Analysis of a Fork/Join Synchronization Station with Coxian Inputs from Finite Populations

Ananth Krishnamurthy and Rajan Suri

Center for Quick Response Manufacturing, University of Wisconsin-Madison,
1513 University Avenue, Madison, WI 53706-1572.
Telephone: 608-262 0921, Fax: 608-265 4017
Email: akrishna@cae.wisc.edu, suri@engr.wisc.edu.

and

Mary Vernon,

Department of Computer Sciences, University of Wisconsin-Madison,
1210 W Dayton Street, Madison, WI-53706.

Abstract

Fork/join stations are commonly used to model the synchronization constraints in queuing models of computer networks, fabrication/assembly systems and material control strategies for manufacturing systems. In many such applications the fork/join station is fed by two or more inputs from finite populations. This paper presents an exact analysis for the case when the input processes are renewal and the inter-arrival times have two-phase Coxian distributions, allowing us to model a wide range of variability in the input processes. The underlying queue length and departure processes are analyzed to determine performance measures such as throughput, mean queue lengths and distribution of inter-departure times from the fork/join station. The results show that, for certain parameter settings, variability in the arrival processes have a significant impact on the performance of the fork/join station. The model is also used to study the sensitivity of performance measures such as throughput, mean queue lengths, and variability of inter-departure times for a wide class of input processes and customer populations.

1 Introduction

Fork/join stations fed by one or more inputs buffers find applications in queuing models of many computer and manufacturing systems. Queuing networks with fork/join stations have been studied in the context of parallel processing, database concurrency control, and communication protocols (Prabhakar, *et al.* 1998, Baccelli *et al*. 1989, Varki 1999). In queuing models of manufacturing assembly systems, the assembly station is typically modeled using a fork/join station (Harrison 1973, Latouche 1981, Hopp and Simon 1989, Rao and Suri 1994, 2000). Recently fork/join stations have also been used model the synchronization constraints in models of material control strategies for multi-stage manufacturing systems (Buzacott and Shanthikumar 1993, Di Mascolo *et al*. 1996).

In order to develop efficient methods to analyze these networks, researchers have analyzed different aspects of the performance of fork/join stations in *isolation*. These efforts have focused on studying the impact of arrival rates and queue capacities on performance measures such as throughput, synchronization delays, and queue lengths at the different input buffers. However for the sake of analytical tractability, a majority of these efforts assume that the fork/join stations are fed by Poisson inputs (Harrison 1973, Bhat 1986, Lipper and Sengupta 1986, Hopp and Simon 1989, Som *et al.* 1994, Takahashi *et al*. 1996,Varki 1999). Although these results are useful, in many of the applications cited above the input processes are not Poisson. Often these input processes have variability quite different from that of a Poisson process. Recently, some studies have been done for fork/join stations fed by input processes that are more general than the Poisson process. In particular, Takahashi *et al*. (2000a, 2000b) use matrix analytical methods to study a fork/join station fed by phase renewal process and Markovian Arrival Process (MAP) processes. However, their work assumes that the arrivals are an uninterrupted process and if upon arrival the input buffer is full, the arrival to the fork/join station is lost.

When the fork/join station is part of a larger *closed* queuing network, (such as in models of multistage kanban systems or closed multi-level fabrication/assembly systems) then once the contents in the input buffer reach a certain level, the arrival process shuts down temporarily. In this paper, we analyze a fork/join station fed by *i.i.d* input processes from a *finite population*. The computational complexity arising from the use of matrix analytical approaches seems to be a high price to pay for analyzing fork/join stations fed by inputs more general that a Poisson process. Keeping in mind that the fork/join stations might be part of a larger queuing network, we propose an alternative approach that would allow us to analyze fork/join stations for a fairly general class of inputs with reduced computational complexity. We study a fork/join station fed by renewal inputs with two-phase Coxian distribution. The choice of two-phase Coxian inputs allows us to model input processes with a wide range of mean $(0,\infty)$ and squared coefficient of variation, SCV $(0.5,\infty)$ without much added computational complexity. We analyze the queue length process as a continuous time Markov chain and estimate the throughput and queue length distributions. Next we analyze the departure process as a semi-Markov process and derive expressions for the distribution of the inter-departure times.

The remainder of this paper is organized as follows. We begin with a description of our model of the fork/join station in section 2. Section 3 provides a summary of the literature to date on the analysis of fork/join stations. Section 4 describe the specific inputs and outputs for our analysis and provide an overview of our analysis approach. Section 5 presents our analysis of the queue length process while section 6 presents our analysis of the departure process from the fork/join station. Section 7 provide some numerical results and our conclusions are presented in section 8.

2 Model Description

Figure 1 describes our model of the fork/join station. The fork/join station has two input buffers, B_1 and B_2. The operation of the fork/join station is as follows. If a part arriving in input buffer B_1 (B_2) finds input buffer B_2 (B_1) empty, it waits for the corresponding part to arrive in input buffer B_2 (B_1). As soon as there is one part in each queue, one part from each buffer B_1 and B_2 are joined together and exit instantaneously. This event corresponds to a departure from the fork/join station and as a result the contents of both input buffers reduce by 1. The external environment from which parts arrive to input buffer B_1 (B_2) is modeled as a queuing network SN1 (SN2) with a finite population K_1 (K_2). Upon departure from the fork/join station, the part forks into two identical entities that get routed back to the SN1 and SN2 respectively. In SN1 and SN2 these entities are subjected to independent random delays before they revisit input buffers B_1 and B_2 respectively. Consequently, the number of parts in input buffer B_1 (B_2) and queuing network SN1 (SN2) always sum up to K_1 (K_2). Additionally, the arrival process to buffer B_1 (B_2) shuts down when the content in buffer B_1 (B_2) increases to K_1 (K_2).

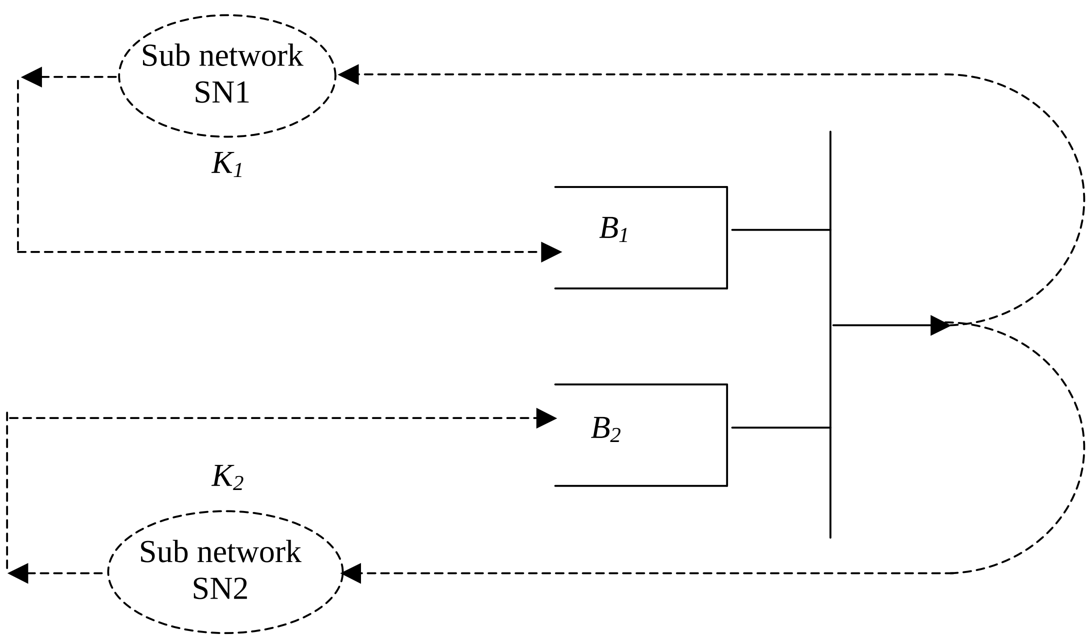

Figure 1. Fork/join station with arrivals from a finite population

Fork/join stations with such characteristics are found in models of multistage kanban systems, closed multi-level fabrication/assembly systems, and communication systems. Models of such systems usually consist of closed queuing networks in which the fork/join stations model the synchronization constraints imposed on the traffic processes in the networks. Since the queuing network SN1 and SN2 from which parts arrive to input buffers B_1 and B_2 can have arbitrary topology, the arrival processes to the fork/join stations can have general characteristics. However, analysis of fork/join stations for general arrival processes can be quite complicated. To simplify analysis previous researchers have assumed Poisson inputs. As a step towards extending these results, we will assume that the arrival processes are independent renewal processes and that the inter-arrival times to the input queues are independent and identically distributed (*i.i.d*) having distinct two-phase Coxian distributions. For such a fork/join station we obtain exact analytical expressions for performance characteristics such as throughput (Section 5), queue length distributions at the input buffers B_1 and B_2 (Section 5), and marginal distribution of inter-departure times (Section 6).

3 Brief Review of Relevant Literature

Fork/join stations fed by one or more inputs have been commonly used to model synchronization constraints in queuing models of many computer and manufacturing systems. Harrison (1973) analyzed a fork/join station fed by renewal input streams and showed that when there was no capacity limit for the buffers, the fork/join station was unstable. However, if the buffers were limited, then the fork/join station was stable. Bhat (1986) analyzed a fork/join station with limited buffers and Poisson inputs and derived expressions for the queue length distributions at the input buffers. Kashyap (1965) studied a kitting process fed by input queues of components. The kitting process was modeled as a double-ended queue and expressions for the waiting time distributions were derived for the case of Poisson inputs. Som *et. al* (1994) and Takahashi *et al*. (1996) studied the departure process from a fork/join station fed by Poisson inputs and finite buffers and derived expressions for the marginal distribution of the inter-departure times. However, in many applications, the input processes to the fork/join stations are not Poisson. Very few studies have been done for fork/join stations fed by input processes that are more general than the Poisson process. In particular, Takahashi *et al*. (2000, 2001) use matrix analytical methods to study a fork/join station fed by phase renewal process and Markovian Arrival Process (MAP) processes.

Our analysis of fork/join stations compares to these previous studies in the following ways. The works of Som *et al.* (1994), and Takahashi *et al.* (1996) assume Poisson inputs while Takahashi *et al.* (2000a, 2000b) assume that arrivals are an uninterrupted process and that if the finite input buffers are full, the arrivals to the fork/join station are lost. When the fork/join station is a part of a larger closed queuing network, then once the contents in the input buffer reach a certain level, the arrival process shuts down temporarily. Varki (1999) assumes a finite customer population in the fork/join queuing network but restricts all service times to be exponentially distributed. In this paper, we analyze a fork/join station fed by *i.i.d* input processes from a *finite population* and having two-phase Coxian distributions. The choice of two-phase Coxian distributions allows us to model a wide range of renewal input processes, namely, input processes with mean inter-renewal times ranging over $(0,\infty)$ and with squared coefficient of variation (SCV) in the range of

$[0.5, \infty)$. This range covers the values typically expected of traffic processes in many practical systems.

4 Outline of our Analysis

4.1 Inputs

We assume that the fork/join station is fed by two renewal input processes. As shown in Figure 2, the distribution of inter-arrival times for the input buffers B_j, $j = 1,2$ is modeled by a two-phase Coxian distribution defined by three parameters, μ_{j1}, μ_{j2} and p_j, with cumulative distribution function

$$G_j(t) = 1 - C_{j1} e^{-\mu_{j1} t} - C_{j2} e^{-\mu_{j2} t} \qquad \text{for } t \geq 0 \text{ and } j = 1,2 \tag{1}$$

where

$$C_{j1} = \frac{\mu_{j1}(1 - p_j) - \mu_{j2}}{\mu_{j1} - \mu_{j2}} \text{ and } C_{j2} = 1 - C_{j1}, \text{ with } \mu_{j1} \neq \mu_{j2} \tag{2}$$

Note that this implies that the arrival rate to buffer B_j is $\lambda_j = \dfrac{\mu_{j1}\mu_{j2}}{p_j \mu_{j1} + \mu_{j2}}$ and the SCV of the inter-arrival times at buffer B_j is $c_j^2 = 1 - \dfrac{2 p_j \mu_{j1}\left(\mu_{j2} - \mu_{j1}(1 - p_j)\right)}{\left(p_j \mu_{j1} + \mu_{j2}\right)^2}$ for $j = 1,2$ (Altiok 1996). However, the arrivals to buffer B_1 (B_2) shut down once it is has K_1 (K_2) units and resumes when the respective buffer drops below the limit.

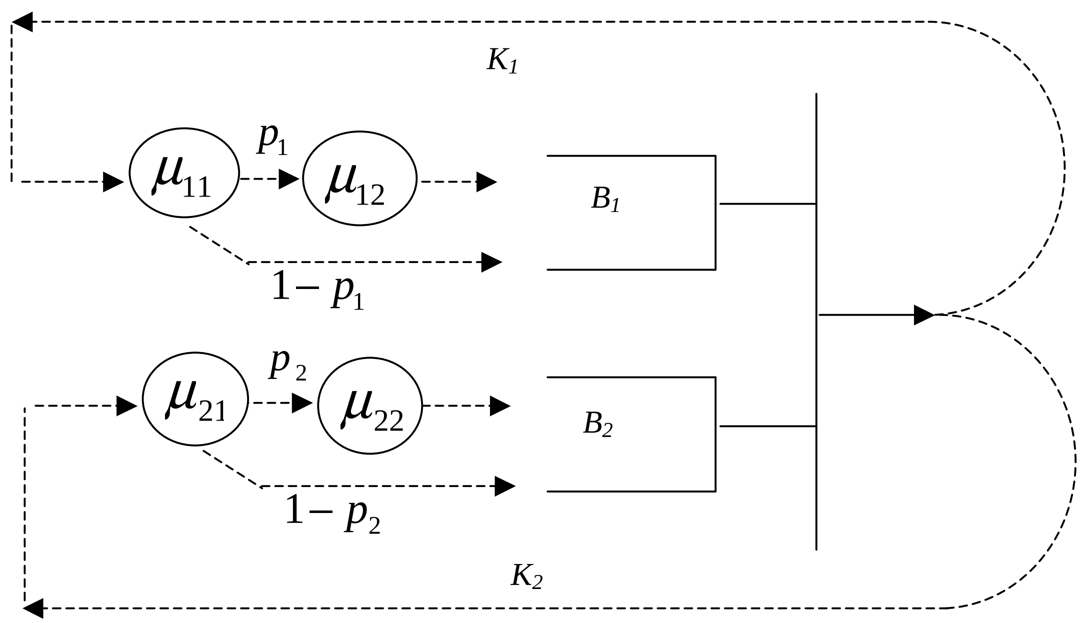

Figure 2. Fork/join station with two-phase Coxian arrivals from a finite population

4.2 Outputs

We compute the throughput χ_D and the mean queue lengths $\overline{L}_1$ and $\overline{L}_2$ at the input buffers B_1 and B_2. In addition, we compute the marginal distribution $G_D(t)$ of the inter-departure times from the fork/join station. Correspondingly, our analysis of the fork/join station consists of two parts: (1) the analysis of the *queue length process* and (2) the analysis of the *departure process*. We briefly summarize the analysis approach in the following paragraphs.

4.3 Overview of Analysis

Analysis of the queue length process: To analyze the queue length process, we first define the state space for the queue length process. Since the arrival processes are phase renewal processes with the underlying distributions having two exponential phases, the queue length process is studied as a simple continuous time Markov process. We solve the continuous time Markov chain to obtain the steady state probability distributions of the queue lengths at buffers B_1 and B_2. From these we obtain different performance measures such as throughput and the mean queue lengths, $\overline{L}_1$ and $\overline{L}_2$. The details are described in Section 5.

Analysis of the departure process: We note that the output process is a Markov renewal process. In analyzing this Markov renewal process, we make use of the special structure of the Markov process embedded at departure instants to obtain the transition probability matrix and stationary probabilities of the Markov chain embedded at departure instants. Using the stationary probabilities, we obtain the marginal distribution of inter-departure times from the fork/join station. The details are given in Section 6.

5 Analysis of the Queue Length Process

In this section, we study the queue length process as a continuous time Markov process. Table 1 summarizes the notation used in the analysis.

Let $N_1(t)$ and $N_2(t)$ denote the number of units in buffers B_1 and B_2 respectively at time t. From the operational characteristics of the fork/join station we note that, it is not possible for both the buffers B_1 and B_2 to be non-empty for any finite time. Departures occur instantaneously from the fork/join station whenever both buffers are non-empty. Therefore, the number of units in both buffers at time t can be described uniquely using the one-dimensional random variable $N(t) = N_1(t) - N_2(t)$. $N(t)$ takes on values $-K_2, \ldots -1, 0, 1, \ldots, K_1$. For instance, $N(t) = -K_2$ implies that the buffer B_1 is empty and the buffer B_2 has K_2 units. If both input buffers are empty, $N(t)$ =0.

Table 1. Notation used for analysis of queue length process

Symbol	*Description*
μ_{j1}, μ_{j2} and p_j	Parameters of the two-phase Coxian distribution for the arrival process at buffer B_j, j=1,2
K_j	Size of the finite population from which arrivals occur to input buffer B_j, j=1,2
λ_j	Rate of arrivals to buffer B_j, when it is not shut down, j=1,2
c_j^2	SCV of inter-arrival times at buffer B_j, when it is not shut down, j=1,2
$N_j(t)$	Number of units in buffer B_j at time t, j=1,2
$J_j(t)$	Phase of unit arriving to buffer B_j time t, j=1,2
$(N(t), J_1(t), J_2(t))$	State of the fork/join station at time t, $N(t) = N_1(t) - N_2(t)$
$\overline{L}_j$	Average queue length at buffer B_j, j=1,2
χ_D	Throughput of the fork/join station

To describe the state of the system at any time t, we need to consider both, the number of units present in the input buffers as well as the phases of the arriving units. Then at time t, each buffer can be in one of three distinct states as defined below:

$J_1(t)$ [$J_2(t)$] = 1 if the arriving unit to buffer B_1 [B_2] is in phase 1

= 2 if the arriving unit to buffer B_1 [B_2] is in phase 2

= 0 if the buffer B_1 [B_2] is full i.e. has K_1 [K_2] units and the arrival process to the buffer has shut down.

(3)

The state of the system is then described as the process $(N, J_1, J_2) = \{(N(t), J_1(t), J_2(t)), t \ge 0\}$. Clearly, (N, J_1, J_2) is a continuous time Markov chain and describes the stochastic behavior of the fork/join station. The state space of state of (N, J_1, J_2) is given by

$$S_Q = \{(n, j_1, j_2) : n = -K_2 + 1, \ldots -1, 0, 1, \ldots, K_1 - 1;\ j_1 = 1,2;\ j_2 = 1,2\}$$
$$\cup \{(-K_2, 1, 0), (-K_2, 2, 0), (K_1, 0, 1), (K_1, 0, 2)\} \qquad (4)$$

Note that the Markov chain has $4(K_1+K_2)$ states. The state transition rates for the continuous time Markov chain representing the queue length process is illustrated in Figure 3. These transition rates are easily derived from the parameters of the two-phase Coxian distribution for the arrival processes, μ_{j1}, μ_{j2} and p_j, j=1,2.

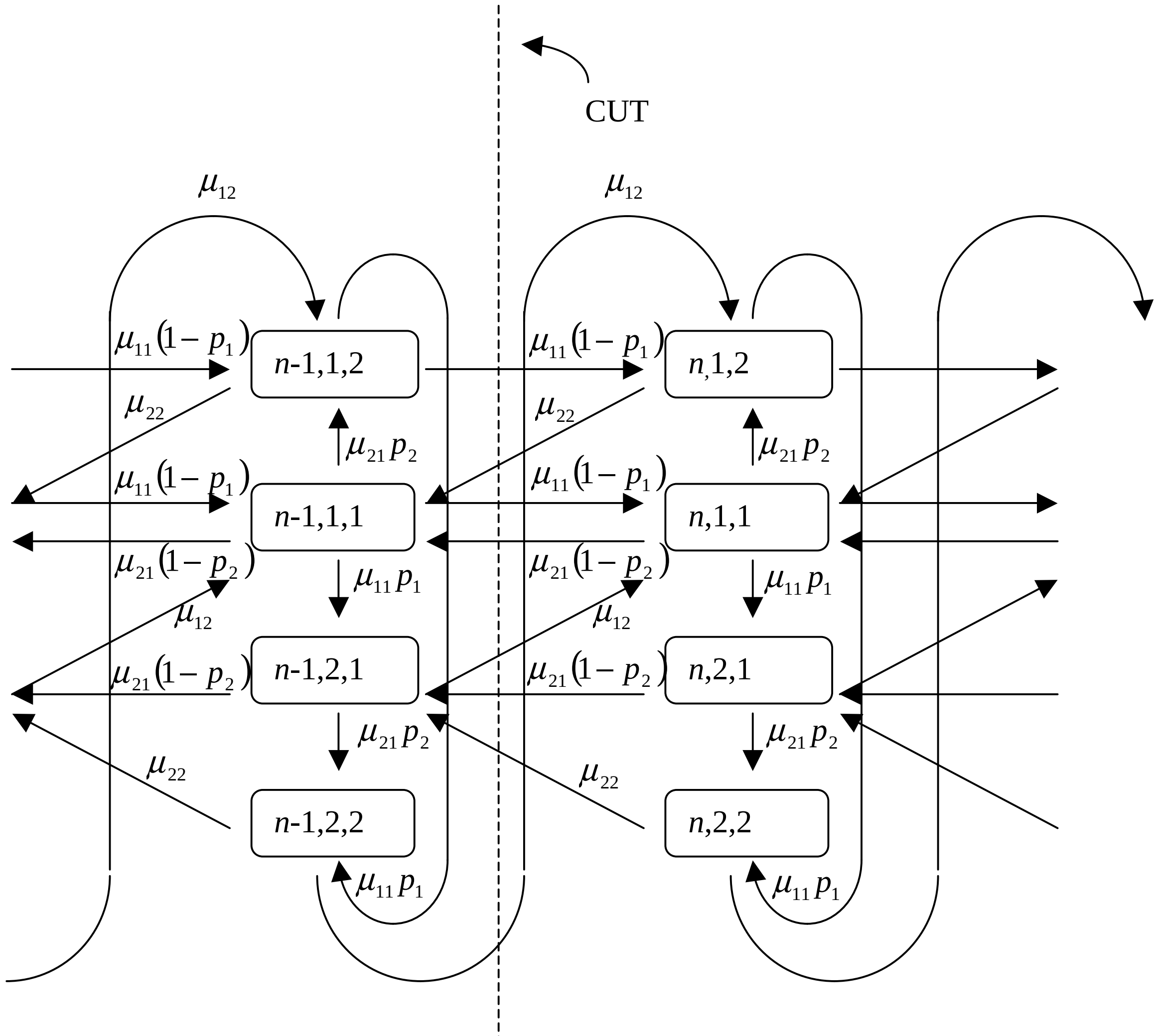

Figure 3. State transitions diagram when $-K_2+2 < n < K_1-2$

5.1 *Transition Equations*

For each $(n, j_1, j_2) \in S_Q$, let $P_Q(n, j_1, j_2)$ denote the steady state probability corresponding to the continuous time Markov chain (N, J_1, J_2). The steady state probabilities can be computed by solving the corresponding state transition equations. Corresponding to whether the arrival process to either buffer has *shut down* (*SD*) or *not shut down* (*NSD*) we partition the state space S_Q into S_Q^{SD} and S_Q^{NSD} where

$$S_Q^{NSD} = \{(n, j_1, j_2) : n = -K_2+1, \ldots -1, 0, 1, \ldots, K_1-1;\ j_1 = 1,2;\ j_2 = 1,2\} \text{ and}$$
$$S_Q^{SD} = \{(-K_2,1,0), (-K_2,2,0), (K_1,0,1), (K_1,0,2)\} = S_Q - S_Q^{NSD} \qquad (5)$$

Then considering the partition of S_Q^{NSD}, we have that in steady state, for each $(n, j_1, j_2) \in S_Q^{NSD}$ and $\mathbf{P_Q}(n) = [P_Q(n,1,2) \quad P_Q(n,1,1) \quad P_Q(n,2,1) \quad P_Q(n,2,2)]^T$:

$$\begin{bmatrix} \mu_{11}(1-p_1) & \mu_{11}(1-p_1) & \mu_{12} & \mu_{12} \\ \mu_{11}(1-p_1) & 0 & 0 & \mu_{12} \\ 0 & -\mu_{11}p_1 & \mu_{12}+\mu_{21} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{P_Q}(n-1) = \begin{bmatrix} \mu_{22} & \mu_{21}(1-p_2) & \mu_{21}(1-p_2) & \mu_{22} \\ \mu_{11}+\mu_{22} & -\mu_{21}p_2 & 0 & 0 \\ 0 & 0 & \mu_{21}(1-p_2) & \mu_{22} \\ -\mu_{11}p_1 & 0 & -\mu_{21}p_2 & \mu_{12}+\mu_{22} \end{bmatrix} \mathbf{P_Q}(n)$$

(6)

For $(K_1,0,1)$ and $(K_1,0,2) \in S_Q^{SD}$ we have:

$$\begin{bmatrix} \mu_{11}(1-p_1) & 0 & 0 & \mu_{12} \\ 0 & \mu_{11}(1-p_1) & \mu_{12} & 0 \end{bmatrix} \mathbf{P_Q}(K_1-1) = \begin{bmatrix} -\mu_{21}p_2 & -\mu_{22} \\ \mu_{21}(1-p_2) & 0 \end{bmatrix} \begin{bmatrix} P_Q(K_1,0,1) \\ P_Q(K_1,0,2) \end{bmatrix} \quad (7)$$

For $(-K_2,1,0)$ and $(-K_2,2,0) \in S_Q^{SD}$ we have:

$$\begin{bmatrix} \mu_{22} & \mu_{21}(1-p_2) & 0 & 0 \\ 0 & 0 & \mu_{21}(1-p_2) & \mu_{22} \end{bmatrix} \mathbf{P_Q}(-K_2+1) = \begin{bmatrix} \mu_{11} & 0 \\ \mu_{11}p_1 & \mu_{12} \end{bmatrix} \begin{bmatrix} P_Q(-K_2,1,0) \\ P_Q(-K_2,2,0) \end{bmatrix} \quad (8)$$

Additionally we have the following:

$$\sum_{(n,j_1,j_2)\in S_Q} P_Q(n,j_1,j_2) = 1 \quad (9)$$

Solving equations 6-9 above, for each $(n, j_1, j_2) \in S_Q$, we can obtain expressions for the steady state probabilities $P_Q(n, j_1, j_2)$.

5.2 Average Queue Length and Throughput

Given the steady state probabilities of the queue length process, the average queue lengths $\overline{L}_1$ and $\overline{L}_2$ at the buffers B_1 and B_2 are obtained using equations 10 and 11 below.

$$\overline{L}_1 = \sum_{n=1}^{K_1-1}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2} nP_Q(n,j_1,j_2) + \sum_{j_2=1}^{2} K_1 P_Q(K_1,0,j_2)$$

(10)

$$\overline{L}_2 = \sum_{n=-K_2+1}^{-1}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2} |-n| P_Q(n,j_1,j_2) + \sum_{j_1=1}^{2} K_2 P_Q(-K_2,j_1,0) \quad (11)$$

Thus we have obtained the probability distribution of the queue length process as well as the mean queue lengths at both the input buffers. The throughput χ_D of the fork/join station is computed in a similar manner from the steady state probabilities of the queue length process as follows:

$$\chi_D = \begin{cases} \sum_{n=-K_2+1}^{0} \left(P_Q(n,1,2) + P_Q(n,1,1)\right)\mu_{11}(1-p_1) + \left(P_Q(n,2,2) + P_Q(n,2,1)\right)\mu_{12} + \\ \sum_{n=0}^{K_1-1} \left(P_Q(n,2,1) + P_Q(n,1,1)\right)\mu_{21}(1-p_2) + \left(P_Q(n,1,2) + P_Q(n,2,2)\right)\mu_{22} + \\ P_Q(-K_2,1,0)\mu_{11}(1-p_1) + P_Q(-K_2,2,0)\mu_{12} + P_Q(K_1,0,1)\mu_{21}(1-p_2) + P_Q(K_1,0,2)\mu_{22} \end{cases} \quad (12)$$

In section 7 we provide some numerical results on our queue length analysis. In the next section, we analyze the departure process of the fork/join station.

6 Analysis of the Departure Process

As noted in Section 4.3, when the arrival processes to both input buffers are phase renewal, the output process is a Markov renewal process. To analyze this Markov renewal process, we first identify the states in the Markov chain embedded at departure instants. Next, we recognize that this embedded Markov process has a special structure and use this information to identify the possible sample paths between successive departures. By deriving the conditional probability

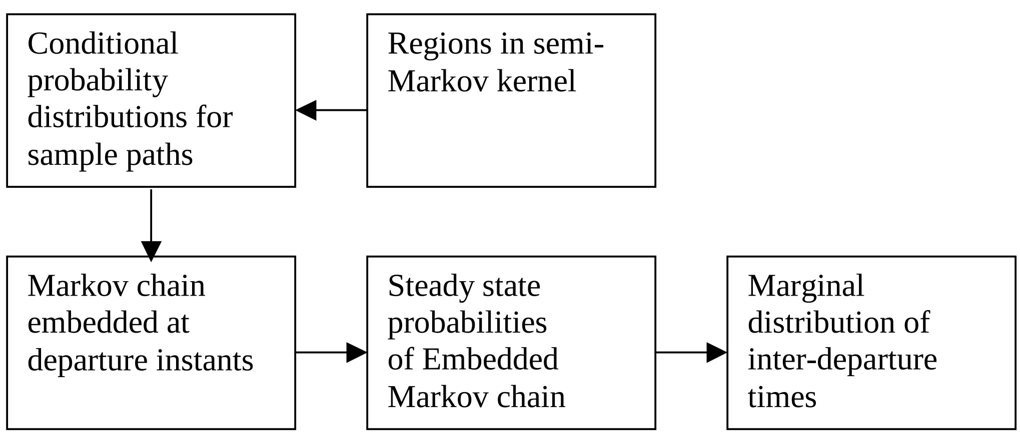

Figure 4. Analysis of the departure process at the fork/join station

distributions for each sample path, we obtain the transition probability matrix P_D, of the Markov chain embedded at instants of departure from the fork/join station. We solve the Markov chain to obtain the stationary probability vector $\prod$. Using $\prod$ and the distributions of the inter-arrival times at the two buffers, we obtain $G_D(t)$, the marginal distribution of inter-departure times from the fork/join station. The approach is summarized in Figure 4.

6.1 Departure Process as a Semi-Markov Process

A departure occurs (simultaneously from both the queues) whenever at least on unit is available in both the input buffers, B_1 and B_2. The sequence of states of the fork/join station and the corresponding departure times describe a two-tuple random process (X, D). The random process $D = \{\tau_m : m \in N\}$, where N is the set of all nonnegative integers, consists of a sequence of real valued random variables τ_m, where τ_m is the time of the m^{th} departure. Also, each departure time for the fork/join station always coincides with either an arrival time in buffer B_1 or an arrival time in buffer B_2. Associated with the random process D is the incremental process $T = \{T_m : m \in N\}$ where T_m is the time interval between consecutive departures τ_m and τ_{m-1}, i.e., $T_m = \tau_m - \tau_{m-1}$, with $T_0 = 0$. The random process $X = \{X_m : m \in N\}$ is vector valued and consists of the triple (N_m, J_{1m}, J_{2m}) where $N_m = N(\tau_m^+)$, $J_{1m} = J_1(\tau_m^+)$ and $J_{2m} = J_2(\tau_m^+)$. τ_m^+ denotes the time instant just after a departure at τ_m, $N(\tau_m^+)$ denotes the number of units at the fork/join station, and $J_1(\tau_m^+)$ and $J_2(\tau_m^+)$ denote the phases of the arrivals to input buffers B_1 and B_2 just after the m^{th} departure.

We study the departure process by analyzing the two-tuple random process (X, T). T_{m+1} depends on the present state X_m and the next state X_{m+1}. However, given these states, T_{m+1} is independent of the previous states $X_0, ..., X_{m-1}$ and $T_0, ..., T_m$. That is , the following relation holds

$$P(X_{m+1}, T_{m+1} \mid X_0, ..., X_m, T_0, ..., T_m) = P(X_{m+1}, T_{m+1} \mid X_m) \text{ for all } m \in N \tag{13}$$

and the output process (X, T) is a Markov renewal process. The state space of the output process (X, T) is $S_D \times R$, where $R = (0, \infty)$, and

$$\begin{aligned} S_D = {} & \{(-K_2 + 1,1,1)\} \cup \{(n,1,1), (n,1,2), -K_2 + 1 < n < 0\} \cup \\ & \{(0,1,1), (0,1,2), (0,2,1)\} \cup \{(n,1,1), (n,2,1), 0 < n < K_1 - 1\} \cup \{(K_1 - 1,1,1)\} \end{aligned} \tag{14}$$

In Figure 5 the set of feasible states in S_D are shown in gray. It can be easily verified that for each state in S_D all transitions lead back to states in S_D.

To see this more precisely, we note that $S_D \subset S_Q$ and define $S_{Q-D} = S_Q - S_D$. States in S_{Q-D} are shown in dotted boxes in Figure 5. These states are not in S_D for the following reasons. For example, it is obvious that $S_Q^{SD} \subset S_{Q-D}$. For $X_m \in S_Q^{NSD}$ we reason in the following manner.

When $N_m = -K_2 + 1$, buffer B_2 is shut down prior to the departure instant implying that $\{(-K_2+1,1,2),(-K_2+1,2,1),(-K_2+1,2,2)\} \subset S_{Q-D}$. Simlarly, when $N_m = K_1 - 1$, buffer B_1 is shut down prior to the departure instant implying that $(K_1-1,1,2),\{(K_1-1,2,1),(K_1-1,2,2)\} \subset S_{Q-D}$. Also, each departure time from the fork/join station always coincides with an arrival time in either buffer B_1 or buffer B_2. Therefore, when $-K_2+1 < N_m < 0$, the departure time coincides with an arrival time in buffer B_1 and hence $\{(N_m,2,1),(N_m,2,2)\} \subset S_{Q-D}$. Simlarly when $0 < N_m < K_1 - 1$, the departure time coincides with an arrival time in buffer B_2 and hence $\{(N_m,1,2),(N_m,2,2)\} \subset S_{Q-D}$. Finally when $N_m = 0$, we have $\{(0,2,2)\} \subset S_{Q-D}$.

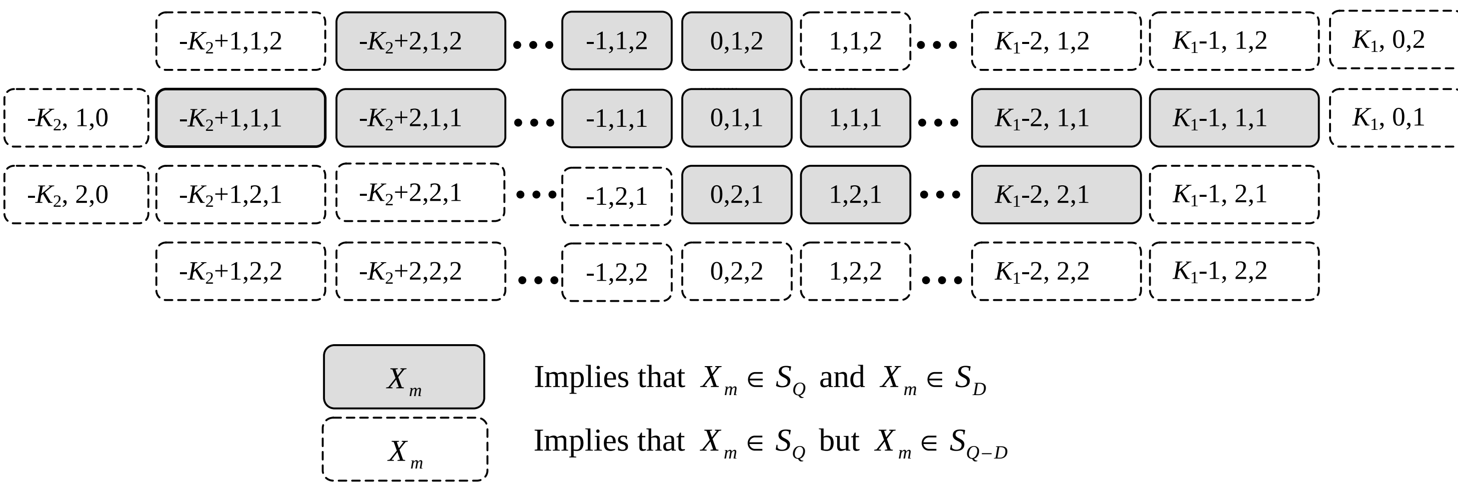

Figure 5. Identifying the states of the Markov chain embedded at departure instants

From this we see that S_D has $2(K_1 + K_2) - 3$ states. Since the output process (X,T) is a Markov renewal process it is completely characterized by its semi-Markov kernel Q. The semi-Markov kernel Q of the output process (X,T) is expressed as

$$Q(X_{m+1}, X_m, t) = P(X_{m+1}, T_{m+1} \le t \mid X_m) \tag{15}$$

For convenience the elements of the semi-Markov kernel can be expressed in the Laplace transform domain as

$$Q^*(X_{m+1}, X_m, ds) = L\{Q(X_{m+1}, X_m, dt)\} \tag{16}$$

Section 6.2 below describes how to compute the values of $Q^*(X_{m+1}, X_m, ds)$. Using $Q^*(X_{m+1}, X_m, ds)$ several characteristics of the output process (X,T) can be obtained (Disney and Kiessler 1987). For example, the state transition matrix P_D of the underlying Markov chain X embedded at times $\tau_m, m \in N$ is obained by setting $s = 0$ in equation 16 above, i.e.:

$$P_D(X_{m+1}, X_m) = P(X_{m+1} = (N_{m+1}, J_{1m+1}, J_{2m+1}) \mid X_m = (N_m, J_{1m}, J_{2m})) \\ = Q^*(X_{m+1}, X_m, ds)|_{s=0} \quad (17)$$

Let $\Pi = \{\Pi(X_k) : X_k \in S_D\}$, where $\Pi(X_k)$ is the steady state probability that fork/join station is in state X_k at a departure instant. The stationary probability vector Π of the underlying Markov chain is obtained as follows:

$$\Pi = \Pi P_D \quad (18)$$

$$\sum_{k \in S_D} \Pi(X_k) = 1 \quad (19)$$

Equations 18 and 19 are solved to obtain Π. Section 6.3 describes how to compute $G_D(t)$, the cumulative distribution function of the inter-departure times using Π.

It is evident from the above discussion that computation of P_D and hence Π is the main requirement for estimating the cumulative distribution function $G_D(t)$ of the inter-departure times. In the next section, we construct P_D using the semi-Markov kernel Q of the output process (X, T) and use P_D to determine Π.

6.2 *Construction of Semi-Markov Kernel* Q *and State Transition Matrix* P_D

To construct P_D, we note that between two successive departure states X_m and X_{m+1}, the fork/join station visits a finite sequence of intermediate states $Z_k, k = 1,2,...$ where $Z_k \in S_Q$. Each such sequence corresponds to a *sample path* leading the fork/join station from departure state X_m at time τ_m to state X_{m+1} at time τ_{m+1}. Let $SP(X_{m+1}, X_m)$ be the set containing all such sequences or sample paths, and $|SP(X_{m+1}, X_m)|$ denote the number of sample paths for the pair (X_{m+1}, X_m). Let $SP_j(X_{m+1}, X_m) \in SP(X_{m+1}, X_m)$ be the j^{th} sample path in this set. Then $SP_j(X_{m+1}, X_m)$ can be written as the ordered sequence $(Z_0, Z_1, ..., Z_{l(j)-1}, Z_{l(j)})$ where $l(j)$ is the length of the sample path $SP_j(X_{m+1}, X_m), Z_0 \equiv X_m, Z_{l(j)} \equiv X_{m+1}$ and $Z_k, k = 1,2,..., l(j)-1$ correspond to the intermediate states. Additionally, let $t_0 \le t_1 \le ... \le t_{l(j)-1} \le t_{l(j)}$ be the times when the fork/join station transitions to states $Z_0, Z_1, ..., Z_{l(j)-1}, Z_{l(j)}$ respectively. Note $t_0 \equiv \tau_m$ and $t_{l(j)} \equiv \tau_{m+1}$. See figure 6 below.

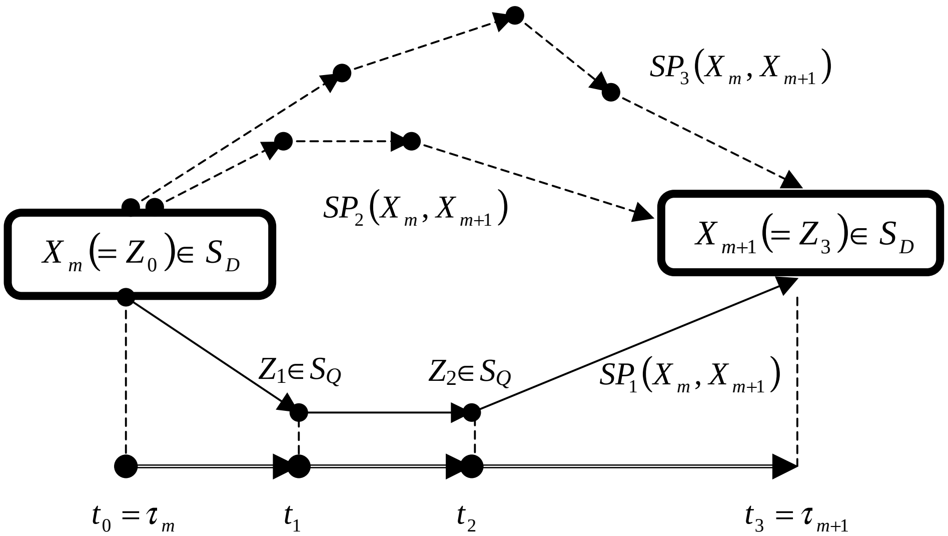

Figure 6. Possible sample paths between departures

Since the sample paths $SP_j(X_{m+1}, X_m)$ are independent and equally likely, we have

$$Q^*(X_{m+1}, X_m, ds) = \sum_{j=1}^{|SP(X_{m+1},X_m)|} \left(\prod_{k=1}^{l(j)} Q^*_{j,k}(Z_k, Z_{k-1}, ds) \right) \tag{20}$$

$$P_D(X_{m+1}, X_m) = \sum_{j=1}^{|SP(X_{m+1},X_m)|} \left(\prod_{k=1}^{l(j)} Q^*_{j,k}(Z_k, Z_{k-1}, ds)|_{s=0} \right) \tag{21}$$

It is evident that in order to compute $P_D(X_{m+1}, X_m)$ using equation 21 above, we need to identify all the sample paths $SP_j(X_{m+1}, X_m) \in SP(X_{m+1}, X_m)$ and then compute $Q^*_{j,k}(Z_k, Z_{k-1}, ds)|_{s=0}$ for each $Z_k \in SP_j(X_{m+1}, X_m) \in SP(X_{m+1}, X_m)$. This may appear to be a cumbersome task. However, because arrivals to buffers B_1 and B_2 are independent and composed of exponential phases, the semi-Markov kernel Q and hence, the transition matrix P_D has a special structure. Depending upon the values of N, J_1 and J_2 of states X_m, and X_{m+1} the non-zero portion of P_D can be partitioned into 14 regions. We exploit this special structure to identify the set of sample paths $SP(X_{m+1}, X_m)$ for each pair (X_{m+1}, X_m) and also to compute $Q^*_{j,k}(Z_k, Z_{k-1}, ds)|_{s=0}$ for each $Z_k \in SP_j(X_{m+1}, X_m)$. Figure 7 illustrates these regions (labeled 1 through 14) in P_D using the example of a fork/join station fed by two input streams and with K_1=K_2=5.

$P_D(X_{m+1}, X_m)$		X_{m+1}																
		-4,1,1	-3,1,2	-3,1,1	-2,1,2	-2,1,1	-1,1,2	-1,1,1	0,1,2	0,1,1	0,2,1	1,1,1	1,2,1	2,1,1	2,2,1	3,1,1	3,2,1	4,1,1
	-4,1,1	5	7	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	-3,1,2	8	6	4	2	4	0	0	0	0	0	0	0	0	0	0	0	0
	-3,1,1	5	7	1	3	1	0	0	0	0	0	0	0	0	0	0	0	0
	-2,1,2	8	6	4	2	4	2	4	0	0	0	0	0	0	0	0	0	0
	-2,1,1	5	7	1	3	1	3	1	0	0	0	0	0	0	0	0	0	0
	-1,1,2	8	6	4	2	4	2	4	2	4	0	0	0	0	0	0	0	0
	-1,1,1	5	7	1	3	1	3	1	3	1	0	0	0	0	0	0	0	0
	0,1,2	8	6	4	2	4	2	4	13	13	13	9	11	9	11	9	11	10
X_m	0,1,1	5	7	1	3	1	3	1	14	14	14	1	3	1	3	1	7	5
	0,2,1	10	11	9	11	9	11	9	12	12	12	4	2	4	2	4	6	8
	1,1,1	0	0	0	0	0	0	0	0	1	3	1	3	1	3	1	7	5
	1,2,1	0	0	0	0	0	0	0	0	4	2	4	2	4	2	4	6	8
	2,1,1	0	0	0	0	0	0	0	0	0	0	1	3	1	3	1	7	5
	2,2,1	0	0	0	0	0	0	0	0	0	0	4	2	4	2	4	6	8
	3,1,1	0	0	0	0	0	0	0	0	0	0	0	0	1	3	1	7	5
	3,2,1	0	0	0	0	0	0	0	0	0	0	0	0	4	2	4	6	8
	4,1,1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	7	5

Figure 7. Regions in the semi-Markov kernel Q and state transition matrix P_D

Regions 1-11 correspond to transition between states X_m and X_{m+1} where one of the buffers B_1 or B_2 is non-empty at time instants τ_m and τ_{m+1}. Regions 12-14 correspond to the states when both the buffers B_1 and B_2 were empty at time instants τ_m and τ_{m+1}. Transitions between states in a region share similarity in the structure of the possible sample paths. We exploit this similarity while deriving expressions for $P_D(X_{m+1}, X_m)$. The Appendix lists the formal definitions as well as the expressions for $P_D(X_{m+1}, X_m)$ for each pair (X_{m+1}, X_m). However, we illustrate the procedure for deriving these expressions using Region 11 as an example.

In Region 11 we have two sub regions:

$$X_m = (0,1,2),\ X_{m+1} = (N_{m+1},2,1) \text{ for } 0 < N_{m+1} \le K_1 - 2 \text{ and}$$
$$X_m = (0,2,1),\ X_{m+1} = (N_{m+1},1,2) \text{ for } 0 > N_{m+1} \ge -K_2 + 2$$

Consider $P_D(X_{m+1}, X_m)$ where $X_m = (0,1,2)$ and $X_{m+1} = (N_{m+1},2,1)$ for $0 < N_{m+1} \le K_1 - 2$:
$P_D(X_{m+1}, X_m)$ is the probability of the event that in buffer B_1, exactly $(N_{m+1}+1)$ arrivals and only phase 1 of the $(N_{m+1}+2)^{nd}$ arrival are completed before the completion of phase 2 of the arrival in buffer B_2. Since arrivals to buffers B_1 and B_2 are independent and composed of exponential phases, for (X_{m+1}, X_m) we can write:

$P_D\left(X_{m+1}, X_m\right)=$
P(Exactly $(N_{m+1}+1)$ arrivals in B_1 before the completion of phase 2 of arrival in B_2)*
P(Completion of phase 1 of $(N_{m+1}+2)^{nd}$ arrival in buffer B_1 before phase 2 of arrival in B_2)*
P(Completion of phase 2 of arrival in B_2 before phase 2 of $(N_{m+1}+2)^{nd}$ arrival in buffer B_1)
(22)

Since each of the $(N_{m+1}+1)$ arrivals in B_1 could be composed of one or two exponential phases, for $\left(X_{m+1}, X_m\right)$ there are $2^{(N_{m+1}+1)}$ possible sample paths in $SP(X_{m+1}, X_m)$. Using this information and equation 24 above, we obtain the following expression for $P_D\left(X_{m+1}, X_m\right)$:

$$P_D\left(X_{m+1}, X_m\right)=\left[\left(\frac{(1-p_1)\mu_{11}}{\mu_{11}+\mu_{22}}\right)+\left(\frac{p_1\mu_{11}}{\mu_{11}+\mu_{22}}\right)\left(\frac{\mu_{12}}{\mu_{12}+\mu_{22}}\right)\right]^{N_m+1}\left[\frac{p_1\mu_{11}}{\mu_{11}+\mu_{22}}\right]\left[\frac{\mu_{22}}{\mu_{12}+\mu_{22}}\right] \tag{23}$$

Using an approach similar to that described above, we can derive expressions for $P_D\left(X_{m+1}, X_m\right)$ when $X_m=(0,2,1)$ and $X_{m+1}=(N_{m+1},1,2)$ for $0>N_{m+1}\geq -K_2+2$ in Region 11. Similarly we can derive expressions for $P_D\left(X_{m+1}, X_m\right)$ for each pair $\left(X_{m+1}, X_m\right)$. The full set of expressions is in the Appendix.

6.3 *Marginal Distribution of Inter-departure Times*

Using P_D derived in the previous section, we solve equations 18 and 19 for the stationary probability vector Π. We use Π to obtain the marginal distribution of the inter-departure times. Note that since P_D is independent of *m*, the inter-departure times are identically distributed. Let $G_D(t)$ be the distribution function of the inter-departure times i.e., $G_D(t)=P\{T_m\leq t\}$ for any *m*. $G_D(t)$ can be written in terms of Π and the distribution function for inter-arrival times at input buffers 1 and 2, namely, $G_j(t)=1-C_{j1}e^{-\mu_{j1}t}-C_{j2}e^{-\mu_{j2}t}$, $j=1,2$.

We have:

$$\begin{aligned} G_D(t) = {} & \Pi_1\, G_2(t)+\Pi_2\, G_1(t)+\Pi(0,1,1)\, G_1(t)\, G_2(t)+ \\ & \Pi(0,1,2)G_1(t)(1-e^{\mu_{22}t})+\Pi(0,2,1)G_2(t)(1-e^{\mu_{12}t}) \end{aligned}$$

where
$$\Pi_1=\sum_{S_{1D}}\Pi\left(X_m\right)$$
$$\Pi_2=\sum_{S_{2D}}\Pi\left(X_m\right)$$

$$S_D = S_{1D} \cup S_{2D} \cup \{(0,1,2),(0,1,1),(0,2,1)\}$$

$$S_{1D} = \{(n, j_1,1) : 0 < n \le K_1 - 2;\ j_1 = 1,2\} \cup \{(K_1 - 1,1,1)\}$$
$$S_{2D} = \{(n,1, j_2) : -K_2 + 2 \le n < 0;\ j_2 = 1,2\} \cup \{(-K_2 + 1,1,1)\} \tag{24}$$

Π_1 is the steady state probability that a departure results in only buffer B_1 being empty, and Π_2 is the steady state probability that a departure results in only buffer B_2 being empty. From $G_D(t)$, we obtain the mean inter-departure times, $\overline{D}\left(= 1/\chi_D\right)$.

$$\overline{D} = \Pi_2\left(\frac{C_{11}}{\mu_{11}} + \frac{C_{12}}{\mu_{12}}\right) + \Pi_1\left(\frac{C_{21}}{\mu_{21}} + \frac{C_{22}}{\mu_{22}}\right) + \Pi(0,1,2)\left[\frac{1}{\mu_{22}} + \frac{C_{11}\mu_{22}}{\mu_{11}(\mu_{11} + \mu_{22})} + \frac{C_{12}\mu_{22}}{\mu_{12}(\mu_{12} + \mu_{22})}\right] +$$
$$\Pi(0,1,1)\left[\left(\frac{C_{11}}{\mu_{11}} + \frac{C_{12}}{\mu_{12}} + \frac{C_{21}}{\mu_{21}} + \frac{C_{22}}{\mu_{22}}\right) - \left(\frac{C_{11}C_{21}}{\mu_{11} + \mu_{21}} + \frac{C_{11}C_{22}}{\mu_{11} + \mu_{22}} + \frac{C_{12}C_{21}}{\mu_{12} + \mu_{21}} + \frac{C_{12}C_{22}}{\mu_{12} + \mu_{22}}\right)\right] +$$
$$\Pi(0,2,1)\left[\frac{1}{\mu_{12}} + \frac{C_{21}\mu_{12}}{\mu_{21}(\mu_{21} + \mu_{12})} + \frac{C_{22}\mu_{12}}{\mu_{22}(\mu_{22} + \mu_{12})}\right] \tag{25}$$

Similarly, from $G_D(t)$, we can derive expressions for the second moment Λ_D of inter-departure times and hence c_D^2, the SCV of inter-departure times.

7 Numerical Results

In this section we present some numerical examples to demonstrate the usefulness of our analysis. First, we compare the results of our analysis of a fork/join station assuming two-phase Coxian arrivals against results that assume Poisson arrival processes. Second, we study the sensitivity of performance measures such as throughput, mean queue lengths, and variability of inter-departure times for a wide class of input processes and customer populations. In the latter we investigate the impact of (1) mean rates of the input processes, (2) different SCVs of the input processes, and (3) network populations K_1 and K_2 on performance measures such as throughput rate, average queue lengths at the input buffers and SCV of inter-departure times.

Figure 8 compares the values of K_1 and K_2 required to obtain a target throughput from a fork/join station fed by input processes having inter-arrival times with the same mean but with SCVs different from that of the Poisson process. An application where such insights would be useful is closed loop fabrication assembly systems. In such systems, it would be necessary to decide the number of fixed pallets in the networks feeding the fork/join station so as to guarantee a required level of throughput. As seen in Figure 8, the throughput increases monotonically with K_1 and K_2. However, as the SCVs of the input processes increase, the values of K_1 and K_2

required to obtain a given throughput increase significantly. As discussed in Kamath *et al.* (1988), in practical flexible assembly systems, we often find SCVs ranging from 0 to 4.0.

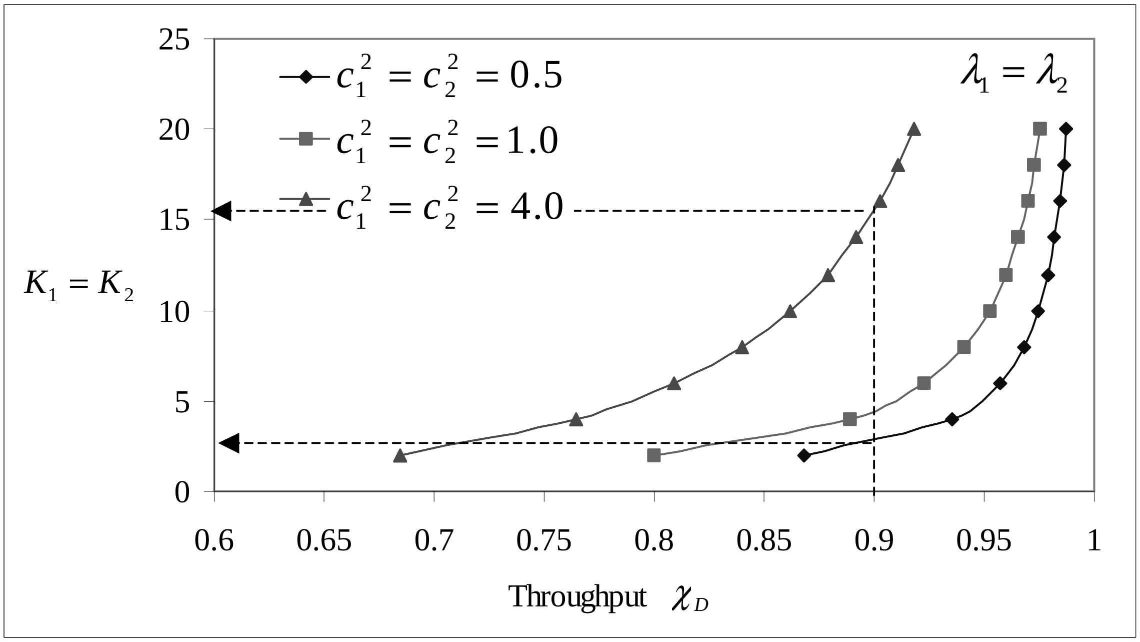

Figure 8. Impact of SCV on K_1 and K_2 required to obtain a given throughput

For example, as SCV increases from 0.5 to 4.0, for a throughput requirement of 0.9 the required values of K_1 and K_2 more than double. For closed loop fabrication assembly systems, this would imply significant investment in pallets to buffer against variability of input processes. Conversely, ignoring the impact of variability in the input processes would result in significantly lower throughput than anticipated otherwise. This implies that analyzing fork/join stations for inputs more general than the Poisson process has important practical implications.

Figure 9 shows the SCV of the inter-departure times for a fork/join station fed by input processes having inter-arrival times with the same mean but different SCVs. As the values of K_1 and K_2 increase, the SCV of the inter-departure times tend to the average of the SCVs of the input processes. Therefore, while adding additional pallets helps improve the throughput performance from the fork/join station, it would not help in reducing the variability of the output process.

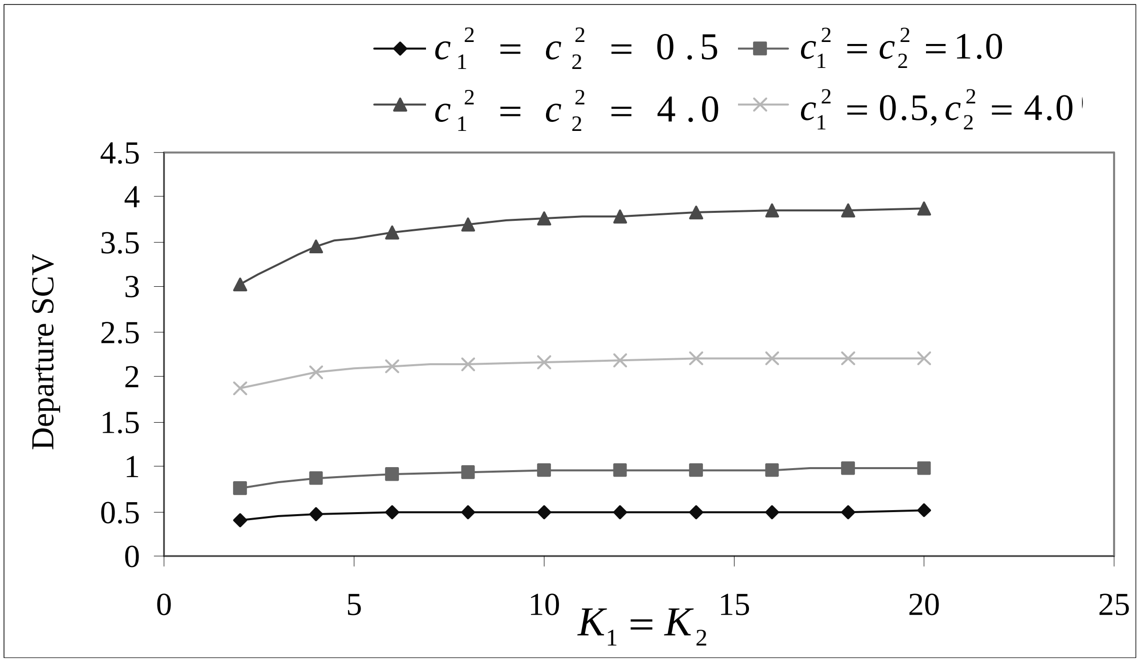

Figure 9. Impact of K_1, K_2 and input SCVs on departure SCVs

In most applications one would expect the rates of the input processes at the fork/join synchronization station to be equal. However, in reality minor imbalances in arrival rates could occur. Figures 10, 11 and 12 indicate that throughput, mean queue length, and SCV of inter-departure times are very sensitive to imbalance in input rates.

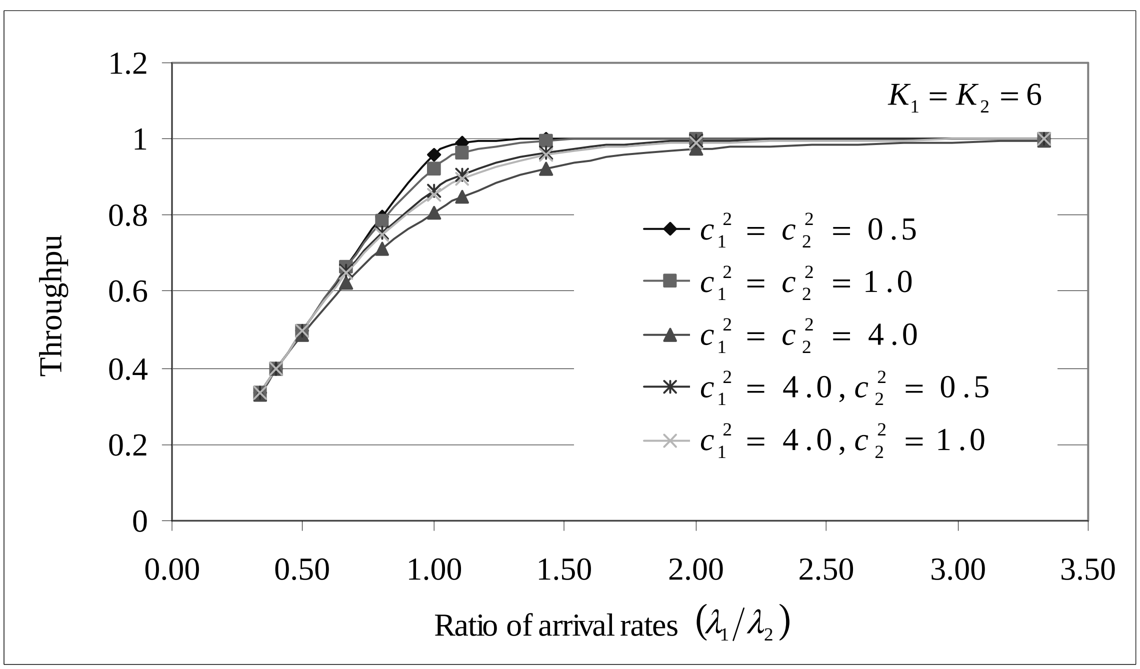

Figure 10. Impact of imbalanced input rates on throughput

In Figure 10 we see that the throughput from the fork/join is highly influenced by the arrival rate of the slower of the two input processes.

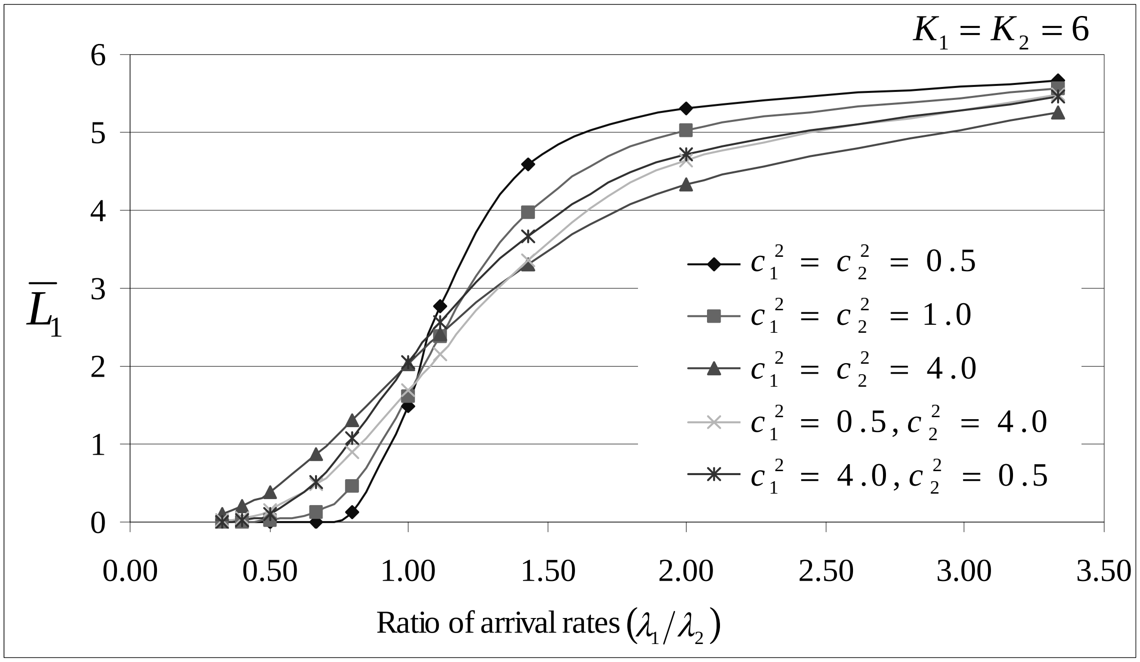

Figure 11. Impact of imbalanced input rates on mean queue length at buffer B_1

In Figure 11 we see that substantial queues are observed at the buffer of the input process with a higher rate of arrivals. In Figure 12 we see that the SCV of the departure process from the fork/join is highly influenced by the SCV of the slower of the two input processes.

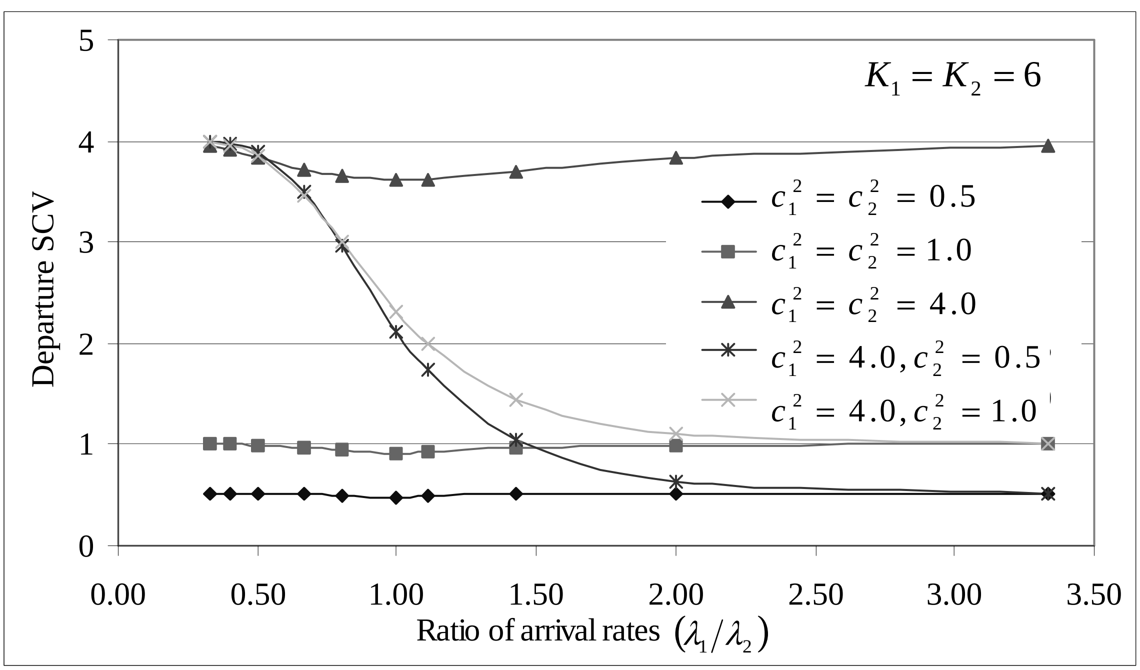

Figure 12. Impact of imbalanced input rates on SCV of inter-departure times

Finally, from all the figures it is clear that the SCV of the input processes have a significant impact on the performance of the fork/join station only when the input processes have nearly equal rates.

Next, we discuss a typical scenario where the analysis presented in this paper could be used to make useful design decisions. We consider a fork/join station fed by input processes having different SCVs and marginally imbalanced inter-arrival times. The input process to buffer B_1 has inter-arrival times with mean 1.1 and SCV equal to 4.0 while input process to buffer B_2 has inter-arrival times with mean 0.9 and SCV equal to 0.5. For a given target throughput, we consider the impact of choosing different values of K_1 and K_2 on the output process. For a closed loop fabrication assembly systems, this would translate into decision of whether to invest in additional pallets of one type or the other. For example, Figure 13 indicates that the same target throughput of 0.87 can be obtained by setting $K_1 = K_2 = 10$ or by setting $K_1 = 18$ and $K_2 = 2$. However, as seen from Figure 14, setting $K_1 = K_2 = 10$ results in higher variability of the inter-departure times. Even setting $K_1 = 18$ and leaving $K_2 = 10$ would not reduce the variability of inter-departure times. To decrease the SCV of the inter-departure times, one would have to not only to set a *high* value of K_1 but also a *low* value of K_2. This large imbalance in K_1 and K_2 setting may not be intuitive to system designers, yet our model points out its benefits.

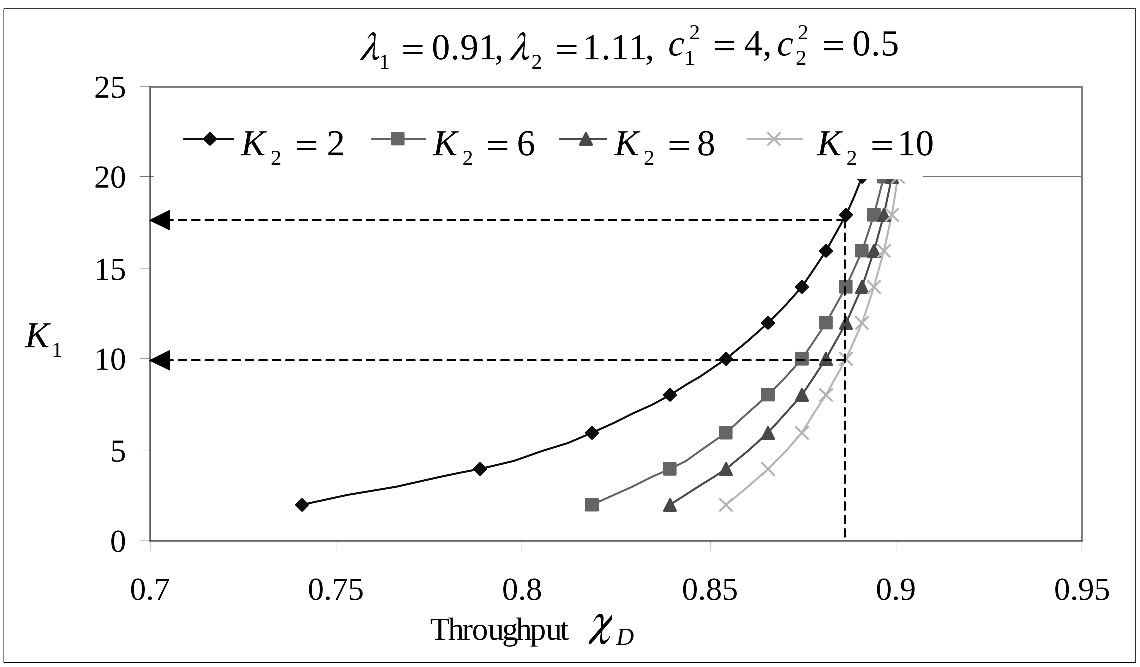

Figure 13. K_1 and K_2 required for a given throughput requirement

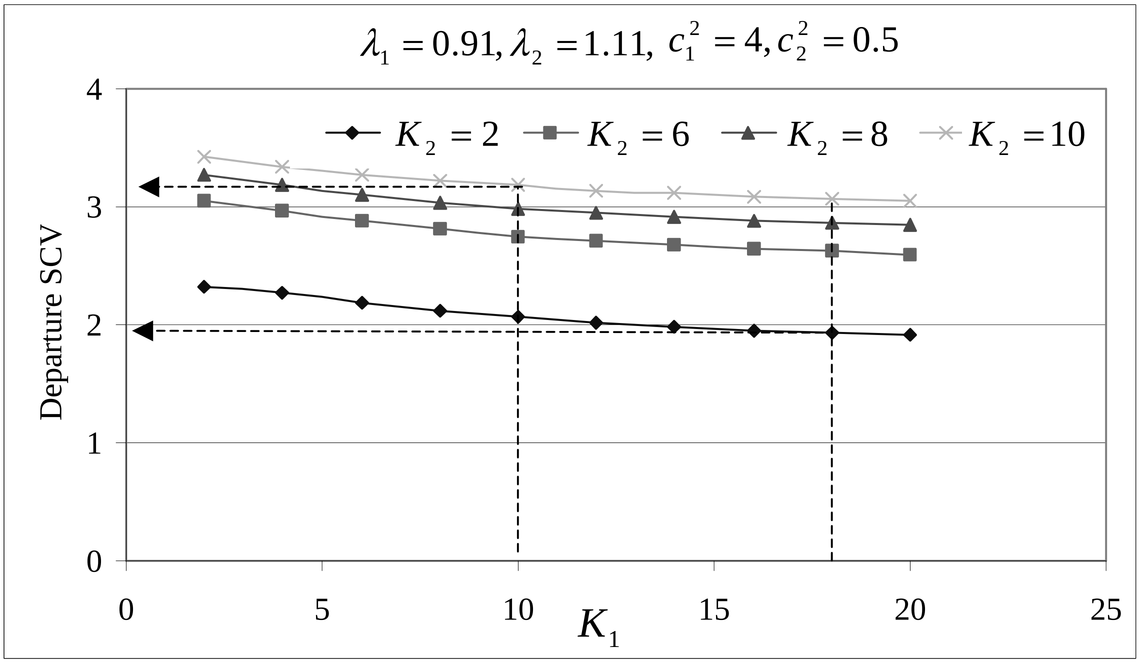

Figure 14. Impact of K_1 and K_2 on SCV of inter-departure times

8 Conclusion

Models for analyzing fork/join stations can be useful in developing efficient decomposition methods for analyzing queuing network models of several manufacturing and computer systems. This paper analyzes a fork/join station with two input buffers fed by arrivals from a finite population. Using an exact analysis for the case when the inter-arrival times have two-phase Coxian distributions we show that when the input processes have equal rates, variability in the arrival processes have a significant impact on the performance of the fork/join station. In addition, we study the sensitivity of performance measures such as throughput, mean queue lengths, and variability of inter-departure times for a wide class of input processes and customer populations. We observe that when the mean rates of arrival to the two input buffers of the fork/join station are nearly equal, variability in the input processes can have significant impact on throughput, queue lengths and variability in the inter-departure times. When the fork/join station is a part of a larger network, this information about the inter-departure times can be useful in developing efficient models to characterize the output process that might in turn be the input process for other stations in the network. Further, our choice of two-phase Coxian distribution allows us to analyze fairly general input processes without much added computational complexity. Existing decomposition methods have been shown to work well with approximations for individual stations in the network. This analysis could also be used in the development and validation of approximate expressions for different performance measures of a fork/join station, which would in turn help extend decomposition methods to networks with fork/join stations.

Appendix

We list the expressions for $P_D(X_{m+1}, X_m)$ for each pair (X_{m+1}, X_m) in Table 1 below. To simplify the notation in the table, we define the following quantities:

$$a_1 = \frac{(1-p_1)\mu_{11}}{\mu_{11}+\mu_{21}} \qquad \tilde{a}_1 = \frac{(1-p_1)\mu_{11}}{\mu_{11}+\mu_{22}}$$

$$b_1 = \frac{p_1\mu_{11}}{\mu_{11}+\mu_{21}} \qquad \tilde{b}_1 = \frac{p_1\mu_{11}}{\mu_{11}+\mu_{22}}$$

$$c_1 = \frac{\mu_{12}}{\mu_{12}+\mu_{21}} \qquad \tilde{c}_1 = \frac{\mu_{12}}{\mu_{12}+\mu_{22}}$$

$$a_2 = \frac{(1-p_2)\mu_{21}}{\mu_{11}+\mu_{21}} \qquad \tilde{a}_2 = \frac{(1-p_2)\mu_{21}}{\mu_{12}+\mu_{21}}$$

$$b_2 = \frac{p_2\mu_{21}}{\mu_{11}+\mu_{21}} \qquad \tilde{b}_2 = \frac{p_2\mu_{21}}{\mu_{12}+\mu_{21}}$$

$$c_2 = \frac{\mu_{22}}{\mu_{12}+\mu_{22}} \qquad \tilde{c}_2 = \frac{\mu_{22}}{\mu_{12}+\mu_{22}}$$

Additionally we define:

$$s_j = a_j + b_j c_j, \; j = 1,2$$

$$\tilde{s}_j = \tilde{a}_j + \tilde{b}_j \tilde{c}_j, \; j = 1,2$$

$$v = \left| N_{m+1} - N_m + 1 \right|$$

and

$$H(r_j, v) = \sum_{i=0}^{i=v} r_j^i \text{ for } r_j = \frac{\tilde{s}_j}{s_j}, \; j = 1,2$$

Table A. Expressions for $P_D(X_{m+1}, X_m)$ *for each pair* (X_{m+1}, X_m)

Region	X_m	X_{m+1}	$P(X_m, X_{m+1})$
1	$(N_m,1,1)$ for $0 < N_m \le K_1 - 1$	$(N_{m+1},1,1)$ for $N_m - 1 \le N_{m+1} \le K_1 - 2$	$a_2 s_1^{\ v} + b_2 c_2 G(\tilde{s}_1/s_1, v) + b_1 \tilde{c}_1 \tilde{b}_2 c_2 G(\tilde{s}_1/s_1, v\text{-}1)$
	$(N_m,1,1)$ for $N_m = 0$	$(N_{m+1},1,1)$ for $0 < N_{m+1} \le K_1 - 2$	Same as above
	$(N_m,1,1)$ for $0 > N_m \ge -K_2 + 1$	$(N_{m+1},1,1)$ for $N_m + 1 \ge N_{m+1} \ge -K_2 + 2$	$a_1 s_2^{\ v} + b_1 c_1 G(\tilde{s}_2/s_2, v) + \tilde{b}_1 c_1 b_2 \tilde{c}_2 G(\tilde{s}_2/s_2, v\text{-}1)$
	$(N_m,1,1)$ for $N_m = 0$	$(N_{m+1},1,1)$ for $0 > N_{m+1} \ge -K_2 + 2$	Same as above
2	$(N_m,2,1)$ for $0 < N_m \le K_1 - 2$	$(N_{m+1},2,1)$ for $N_m - 1 < N_{m+1} \le K_1 - 3$	$b_1 c_1 \tilde{a}_2 s_1^{\ v-1} + \tilde{b}_1 \tilde{c}_1 \tilde{b}_2 \tilde{c}_2 \tilde{s}_1^{\ v-1}$ $+ \tilde{b}_1 c_1 \tilde{c}_2 \left(b_2 G(\tilde{s}_1/s_1, v\text{-}1) + b_1 \tilde{c}_1 \tilde{b}_2 G(\tilde{s}_1/s_1, v\text{-}2)\right)$
	$(N_m,2,1)$ for $N_m = 0$	$(N_{m+1},2,1)$ for $0 < N_{m+1} \le K_1 - 3$	Same as above
	$(N_m,2,1)$ for $0 < N_m \le K_1 - 2$	$(N_{m+1},2,1)$ for $N_{m+1} = N_m - 1$	$\tilde{s}_2$
	$(N_m,1,2)$ for $0 > N_m \ge -K_2 + 2$	$(N_{m+1},1,2)$ for $N_m + 1 > N_{m+1} \ge -K_2 + 3$	$c_2 b_2 \tilde{s}_1 s_2^{\ v-1} + \tilde{b}_1 \tilde{c}_1 \tilde{b}_2 \tilde{c}_2 \tilde{s}_2^{\ v-1}$ $+ \tilde{c}_1 \tilde{b}_2 c_2 \left(b_1 G(\tilde{s}_2/s_2, v\text{-}1) + \tilde{b}_1 b_2 \tilde{c}_2 G(\tilde{s}_2/s_2, v\text{-}2)\right)$
	$(N_m,1,2)$ for $N_m = 0$	$(N_{m+1},1,2)$ for $0 > N_{m+1} \ge -K_2 + 3$	Same as above
	$(N_m,1,2)$ for $0 > N_m \ge -K_2 + 2$	$(N_{m+1},1,2)$ for $N_{m+1} = N_m + 1$	$\tilde{s}_1$
3	$(N_m,1,1)$ for $0 < N_m \le K_1 - 1$	$(N_{m+1},2,1)$ for $N_m - 1 < N_{m+1} \le K_1 - 3$	$b_1 \tilde{s}_2 s_1^{\ v} + \tilde{b}_1 \tilde{c}_2 \left(b_2 G(\tilde{s}_1/s_1, v) + b_1 \tilde{c}_1 \tilde{b}_2 G(\tilde{s}_1/s_1, v\text{-}1)\right)$
	$(N_m,1,1)$ for $N_m = 0$	$(N_{m+1},2,1)$ for $0 < N_{m+1} \le K_1 - 3$	Same as above
	$(N_m,1,1)$ for $0 > N_m \ge -K_2 + 1$	$(N_{m+1},1,2)$ for $N_m + 1 > N_{m+1} \ge -K_2 + 3$	$b_2 \tilde{s}_1 s_2^{\ v} + \tilde{c}_1 \tilde{b}_2 \left(b_1 G(\tilde{s}_2/s_2, v) + \tilde{b}_1 b_2 \tilde{c}_2 G(\tilde{s}_2/s_2, v\text{-}1)\right)$
	$(N_m,1,1)$ for $N_m = 0$	$(N_{m+1},1,2)$ for $0 > N_{m+1} \ge -K_2 + 3$	Same as above

Table A. Expressions for $P_D\left(X_{m+1}, X_m\right)$ for each pair $\left(X_{m+1}, X_m\right)$contd.

Region	X_m	X_{m+1}	$P\left(X_m, X_{m+1}\right)$
4	$(N_m,2,1)$ for $0 < N_m \le K_1 - 2$	$(N_{m+1},1,1)$ for $N_m - 1 < N_{m+1} \le K_1 - 2$	$\left(c_1 a_2 + \tilde{c}_1 \tilde{b}_2 c_2\right) s_1^{v-1} + c_1 c_2 b_2 G\left(\tilde{s}_1/s_1, v-1\right)$ $+ c_1 b_1 \tilde{c}_1 \tilde{b}_2 c_2 G\left(\tilde{s}_1/s_1, v-2\right)$
	$(N_m,2,1)$ for $N_m = 0$	$(N_{m+1},1,1)$ for $0 < N_{m+1} \le K_1 - 2$	Same as above
	$(N_m,2,1)$ for $0 < N_m \le K_1 - 2$	$(N_{m+1},1,1)$ for $N_{m+1} = N_m - 1$	0
	$(N_m,1,2)$ for $0 > N_m \ge -K_2 + 2$	$(N_{m+1},1,1)$ for $N_m + 1 > N_{m+1} \ge -K_2 + 2$	$a_1 c_2 s_2^{v-1} + c_1 \tilde{c}_2 \tilde{b}_1 \tilde{s}_2^{v-1} +$ $c_1 c_2 \left(b_1 G\left(\tilde{s}_2/s_2, v-1\right) + \tilde{c}_2 \tilde{b}_1 b_2 G\left(\tilde{s}_2/s_2, v-2\right)\right)$
	$(N_m,1,2)$ for $N_m = 0$	$(N_{m+1},1,1)$ for $0 > N_{m+1} \ge -K_2 + 2$	Same as above
	$(N_m,1,2)$ for $0 > N_m \ge -K_2 + 2$	$(N_{m+1},1,1)$ for $N_{m+1} = N_m + 1$	0
5	$(N_m,1,1)$ for $0 \le N_m \le K_1 - 1$	$(K_1 - 1,1,1)$	$\left(1-b_2\right) s_1^{v} + b_2 G\left(\tilde{s}_1/s_1, v\right) + b_1 \tilde{c}_1 \tilde{b}_2 G\left(\tilde{s}_1/s_1, v-1\right)$
	$(N_m,1,1)$ for $0 \ge N_m \ge -K_2 + 1$	$(-K_2 + 1,1,1)$	$\left(1-b_1\right) s_2^{v} + b_1 G\left(\tilde{s}_2/s_2, v\right) + \tilde{b}_1 b_2 \tilde{c}_2 G\left(\tilde{s}_2/s_2, v-1\right)$
6	$(N_m,2,1)$ for $0 \le N_m \le K_1 - 2$	$(K_1 - 2,2,1)$	$c_1 b_1 \tilde{s}_2 s_1^{v-1} + \tilde{b}_1 \tilde{c}_1 \tilde{b}_2 \tilde{c}_2 \tilde{s}_1^{v-1}$ $+ \tilde{b}_1 c_1 \tilde{c}_2 \left(b_2 G\left(\tilde{s}_1/s_1, v-1\right) + b_1 \tilde{c}_1 \tilde{b}_2 G\left(\tilde{s}_1/s_1, v-2\right)\right)$
	$(N_m,1,2)$ for $0 \ge N_m \ge -K_2 + 2$	$(-K_2 + 2,1,2)$	$b_2 c_2 \tilde{s}_1 s_2^{v-1} + \tilde{b}_1 \tilde{c}_1 \tilde{b}_2 \tilde{c}_2 \tilde{s}_2^{v-1} +$ $\tilde{c}_1 \tilde{b}_2 c_2 \left(b_1 G\left(\tilde{s}_2/s_2, v-1\right) + \tilde{b}_1 b_2 \tilde{c}_2 G\left(\tilde{s}_2/s_2, v-2\right)\right)$
7	$(N_m,1,1)$ for $0 \le N_m \le K_1 - 1$	$(K_1 - 2,2,1)$	$b_1 \tilde{s}_2 s_1^{v} + \tilde{b}_1 \tilde{c}_2 \left(b_2 G\left(\tilde{s}_1/s_1, v\right) + \tilde{c}_1 \tilde{b}_2 b_1 G\left(\tilde{s}_1/s_1, v-1\right)\right)$
	$(N_m,1,1)$ for $0 \ge N_m \ge -K_2 + 1$	$(-K_2 + 2,1,2)$	$b_2 \tilde{s}_1 s_2^{v} + \tilde{c}_1 \tilde{b}_2 \left(b_1 G\left(\tilde{s}_2/s_2, v\right) + \tilde{b}_1 b_2 \tilde{c}_2 G\left(\tilde{s}_2/s_2, v-1\right)\right)$
8	$(N_m,2,1)$ for $0 \le N_m \le K_1 - 2$	$(K_1 - 1,1,1)$	$\left(1-b_2\right) c_1 s_1^{v-1} + \tilde{c}_1 \tilde{b}_2 \tilde{s}_1^{v-1}$ $+ c_1 b_2 G\left(\tilde{s}_1/s_1, v-1\right) + \tilde{c}_1 \tilde{b}_2 b_1 c_1 G\left(\tilde{s}_1/s_1, v-2\right)$
	$(N_m,1,2)$ for $0 \ge N_m \ge -K_2 + 2$	$(-K_2 + 1,1,1)$	$\left(1-b_1\right) c_2 s_2^{v-1} + \tilde{b}_1 \tilde{c}_2 \tilde{s}_2^{v-1} +$ $b_1 c_2 G\left(\tilde{s}_2/s_2, v-1\right) + \tilde{b}_1 b_2 c_2 \tilde{c}_2 G\left(\tilde{s}_2/s_2, v-2\right)$

Table A. Expressions for $P_D(X_{m+1}, X_m)$ for each pair (X_{m+1}, X_m)contd.

Region	X_m	X_{m+1}	$P(X_m, X_{m+1})$
9	(0,1,2)	$(N_{m+1},1,1)$ for $0 < N_{m+1} \le K_1 - 2$	$c_2 \tilde{s}_1^{\,v}$
	(0,2,1)	$(N_{m+1},1,1)$ for $0 > N_{m+1} \ge -K_2 + 2$	$c_1 \tilde{s}_2^{\,v}$
10	(0,1,2)	$(K_1 - 1,1,1)$	$\tilde{s}_1^{\,v}$
	(0,2,1)	$(-K_2 + 1,1,1)$	$\tilde{s}_2^{\,v}$
11	(0,1,2)	$(N_{m+1},2,1)$ for $0 < N_{m+1} \le K_1 - 2$	$\tilde{b}_1 \tilde{c}_2 \tilde{s}_1^{\,v}$
	(0,2,1)	$(N_{m+1},1,2)$ for $0 > N_{m+1} \ge -K_2 + 2$	$\tilde{b}_2 \tilde{c}_1 \tilde{s}_2^{\,v}$
12	(0,2,1)	(0,2,1)	$b_1 c_1 \tilde{s}_2 + \tilde{b}_1 \tilde{c}_2 \left(\tilde{b}_2 \tilde{c}_1 + b_2 c_1\right)$
	(0,2,1)	(0,1,2)	$\tilde{b}_2 \tilde{c}_1 \tilde{s}_2$
	(0,2,1)	(0,1,1)	$c_1 (s_2 + \tilde{s}_2) + \tilde{b}_2 \tilde{c}_1 c_2$
13	(0,1,2)	(0,1,2)	$\tilde{b}_2 \tilde{c}_1 \left(b_1 c_2 + \tilde{b}_1 \tilde{c}_2\right) + b_2 c_2 \tilde{s}_1$
	(0,1,2)	(0,2,1)	$\tilde{b}_1 \tilde{c}_2 \tilde{s}_2$
	(0,1,2)	(0,1,1)	$c_2 (s_1 + \tilde{s}_1) + \tilde{b}_1 \tilde{c}_2 c_1$
14	(0,1,1)	(0,1,1)	$2 s_1 s_2 + b_1 c_1 \tilde{s}_2 + c_2 b_2 \tilde{s}_1 + \tilde{b}_2 \tilde{c}_1 b_1 c_2 + \tilde{b}_1 c_1 b_2 \tilde{c}_2$
	(0,1,1)	(0,2,1)	$b_1 s_1 \tilde{s}_2 + b_2 \tilde{b}_1 \tilde{c}_2 (s_1 + \tilde{s}_1) + b_1 \tilde{b}_1 \tilde{c}_1 \tilde{b}_2 \tilde{c}_2$
	(0,1,1)	(0,1,2)	$b_2 \tilde{s}_1 s_2 + b_1 \tilde{b}_2 \tilde{c}_1 (s_2 + \tilde{s}_2) + b_2 \tilde{b}_1 \tilde{b}_2 \tilde{c}_1 \tilde{c}_2$

References

Altiok, T., 1996, "*Performance Analysis of Manufacturing Systems*", Springer Series in Operations Research, Springer, New York.

Bacelli, F., Massey, W.A., and Towsley, 1989, D., "Acyclic fork/join queuing networks", *Journal of ACM*, 36, 615-642.

Bhat, U.N., 1986, "Finite capacity assembly like queues", *Queuing Systems*, 1, 185-101.

Buzacott, J.A., and Shanthikumar, J. G., 1993, *Stochastic Models of Manufacturing Systems*, Prentice-Hall, Englewood Cliffs, NJ.

Di Mascolo, M., Frein, Y., and Dallery, Y., 1996, "An Analytical Method for Performance Evaluation of Kanban Controlled Production Systems", *Operations Research*, 44(1), 50-64.

Disney, R.L., and Kiessler, R.L., *Traffic Processes in Queuing Networks: a Markov Renewal Approach*, Johns Hopkins University Press, 1987.

Harrison, J.M., 1973, "Assembly-like queues", *Journal of Applied Probability*, 10(2), 354-367.

Hopp, W.J., and Simon, J.T., 1989, "Bounds and heuristics for assembly-like queues", *Queuing Systems*, 4,137-156.

Kamath, M., Suri, R., Sanders, J.L., 1988, "Analytical Performance Models for Closed – Loop Flexible Assembly Systems", *International Journal of Flexible Manufacturing Systems*, 1(1), 51-84.

Kashyap, B.R.K., 1965, "A double ended queuing system with limited waiting space", *Proceedings of the National Institute of Science (India)*, 31, 559-570.

Latouche, G., 1981, "Queues with paired customers", *Journal of Applied Probability*, 18, 684-696.

Lipper, E.H., and Sengupta, B., 1986, "Assembly like queues with finite capacity: bounds, asymptotic and approximations", *Queuing Systems*, 1, 67-83.

Rao, P.C. and Suri, R., 1994, "Approximate queuing network models of fabrication/assembly systems: Part I-Single level systems", *Production and Operations Management*, 3, 244-275.

Rao, P.C. and Suri, R., 2000, "Performance analysis of an assembly station with input from multiple fabrication lines", *Production and Operations Management*, 9(3), 283-302.

Som, P., Wilhelm, W.E., and Disney, R.L., 1994, “Kitting process in a stochastic assembly system”, *Queuing Systems*, 17, 471-490.

Takahashi, M., Osawa, H., Fujisawa, T., 1996, “A stochastic assembly system with resume levels”, *Asia-Pacific Journal of Operations Research*, 15, 127-146.

Takahashi, M., Osawa, H., Fujisawa, T., 2000a, “On a synchronization queue with two finite buffers”, *Queuing Systems*, 36, 107-123.

Takahashi, M., Takahashi, Y., T., 2000b, “Synchronization queue with two MAP inputs and finite buffers”, *Proceedings of the Third International Conference on Matrix Analytical Methods in Stochastic Models*, *Leuven (Belgium)*, 375-390.

Varki, E., 1999, “Mean value technique for closed fork-join networks”, *Proceedings of ACM SIGMETRICS Conference on Measurement and Modeling of Computer Systems, Atlanta, GA*, 103-112.