Applied Categorical Structures **9:** 505–516, 2001.

Another Approach to Topological Descent Theory

MANUELA SOBRAL
Departamento de Matemática, Universidade de Coimbra, 3000 Coimbra, Portugal

(Received: 8 May 1998; accepted: 20 November 1999)

To Horst Herrlich, on his sixtieth birthday

Abstract. In the category $\mathcal{T}op$ of topological spaces and continuous functions, we prove that surjective maps which are descent morphisms with respect to the class $\mathbb{E}$ of continuous bijections are exactly the descent morphisms, providing a new characterization of the latter in terms of subfibrations $\mathbb{E}(X)$ of the basic fibration given by $\mathcal{T}op/X$ which are, essentially, complete lattices. Also effective descent morphisms are characterized in terms of effective morphisms with respect to continuous bijections. For classes $\mathbb{E}$ satisfying suitable conditions, we show that the class of effective descent morphisms coincides with the one of effective $\mathbb{E}$-descent morphisms.

Mathematics Subject Classifications (2000): 54B30, 18A30, 18C20, 18A40, 54C10.

Key words: descent data, (effective) descent map, monad, monadic functor, universal regular epimorphism, (effective) étale-descent.

1. Introduction

Let $\mathcal{U}$, $\mathcal{R}$, and $\mathcal{E}$ denote the classes of universal regular epimorphisms, regular epimorphisms and effective descent morphisms, respectively.

In $\mathcal{T}op$, descent morphisms are exactly the universal regular epimorphisms ([7], 2.2) and one has the following inclusions $\mathcal{E} \subseteq \mathcal{U} \subseteq \mathcal{R}$.

It is well-known that the second inclusion is strict. In [10], J. Reiterman and W. Tholen gave a filter-theoretic characterization of effective descent maps as well as an example to show that $\mathcal{E}$ is properly contained $\mathcal{U}$. Aiming to understand better the first we looked for an easier example of a descent map which is not effective for descent.

Using a criterion presented in [11], we give a very simple example involving bijective bundles over finite and quite small spaces.

The subfibration given by the bijective maps over some space X of the basic fibration given by $\mathcal{T}op/X$ is essentially a complete lattice: it is equivalent to a small complete category with at most one morphism between each pair of objects. Such categories are called thin in [1].

In this context, these are relevant subcategories. Indeed, surjections which are descent morphisms with respect to bijective maps (bijective-descent) are exactly the descent morphisms. Also effective descent maps can be characterized in terms of effective maps for bijective-descent.

Furthermore, for regular epimorphisms $p : E \to B$, bijective bundles occur in a natural way in the category $Des(p)$ of bundles over E equipped with descent data and morphisms compatible with it: for each object $(C, \gamma; \xi)$, the morphism γ from $(C, \gamma; \xi)$ to the terminal object $(E, 1_E; p_1)$ has a (bijective, $\mathcal{M}$)-factorization, which plays an important role here. It is the factorization induced in $Des(p)$ by the comparison adjunction, as defined in Theorem 3.3 of [3].

A closer look at the me meaning of descent data suggests a formulation of effective-global descent in terms of effective descent with respect to surjective maps (surjective-descent). We prove that, not only for surjective maps but also for classes $\mathbb{E}$ containing these maps and satisfying suitable conditions, the (effective) descent morphism are exactly the (effective) $\mathbb{E}$-descent maps.

2. Notation and Definitions

For a continuous map $p : E \to B$, let $\mathbb{T} = (T, \nu, \mu)$ be the monad induced in $\mathcal{T}op/E$ by the adjunction

$$p! \dashv p^* : \mathcal{T}op/B \to \mathcal{T}op/E,$$

where p^* and $p!$ are defined by pulling back along p and by composition on the left with p, respectively.

Descent data for an object (C, γ), with respect to p, is given by a $\mathbb{T}$-structure map

$$\xi : (E \times_B C, \pi_1) \to (C, \gamma),$$

where $(E \times_B C, \pi_1, \pi_2)$ is the pullback of $(p, p\gamma)$. Indeed, the category $Des(p)$ of bundles over the space E equipped with descent data and maps compatible with it is, up to isomorphism, the Eilenberg–Moore category $(\mathcal{T}op/E)^{\mathbb{T}}$ of $\mathbb{T}$-algebras ([2] and Beck (unpublished)).

If $\mathbb{E}$ is a class of morphisms in $\mathcal{T}op$ which is stable under pullback along p, the restriction of p^* to the full subcategory of $\mathcal{T}op/B$ with objects all $\mathbb{E}$-bundles over the space B, which we denote by $\mathbb{E}(B)$, is a functor $p^* : \mathbb{E}(B) \to \mathbb{E}(E)$. In the commutative diagram

$$\begin{tikzcd}
\mathcal{T}op/B \arrow[rr, "\Phi^p"] \arrow[dr] & & Des(p) \cong (\mathcal{T}op/E)^{\mathbb{T}} \arrow[dl] \\
& \mathcal{T}op/E & \\
\mathbb{E}(B) \arrow[uu] \arrow[rr, "\Phi^p_{\mathbb{E}}"] \arrow[dr, "p^*"'] & & Des_{\mathbb{E}}(p) \arrow[uu] \arrow[dl, "U^p"] \\
& \mathbb{E}(E) \arrow[uu] &
\end{tikzcd}$$

the vertical arrows are full embeddings, Φ^p is the comparison functor and $Des_{\mathbb{E}}(p)$ is the full subcategory of $Des(p)$ with objects all $\mathbb{T}$-algebras $(C, \gamma; \xi)$ such that $\gamma \in \mathbb{E}$.

A map p is $\mathbb{E}$-*descent* if $\Phi^p_{\mathbb{E}}$ is full and faithful and p is *effective* $\mathbb{E}$-*descent* if $\Phi^p_{\mathbb{E}}$ is an equivalence.

In case $\mathbb{E}$ is the class of all continuous maps, the prefix $\mathbb{E}$ is dropped. However, for emphasis, we sometimes use the terminology of [8] and speak of (effective) global-descent. We also speak about open-descent, surjective-descent and bijective-descent when $\mathbb{E}$ is the class of open embeddings, surjective and bijective maps, respectively.

3. Effective Descent versus Descent

In a category $\mathcal{X}$ with pullbacks, a *universal regular epimorphism* is a morphism whose pullback along any morphism is a regular epimorphism.

For an $\mathcal{X}$-morphism $p : E \to B$, the functor Φ^p is full and faithful if and only if the counit of the adjunction

$$p! \dashv p^* : \mathcal{X}/B \to \mathcal{X}/E,$$

is pointwise a regular epimorphism, by a result due to Beck (see, e.g., [5], 3.3, Theorem 9).

Since regular epimorphisms in $\mathcal{X}$ are regular epimorphisms in $\mathcal{X}/B$, for every object B, universal regular epimorphisms are descent morphisms in all categories with pullbacks as observed in [7], Proposition 2.2.

Conversely, pullbacks of a descent morphism $p : E \to B$ along any morphism, being the coequalizers of their kernel pairs in $\mathcal{X}/B$, are the coequalizers of the corresponding pairs in $\mathcal{X}$, provided they exist. In this case, the classes of universal regular epimorphisms and of descent morphisms coincide.

If, furthermore, X has a (Reg Epi, Mono)-factorization of morphisms then p is a descent morphism if and only if p^* reflects isomorphisms as it follows from the proof of Theorem 1.1 of [7].

The universal regular epimorphisms in $\mathcal{T}op$ where characterized by Day and Kelly in [4]. They are the morphisms $p : E \to B$ such that, for each $b \in B$ and directed open cover $\mathcal{D}$ of $p^{-1}(b)$, $p(V)$ is a neighbourhood of b, for some $V \in \mathcal{D}$.

For a morphism $p : E \to B$ in $\mathcal{T}op$ and $(C, \gamma; \xi) \in Des(p)$, let $q = coeq(\pi_2, \xi)$ and δ be the unique morphism such that $\delta \cdot q = p \cdot \gamma$. Then $(Q, \delta) = \Psi^p(C, \gamma; \xi)$, for the left adjoint Ψ^p to $\Phi^p : \mathcal{X}/B \to Des(p)$.

The diagram

$$\begin{tikzcd} E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] & C \arrow[r, "q"] \arrow[d, "\gamma"] & Q \arrow[d, "\delta"] \\ & E \arrow[r, "p"'] & B \end{tikzcd}$$

will be called *a descent situation defining Q*.

We recall that, for the comparison adjunction $\Psi^p \dashv \Phi^p(\alpha, \beta)$, the unit and counit are defined by

$$\alpha_{(C,\gamma;\xi)} = \langle \gamma, q \rangle \quad \text{and} \quad \beta_{(A,f)} \cdot g = \pi_2,$$

for π_2 the pullback of p along f and g the coequalizer of its kernel pair. Furthermore, they are pointwise bijective maps if p is surjective.

The following criterion will be our main tool in the sequel.

THEOREM 3.1 ([11], 2.8). *In $\mathcal{T}op$, p is an effective descent morphism if and only if it is a universal regular epimorphism and, for every descent situation defining Q, the square is a pullback.*

Let $\mathbb{T}$ be the monad defined in the introduction. For a $\mathbb{T}$-algebra $(C, \gamma; \xi)$, we have that

$$\gamma \cdot \xi = \pi_1, \quad \text{because } \xi \text{ is a morphism of } \mathcal{T}op/E,$$
$$\xi \cdot \eta = 1 \quad \text{and} \quad \xi \cdot 1 \times_B \xi = \xi \cdot 1 \times_B \pi_2,$$

because ξ is a $\mathbb{T}$-structure map.

From the equality $\gamma \cdot \xi = \pi_1$ and the fact that (π_2, ξ) is an effective equivalence relation (see, e.g., [11], 2.2 and 2.4), it is easy to prove the following:

PROPOSITION 3.2. *For a morphism $p : E \to B$ in $\mathcal{T}op$ and a descent situation as above, the following holds*:

(i) *If $p^{-1}(b) \cap \gamma(C) \neq \emptyset$ then $p^{-1}(b) \subseteq \gamma(C)$;*
(ii) *For $c, c' \in C$, $q(c) = q(c')$ if and only if $\xi(\gamma(c), c') = c$ or, equivalently, $\xi(\gamma(c'), c) = c'$.*

From (i), we conclude that, for $(C, \gamma; \xi) \in Des(p)$, the subspace $\gamma(C)$ of E is the pullback along p of a subspace of B.

The second item tells us how to define the coequalizer of the pair (π_2, ξ).

We present now a very simple example of a non-effective descent map.

EXAMPLE 3.3. Let E be the set $\{e_{11}, e_{12}, e_{21}, e_{22}, e_{31}, e_{32}\}$ with the topology generated by the subsets $U_1 = \{e_{11}, e_{21}\}$ and $U_2 = \{e_{22}, e_{31}\}$ and B be the set $\{b_1, b_2, b_3\}$ with the indiscrete topology.

The function $p : E \to B$ defined by $p(e_{ir}) = b_i$ is a universal regular epimorphism but it is not effective for global descent. To prove the latter, consider $(C, \gamma; \xi) \in Des(p)$, where C has the same underlying set as E and the topology generated by the topology of E and the open set $\{e_{21}\}$. Then we obtain a bundle (C, γ), with $\gamma(x) = x$, equipped with descent data in the only possible way: the $\mathbb{T}$-structure map is the function $\xi : E \times_B C \to C$ defined by $\xi(x, y) = x$. Indeed, the function ξ satisfies the equalities

$$\gamma \cdot \xi = \pi_1, \quad \xi \cdot \eta = 1 \quad \text{and} \quad \xi \cdot 1 \times_B \xi = \xi \cdot 1 \times_b \pi_2,$$

and is continuous because

$$\xi^{-1}(e_{21}) = U_1 \times_B (U_2 \cup \{e_{21}\}).$$

Hence we have a descent situation defining B

$$\begin{tikzcd} E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] & C \arrow[r, "q"] \arrow[d, "\gamma"] & B \arrow[d, "id"] \\ & E \arrow[r, "p"'] & B \end{tikzcd}$$

in which the square is not a pullback.

We point out that, using results in [6] and the fact that effective descent morphisms are stable under pullback ([11], 3.1), we conclude that there exist non-effective descent maps $p : E \to B$ for every space B which has a two point indiscrete space.

4. Characterizations of Effective Descent Maps

Let $\mathbb{E} \subseteq \mathbb{E}'$ be classes of continuous functions stable under pullback along a morphism p and closed under composition with isomorphisms.

PROPOSITION 4.1 ([8], 2.6). *The map p is $\mathbb{E}$-descent if it is $\mathbb{E}'$-descent. An effective $\mathbb{E}'$-descent map p is effective for $\mathbb{E}$-descent if and only if for each pullback diagram*

$$\begin{tikzcd} E \times_B D \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & D \arrow[d, "\delta"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

if $\pi_1 \in \mathbb{E}$ and $\delta \in \mathbb{E}'$ then $\delta \in \mathbb{E}$.

When this transferability condition holds for $\mathbb{E}'$ the class of all morphisms in $\mathcal{T}op$, we say that $\mathbb{E}$ *descends along p*.

In this case,

- the $\mathbb{E}$-descent maps are exactly the $\mathbb{E}$-*universal regular epimorphisms* (as it follows from Proposition 1.6 in [8]), that is the morphisms whose pullbacks along $\mathbb{E}$-morphisms are regular epimorphisms;
- effective descent morphisms are effective $\mathbb{E}$-descent maps.

THEOREM 4.2. *A map in $\mathcal{T}op$ is global-descent if and only if it is a surjection and a bijective-descent map.*

Proof. One just need to prove that each surjective map $p : E \to B$ that is a bijective-descent morphism is a descent map. Given a directed open cover $\mathcal{D}$ of $p^{-1}(b)$, for some $b \in B$, consider a space B' with the same underlying set as B and the coarsest topology containing the open sets of the space B and the sets of the form

$$\{b\} \cup B \setminus p(V)$$

for all $V \in \mathcal{D}$, such that $V \cap p^{-1}(b) \neq \emptyset$.

The pullback $p' : E' \to B'$ of p along the map $i : B' \to B$, with $i(x) = x$, is a quotient. Indeed, bijective-descent maps are bijective-regular epimorphisms because bijective maps descend along surjections and p is surjective. Consequently, they are quotients because identities are bijective maps.

The set $p^{-1}(b)$ is open in E' because each $x \in p^{-1}(b)$ belongs to

$$V \cap p^{-1}(\{b\} \cup B \setminus p(V)) \subseteq p^{-1}(b),$$

for some $V \in \mathcal{D}$, which is an open subset of E'. Hence $\{b\}$ is open in B' and so

$$\{b\} = U \cap (\{b\} \cup B \setminus p(V))$$

for some U open in E and $V \in \mathcal{D}$. Therefore, $b \in U \subseteq p(V)$, i.e. $b \in int(p(V))$. $\square$

We remark that, since only the neighbourhoods of b in B' are relevant in the proof 4.2, the space defined in the proof of Theorem 1 in [4] can also be used to prove our claim.

From 4.1 and 4.2 one immediatly obtains the result below.

COROLLARY 4.3. *For classes $\mathbb{E}$ stable under pullback and containing continuous bijections, the surjective $\mathbb{E}$-descent maps are exactly the descent maps.*

For $\mathbb{E}$ the class of bijective maps, the subfibration given by $\mathbb{E}(X)$ of the basic fibration given by $\mathcal{T}op/X$ is equivalent to a small category, because $\mathcal{T}op$ is well-powered. Since there is at most one morphism between any two objects, $\mathbb{E}(X)$ is, up to equivalence, the complete lattice of all topologies on spaces X' for which $1 : X' \to X$ is continuous.

Since the functor $p^* : \mathbb{E}(B) \to \mathbb{E}(E)$ preserves all meets it has a left adjoint, which, for surjective p, is defined on objects by

$$L(C, \gamma) = (D, \delta),$$

where $\delta \cdot q$ is the (Reg Epi, Mono)-factorization of $p \cdot \gamma$. The counit $\bar{\epsilon}$ of the adjunction has as components the maps induced by the diagonal property of the factorization: for each $(A, f) \in \mathbb{E}(B)$ the counit $\bar{\epsilon}_{(A,f)}$ is the unique map such that

$\bar{\epsilon}_{(A,f)} \cdot q = \pi_2$ and $f \cdot \bar{\epsilon}_{(A,f)} = \delta$, for $(D, \delta) = L(E \times_B A, \pi_1)$. Then, since p^* restricted to $\mathbb{E}(B)$ is obviously faithful, we have that

$$p \text{ is a descent morphism } \Leftrightarrow p^* : \mathbb{E}(B) \to \mathbb{E}(E) \text{ is full}$$

and this occurs if and only if $\mathbb{E}(B)$ is, up to isomorphism, a sublattice of $\mathbb{E}(E)$.

From Theorem 1.1 in [7], already referred to at the beginning of Section 3, and the above equivalence we conclude that

$$p^* : \mathcal{T}op/B \to \mathcal{T}op/E \text{ reflects isomorphisms } \Leftrightarrow p^* : \mathbb{E}(B) \to \mathbb{E}(E)$$
$$\text{is a full functor.}$$

Also the effective descent maps can be characterized in terms of surjective maps which are effective for bijective-descent, as we show next.

PROPOSITION 4.4. *A surjective map is effective for global-descent if and only if its pullback along an arbitrary morphism is effective for bijective-descent.*

Proof. Let $\mathbb{E}$ denote the class of bijective maps. Then $\mathbb{E}$ descends along surjections.

Since effective descent morphisms are effective for $\mathbb{E}$-descent, the necessity of the condition follows from the fact that pullbacks of effective descent morphisms are effective descent morphisms.

Conversely, if the pullback of p along an arbitrary morphism is an $\mathbb{E}$-descent morphism, then p itself is a descent map.

Let α denote the component of the unit of the $\Psi^p \dashv \Phi^p$ at $(C, \gamma; \xi) \in Des(p)$. It is easy to check that $(C, \alpha; \zeta) \in Des_{\mathbb{E}}(\pi_2)$, for $\zeta = \xi f$, where $f : (E \times_B Q) \times_Q C \to E \times_B C$ is the canonical isomorphism as shown in the diagram

$$\begin{tikzcd}
(E \times_B Q) \times_Q C \arrow[d, "f"] \arrow[dr, shift left] \arrow[dr, shift right] & & \\
E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] & C \arrow[r, "q"] \arrow[d, "\alpha"] & Q \arrow[d, "1"] \\
& E \times_B Q \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Q \arrow[d, "\delta"] \\
& E \arrow[r, "p"] & B
\end{tikzcd}$$

Since α is a bijective map and $\pi_2 : E \times_B Q \to Q$ is effective for $\mathbb{E}$-descent, there exists some $(D, \delta) \in \mathbb{E}(Q)$ such that $\Phi_{\mathbb{E}}^{\pi_2}(D, \delta)$ is, up to isomorphism, $(C, \alpha; \zeta)$. But the pullback of π_2 along δ is the coequalizer of (π_2, ξ), so δ is an isomorphism. Consequently, α is an isomorphism and so p is effective for global-descent. $\square$

With very little change, the same proof works if we consider surjective maps instead of bijective maps. In this case we can consider just pullbacks along subspace embeddings.

PROPOSITION 4.5. *A map is effective for global-descent morphism if and only if its pullbacks along subspace embeddings are effective for surjective-descent.*

Proof. Let now $\mathbb{E}$ denote the class of surjective maps. Also in this case $\mathbb{E}$ descends along surjective maps.

If p is an effective descent map, its pullback along any morphism, being an effective descent map, is effective for surjective-descent.

Now, for $(C, \gamma; \xi) \in Des(p)$ we consider the diagram

$$
\begin{tikzcd}
\gamma(C) \times_A C \arrow[d, "f"] \arrow[dr, shift left] \arrow[dr, shift right] & & \\
E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] & C \arrow[r, "q"] \arrow[d, "g"] & Q \arrow[d, "h"] \\
& \gamma(C) \arrow[r, "p'"] \arrow[d, "m"] & A \arrow[d, "n"] \\
& E \arrow[r, "p"] & B
\end{tikzcd}
$$

where $m \cdot g$ is the (Reg Epi, Mono)-factorization of γ and A is the subspace of B such that $p^*(A, n) = (\gamma(C), m)$, whose existence follows from 3.2(i). If p' is effective for surjective-descent, as in the proof of 4.4, we conclude that the upper square is a pullback. Therefore, also the outer rectangle is a pullback and so p is effective for descent. □

For arbitrary categories with pullbacks and classes $\mathbb{E}$ satisfying suitable conditions, effective $\mathbb{E}$-descent morphisms are stable under pullback along $\mathbb{E}$-morphisms. Indeed, if $\mathbb{E}$ contains Iso$(\mathcal{X})$, is closed under composition and weakly left cancellable (i.e. $gf, g \in \mathbb{E} \Rightarrow f \in \mathbb{E}$), then the class of effective $\mathbb{E}$-descent morphisms is stable under pullback along $\mathbb{E}$-morphisms.

This is Theorem 2.4 in [12], where, though clear in the proof that preceeds it, the restriction to pullbacks along morphisms in $\mathbb{E}$, instead of along arbitrary morphisms, is not stated.

We are going to show that effective surjective-descent maps are stable under pullback along arbitrary maps and so, by 4.5, that effective global-descent maps are exactly the maps that are effective for $\mathbb{E}$-descent, for the class $\mathbb{E}$ of surjective maps.

First we prove an auxiliary result.

LEMMA 4.6. *For the pullback* $(E \times_B A, \pi_1, \pi_2)$ *of the pair* (p, f)*, let* $(C, \gamma; \xi)$ *be an object of* $Des(\pi_2)$ *and* D *be the complement of* $\pi_1\gamma(C)$ *in* E*. Then the bundle* $(C \amalg D, \sigma)$*, where* $\sigma : C \amalg D \to E$ *is the map induced by* $\pi_1 \cdot \gamma$ *and the subspace embedding of* D *in* E*, is equipped with descent data with respect to* p*.*

Proof. We define a function

$$\zeta : E \times_B (C \amalg D) \to C \amalg D$$

by

$$\zeta(e, x) = \begin{cases} \xi(e, x) & \text{if } x \in C, \\ e & \text{otherwise} \end{cases}$$

and prove that it is continuous.

Identifying $(E \times_B A) \times_A C$ with $E \times_B C$ and denoting by τ_C the coproduct injection, the following

$$\begin{tikzcd}
E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] \arrow[d, "1\times_B \tau_C"'] & C \arrow[d, "\tau_C"] & \\
E \times_B (C \amalg D) \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\zeta"'] & C \amalg D \arrow[d, "\sigma"] & \\
& E \arrow[r, "p"] & B
\end{tikzcd}$$

is a commutative diagram in $\mathcal{T}op$ as we show next.

The morphism $1 \times_B \tau_C$ is an open embedding, because it is the pullback of the open embedding τ_C along π_2. Also $\zeta \cdot 1 \times_B \tau_C = \tau \cdot \xi$.

For open subsets U of C,

$$\zeta^{-1}(U) = \zeta^{-1}(\tau_C(U)) = 1 \times_B \tau_C(\xi^{-1}(U))$$

which is open in $E \times_B (C \amalg D)$.

By 3.2(i),

$$\zeta^{-1}(D) = D \times_B (C \amalg D) = E \times_B D$$

and for $U = V \cap D$, with V open in E,

$$\zeta^{-1}(U) = U \times_B (C \amalg D) = V \times_B D,$$

which are open sets. Hence ζ is a continuous function.

It is easy to check that ζ is a $\mathbb{T}$-structure map and this completes the proof of the lemma. $\square$

PROPOSITION 4.7. *Effective maps for surjective-descent are pullback stable.*

Proof. With the notation of the previous lemma, let $(C, \gamma; \xi) \in Des_{\mathbb{E}}(\pi_2)$, where $\mathbb{E}$ denotes the class of surjective maps.

Identifying again $(E \times_B A) \times_A C$ with $E \times_B C$, we have that $(C, \gamma'; \xi) \in Des(p)$ for $\gamma' = \pi_1\gamma$. Let $q = coeq(\pi_2, \xi)$ and consider the diagram

$$\begin{tikzcd}
E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] & C \arrow[r, "q"] \arrow[d, "\gamma"] & Q \arrow[d, "h"] \\
& E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & A \arrow[d, "f"] \\
& E \arrow[r, "p"] & B
\end{tikzcd}$$

For $D = E \setminus \pi_1\gamma(C)$, let $\sigma : C \amalg D \to E$ be the morphism induced by γ' and by the subspace embedding of D in E. By 4.6, $(C \amalg D, \sigma; \zeta) \in Des(p)$ for the map ζ C defined there. Furthermore, since σ is surjective, $(C \amalg D, \sigma; \zeta)$ belongs to $Des_{\mathbb{E}}(p)$.

In the diagram

$$\begin{tikzcd}
E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] \arrow[d, "1\times_B\tau_C"'] & C \arrow[r, "q"] \arrow[d, "\tau_C"] & Q \arrow[d, "g"] \\
E \times_B (C \amalg D) \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\zeta"'] & C \amalg D \arrow[r, "q'"] \arrow[d, "\sigma"] & Q' \arrow[d, "\delta'"] \\
& E \arrow[r, "p"] & B
\end{tikzcd}$$

where $q' = coeq(\pi_2, \zeta)$ the bottom square is a pullback, because p is effective for surjective-descent.

For each subset U of Q

$$\tau_C(q^{-1}(U)) = q'^{-1}(g(U))$$

and so g is an open embedding because τ_C is an open embedding and q' is a quotient. Since open embeddings are stable under pullback and weakly left cancellable, also the right-upper square is a pullback.

Now, since $\sigma \cdot \tau_C = \pi_1 \cdot \gamma$ and $\delta' \cdot g = f \cdot h$, the outer rectangle in

$$\begin{tikzcd}
C \arrow[r, "q"] \arrow[d, "\gamma"'] & Q \arrow[d, "h"] \\
E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\
E \arrow[r, "p"] & B
\end{tikzcd}$$

is a pullback. Since the bottom square is a pullback, the same holds for the upper square. Thus, π_2 is effective for surjective-descent as claimed. □

Combining 4.7 and 4.5 we conclude the following:

THEOREM 4.8. *Effective descent morphism in $\mathcal{Top}$ are exactly the maps which are effective for surjective-descent.*

COROLLARY 4.9. *A map is effective for descent if and only if it is effective for $\mathbb{E}$-descent, for each class $\mathbb{E}$ stable under pullback which contains the surjective maps and descends along universal quotients.*

Proof. Under each one of the conditions, the morphism is a universal quotient.

Effective descent morphisms are effective $\mathbb{E}$-descent morphisms, because $\mathbb{E}$ descends along universal quotients, and, by the same reason, effective $\mathbb{E}$-descent morphisms are effective for surjective-descent. Now the conclusion follows from the previous result. □

Acknowledgements

Partial financial support by CMUC/FCT, PRAXIS Projects 2/2.1/MAT/46/94 and PCEX/P/MAT/46/96 is gratefully acknowledged.

I gratefully acknowledge very useful discussions with George Janelidze and Walter Tholen on the subject of this paper. I am also grateful to Maria Cristina Pedicchio for inviting me to visit the University of Trieste, where the conversations with George Janelidze took place. Thanks are due to J. Bénabou for helpful comments, namely for pointing out that the surjectivity condition cannot be avoided in Theorem 4.2.

References

1. Adámek, J., Herrlich, H. and Strecker, G. E.: *Abstract and Concrete Categories*, Wiley, New York, 1990.
2. Bénabou, J. and Roubaud, J.: Monades et descente, *C.R. Acad. Sci.* (1970), 96–98.

3. Cassidy, C., Hébert, M. and Kelly, G. M.: Reflective subcategories, localizations and factorization systems, *J. Austral. Math. Soc. (Series A)* **38** (1985), 287–329.
4. Day, B. J. and Kelly, G. M.: On topological quotient maps preserved by pullbacks or products, *Proc. Cambridge Phil. Soc.* **67** (1970), 553–558.
5. Barr, M. and Wells, C.: *Toposes, Triples and Theories*, Springer, New York, 1985.
6. Janelidze, G. and Sobral, M.: Finite preorders and topological descent (in preparation).
7. Janelidze, G. and Tholen, W.: How algebraic is the change-of-base functor? *Lect. Notes in Math.* **1448**, Springer, Berlin, 1991, pp. 157–173.
8. Janelidze, G. and Tholen, W.: Facets of descent I, *Applied Cat. Struct.* **2** (1994), 245–281.
9. Janelidze, G. and Tholen, W.: Facets of descent II, *Applied Cat. Struct.* **5** (1997), 229–248.
10. Reiterman, J. and Tholen, W.: Effective descent maps of topological spaces, *Topology Appl.* **57** (1994), 53–69.
11. Sobral, M. and Tholen, W.: Effective descent morphisms and effective equivalence relations, *CMS Conference Proceedings Vol. 13*, AMS, Providence, RI, 1992, pp. 421–433.
12. Sobral, M.: Some aspects of topological descent, *Applied Cat. Struct.* **4** (1996), 97–106.