{-1}(\mathtt{X})$.

Using the same notation, let us treat the case when the last rule is $([\mathtt{a}] \vdash)$. The proof

$$\dfrac{\begin{array}{c}\vdots\ \Pi \\ \psi \vdash \mathtt{X}\end{array}}{[\mathtt{a}]\psi \vdash \bullet_{\mathtt{a}}\mathtt{X}}\ ([\mathtt{a}] \vdash)$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ g_n^{-1}(\Pi) \\ \psi' \vdash \mathtt{X}'\end{array} \qquad \psi' \vdash \psi'}{\triangle(\alpha)\psi' \vdash \bullet_{\alpha}\mathtt{X}' \circ \bullet_{\alpha} * \psi'}\ (\triangle(\alpha) \vdash)}{\triangle(\alpha)\psi' \vdash \bullet_{\alpha} * \psi' \circ \bullet_{\alpha}\mathtt{X}'}\ (com_r)}{* \bullet_{\alpha} * \psi' \vdash \bullet_{\alpha}\mathtt{X}' \circ * \triangle(\alpha)\psi'}\ (dp)}{\psi' \vdash \bullet_{\alpha}\mathtt{X}' \circ * \triangle(\alpha)\psi'}\ (refl_{\alpha})}{\psi' \circ \triangle(\alpha)\psi' \vdash \bullet_{\alpha}\mathtt{X}'}\ (dp)}{\psi' \wedge \triangle(\alpha)\psi' \vdash \bullet_{\alpha}\mathtt{X}'}\ (\wedge \vdash)$$

When the last applied rule is an instance of any of the rules $(refl_{\mathtt{a}})$, $(sym_{\mathtt{a}})$, $(trans_{\mathtt{a}})$ then the corresponding applied rule in $\delta\mathcal{L}$ is an instance of one of $(refl_{\alpha})$, (sym_{α}), $(trans_{\alpha})$ respectively.

In order to conclude let us treat the cases when the last rule is $(uni_{\mathtt{a}})$ or $(weak^1_{\mathtt{a},\mathtt{b}})$ (the other cases can then be easily obtained). The proof

$$\dfrac{\begin{array}{c}\vdots\ \Pi \\ * \bullet_{\mathtt{U}} * \mathtt{X} \vdash \mathtt{Y}\end{array}}{* \bullet_{\mathtt{a}} * \mathtt{X} \vdash \mathtt{Y}}\ (uni_{\mathtt{a}})$$

is transformed into

$$\dfrac{\begin{array}{c}\vdots\ g_n^{-1}(\Pi) \\ * \bullet_{\delta_1 \cap -\delta_1} * g_n^{-1}(\mathtt{X}) \vdash g_n^{-1}(\mathtt{Y})\end{array}}{* \bullet_{g_n^{-1}(\mathtt{a})} * g_n^{-1}(\mathtt{X}) \vdash g_n^{-1}(\mathtt{Y})}\ (\sqsubseteq_{\delta_1 \cap -\delta_1, g_n^{-1}(\mathtt{a})})$$

Similarly, the proof

$$\dfrac{\begin{array}{c}\vdots\ \Pi \\ * \bullet_{\mathtt{a}} * \mathtt{X} \vdash \mathtt{Y}\end{array}}{* \bullet_{\mathtt{a} \cap \mathtt{b}} * \mathtt{X} \vdash \mathtt{Y}}\ (weak^1_{\mathtt{a},\mathtt{b}})$$

is transformed into

$$\dfrac{\begin{array}{c}\vdots\; g_n^{-1}(\Pi)\\ *\bullet_{g_n^{-1}(\mathtt{a})} *g_n^{-1}(\mathtt{X}) \vdash g_n^{-1}(\mathtt{Y})\end{array}}{*\bullet_{g_n^{-1}(\mathtt{a})\cup g_n^{-1}(\mathtt{b})} *g_n^{-1}(\mathtt{X}) \vdash g_n^{-1}(\mathtt{Y})}\ (\sqsubseteq_{g_n^{-1}(\mathtt{a})\cup g_n^{-1}(\mathtt{b}),g_n^{-1}(\mathtt{a})})$$

Q.E.D.

Corollary 4.6. If an $\mathcal{L}$-formula ϕ is $\mathcal{L}$-valid, then $I \vdash \phi$ has a cut-free proof in $\delta\mathcal{L}$.

Corollary 4.7. Let ϕ be an $\mathcal{L}$-formula such that Π is a cut-free proof of $I \vdash g(\phi)$ in $\delta\mathcal{L}_{\cap}$. Then $I \vdash \phi$ has a cut-free proof Π' in $\delta\mathcal{L}$ of size $O(2^{|\Pi|})$.

For any $\alpha, \beta \in \mathtt{A}$, the $(\sqsubseteq_{\beta,\alpha})$-rule can be viewed as a cut-rule in disguise. Indeed, by reading the proof upwards, given $*\bullet_\alpha *\mathtt{X} \vdash \mathtt{Y}$, one has to choose which β' to consider in order to apply the $(\sqsubseteq_{\beta',\alpha})$-rule, for $\beta' \in \mathtt{A}$. Corollary 4.8 states that if $I \vdash \phi$ is derivable in $\delta\mathcal{L}$, then there are at most 2^{2^n} choices for β' where $n = card(\mathtt{A}_0(\phi))$. At first glance, the implicit cut rule may seem problematic. However, simply considering each agent expression to be a representative of the class having the same normal form with respect to $\mathtt{N}$ allows to reduce the non-determinism. Indeed, less choices for β' are then possible.

Corollary 4.8. Let ϕ be an $\mathcal{L}$-formula such that $I \vdash \phi$ admits a derivation in $\delta\mathcal{L}$ (and recall that $\mathtt{N}$ is defined with respect to ϕ). Then $I \vdash \mathtt{N}(\phi)$ [resp. $I \vdash \phi$] admits a cut-free derivation Π [resp. Π'] in $\delta\mathcal{L}$ such that each instance of the $(\sqsubseteq_{\beta,\alpha})$-rule in Π [resp. Π'] satisfies $\mathtt{A}_0(\beta) \subseteq \mathtt{A}_0(\phi)$ and $\mathtt{N}(\beta) = \beta$.

Instead of having a countably infinite set of $\sqsubseteq_{\beta,\alpha}$-rules, let us replace them by a single rule. To do so, consider the $(\sqsubseteq)$-rule below:

$$\dfrac{*\bullet_\beta *\mathtt{X} \vdash \mathtt{Y} \quad \sigma(\alpha) \vdash \sigma(\beta)}{*\bullet_\alpha *\mathtt{X} \vdash \mathtt{Y}}\ (\sqsubseteq)$$

where $\sigma : \mathtt{A} \to \mathtt{For}$ is defined as follows:

$$\begin{array}{ll} \sigma(\delta_i) \stackrel{\text{def}}{=} \mathtt{p}_i \text{ for } i \in \omega & \sigma(-\alpha) \stackrel{\text{def}}{=} \neg\sigma(\alpha) \\ \sigma(\alpha \cup \beta) \stackrel{\text{def}}{=} \sigma(\alpha) \vee \sigma(\beta) & \sigma(\alpha \cap \beta) \stackrel{\text{def}}{=} \sigma(\alpha) \wedge \sigma(\beta) \end{array}$$

Observe that in the $\sqsubseteq$-rule, α and β are agent expression *variables* used partly as indices for modal structural connectives. This kind of rule is out of the scope of the application of the standard conditions (C1)-(C8) in [Bel82]. Let us define the display calculus $\delta^+\mathcal{L}$ from $\delta\mathcal{L}$ by deleting the $\sqsubseteq_{\beta,\alpha}$-rules, and by adding the ($\sqsubseteq$)-rule. The advantage of $\delta^+\mathcal{L}$ over $\delta\mathcal{L}$ is twofold:

1. no *semantic* condition has to be checked since $\vdash$ provides a calculus for checking $\beta \sqsubseteq \alpha$;

2. a single rule replaces an infinite set of rules.

Although $\delta^+\mathcal{L}$ is not an orthodox display calculus, we can now easily show that

Theorem 4.9. $I \vdash \phi$ has a cut-free proof in $\delta^+\mathcal{L}$ iff ϕ is $\mathcal{L}$-valid.

Proof: Each cut-free proof of $I \vdash \phi$ in $\delta\mathcal{L}$ can be simply transformed into a cut-free proof of $I \vdash \phi$ in $\delta^+\mathcal{L}$. **Q.E.D.**

The idea behind the definition of the ($\sqsubseteq$)-rule consists in defining two relations $\vdash_1$ and $\vdash_2$: the relation $\vdash_1$ is dedicated to deducibility in the logic whereas $\vdash_2$ is dedicated to some semantic relation in the algebra of modal terms. In the particular case of $\mathcal{L}$, the relation $\vdash_2$ can be defined from $\vdash_1$ via the map σ.

5 Concluding remarks

For various logics with relative accessibility relations, we have defined cut-free display calculi by taking advantage of semantical relationships with polymodal logics axiomatised by primitive axioms. The cut-elimination and completeness proofs are totally syntactic and use a backward translation. As we mentioned in the introduction, these logics seem to have non-standard properties when viewed from the traditional polymodal viewpoint (for instance the family of relations in the models is indexed by sets). For these reasons, they have remained on the fringe of mainstream research about logics for reasoning about knowledge. We have shown that these logics are amenable to an analysis in terms of Display Logic, bringing them into the field of "displayable logics". Further general results about **DL** are now applicable to these logics. Among the continuations of this work, the definition of decision procedures based on our calculi is first in the list (following [Dem98b], satisfiability for $T_\triangle$, $B_\triangle$ and $S5_\triangle$ can be shown to be **EXPTIME**-complete). This is a non-trivial task since most display calculi enjoy the *subformula property*, but not the *substructure property*. Decision procedures for particular cut-free display calculi are known: for example, the restriction to *flat* structural rules [Res98], or the restriction to calculi without the contraction rule [Wan97][9]. But this is not the case in general, and is an avenue for further work for which the techniques in [Kra96, Res98] could be helpful.

References

[Aum76] R. Aumann. Agreeing to disagree. *Annals of Statistics*, 4(3):1236–1239, 1976.

[Bal97] Ph. Balbiani. Axiomatization of logics based on Kripke models with relative accessibility relations. In *[Oe97]*, pages 553–578, 1997.

[Bel82] N. Belnap. Display logic. *Journal of Philosophical Logic*, 11:375–417, 1982.

[BO98] Ph. Balbiani and E. Orlowska. A hierarchy of modal logics with relative accessibility relations. *Journal of Applied Non-Classical Logics, special issue in the Memory of George Gargov*, 1998. To appear.

[Dem98a] S. Demri. A class of decidable information logics. *Theoretical Computer Science*, 195(1):33–60, 1998.

[Dem98b] S. Demri. A logic with relative knowledge operators. *Journal of Logic, Language and Information*, 1998. To appear.

[DG98a] J. Dawson and R. Goré. A mechanised proof system for relation algebra using display logic. In *JELIA'98*, pages 264-278. LNAI 1489, Springer-Verlag, 1998.

[9] The decision procedure for PDL^- in [Wan97] can be extended in order to prove that several polymodal logics using constants as modal indices are decidable, when restricted to formulae with modal operators of *possibility force* only. In the case of PDL^-, modal operators of possibility force are logically equivalent to modal operators of *necessity force*.

[DG98b] S. Demri and R. Goré. Cut-free display calculi for nominal tense logics. Technical Report TR-ARP-07-98, Automated Reasoning Project, Australian National University, June 1998.

[DK98] S. Demri and B. Konikowska. Relative similarity logics are decidable: reduction to FO^2 with equality. In *JELIA'98*, pages 279-293. LNAI 1489, Springer-Verlag, 1998.

[FdCO85] L. Fariñas del Cerro and E. Orłowska. DAL - A logic for data analysis. *Theoretical Computer Science*, 36:251–264, 1985.

[FHMV95] R. Fagin, J. Halpern, Y. Moses, and M. Vardi. *Reasoning about Knowledge*. The MIT Press, 1995.

[Gab96] D. Gabbay. *Labelled Deductive Systems*. Oxford University Press, 1996.

[Gar86] G. Gargov. Two completeness theorems in the logic for data analysis. Technical Report 581, ICS, Polish Academy of Sciences, Warsaw, 1986.

[Gor95] R. Goré. Intuitionistic logic redisplayed. Technical Report TR-ARP-1-95, ARP, RSISE, Australian National University, January 1995, `http://arp.anu.edu.au/~rpg`.

[Gor96] R. Goré. On the completeness of classical modal display logic. In H. Wansing, editor, *Proof theory of modal logic*, pages 137–140. Kluwer Academic Publisher, 1996.

[Gor97] R Goré. Cut-free display calculi for relation algebras. In D. van Dalen and M. Bezem, editors, *CSL96*, pages 198–210. Springer-Verlag, LNCS 1258, 1997.

[Gor98] R. Goré. Substructural logics on display. *Logic Journal of the IGPL*, 6(3):451–504, 1998.

[GP92] V. Goranko and S. Passy. Using the universal modality: gains and questions. *Journal of Logic and Computation*, 2(1):5–30, 1992.

[GT75] R. Goldblatt and S. Thomason. Axiomatic classes in propositional modal logic. In J. Crossley, editor, *Algebra and Logic*, pages 163–173. Springer-Verlag, Lecture Notes in Mathematics 450, 1975.

[Hal87] J. Halpern. Using reasoning about knowledge to analyze distributed systems. *Ann. Rev. Comput. Sci.*, 2:37–68, 1987.

[Hem96] E. Hemaspaandra. The price of universality. *Notre Dame Journal of Formal Logic*, 37(2):173–203, 1996.

[Hin62] J. Hintikka. *Knowledge and Belief.* Cornell University Press, 1962.

[Hum95] L. Humberstone. The logic of non-contingency. *Notre Dame Journal of Formal Logic*, 36(2):214–229, 1995.

[Kon97a] B. Konikowska. A logic for reasoning about relative similarity. *Studia Logica*, 58(1):185–226, 1997.

[Kon97b] B. Konikowska. A logic for reasoning about similarity. In *[Oe97]*, pages 462–491, 1997.

[Kra96] M. Kracht. Power and weakness of the modal display calculus. In H. Wansing, editor, *Proof theory of modal logic*, pages 93–121. Kluwer Academic Publisher, 1996.

[Kuh95] S. Kuhn. Minimal logic of non-contingency. *Notre Dame Journal of Formal Logic*, 36(2):230–234, 1995.

[Lem65] E. Lemmon. *Beginning Logic.* Chapman and Hall, 1965.

[Min88] G. Mints. Gentzen-type and resolution rules part I: propositional logic. In P. Martin-Löf and G. Mints, editors, *International Conference on Computer Logic, Tallinn*, pages 198–231. Springer Verlag, LNCS 417, 1988.

[Min97] G. Mints. Indexed systems of sequents and cut-elimination. *Journal of Philosophical Logic*, 26(6):671–696, 1997.

[MR66] H. Montgomery and R. Routley. Contingency and non-contingency bases for normal modal logics. *Logique et Analyse*, 9:318–328, 1966.

[Oe97] E. Orlowska (ed.). *Incomplete Information: Rough Set Analysis.* Studies in Fuzziness and Soft Computing. Physica-Verlag, Heidelberg, 1997.

[Orło88] E. Orłowska. Relational interpretation of modal logics. In H. Andréka, D. Monk, and I. Németi, editors, *Algebraic logic.*

Colloquia Mathematica Societatis Janos Bolyai 54, pages 443–471. North Holland, 1988.

[Orło89] E. Orłowska. Logic for reasoning about knowledge. *Zeitschr. f. Math. Logik und Grundlagen d. Math.*, 35:559–568, 1989.

[Orło91] E. Orłowska. Relational proof systems for some AI logics. In Ph. Jorrand and J. Kelemen, editors, *FAIR'91*, pages 33–47. Springer-Verlag, LNAI 535, 1991.

[Orło92] E. Orłowska. Relational proof system for relevant logics. *The Journal of Symbolic Logic*, 57(4):1425–1440, December 1992.

[Paw81] Z. Pawlak. Information systems theoretical foundations. *Information Systems*, 6(3):205–218, 1981.

[Paw82] Z. Pawlak. Rough sets. *International Journal of Information and Computer Sciences*, 11:341–356, 1982.

[PT91] S. Passy and T. Tinchev. An essay in combinatory dynamic logic. *Information and Computation*, 93:263–332, 1991.

[Res98] G. Restall. Displaying and deciding substructural logics 1: logics with contraposition. *Journal of Philosophical Logic*, 27(2):179–216, 1998.

[Sah75] H. Sahlqvist. Completeness and correspondence in the first and second order semantics for modal logics. In S. Kanger, editor, *3rd Scandinavian Logic Symposium*, pages 110–143. North Holland, 1975.

[Vak91] D. Vakarelov. Modal logics for knowledge representation systems. *Theoretical Computer Science*, 90:433–456, 1991.

[Wan94] H. Wansing. Sequent calculi for normal modal propositional logics. *Journal of Logic and Computation*, 4(2):125–142, April 1994.

[Wan97] H. Wansing. Proof-theoretic aspects of intensional and non-classical logics, December 1997. Habilitation Thesis.

[Wan98] H. Wansing. Translation of hypersequents into display sequents. *Logic Journal of the IGPL*, 6(5):719–733, 1998.