Gap functions and Lyapunov functions*

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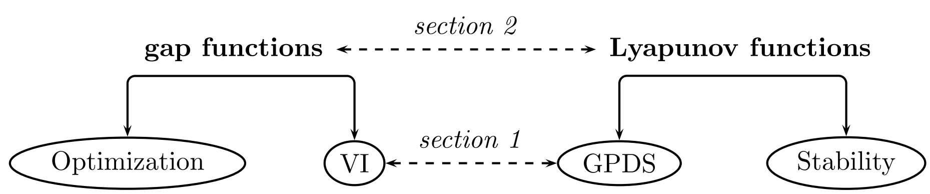
Abstract: Equilibrium problems play a central role in the study of complex and competitive systems. Many variational formulations of these problems have been presented in these years. So, variational inequalities are very useful tools for the study of equilibrium solutions and their stability. More recently a dynamical model of equilibrium problems based on projection operators was proposed. It is designated as globally projected dynamical system (GPDS). The equilibrium points of this system are the solutions to the associated variational inequality (VI) problem. A very popular approach for finding solution of these VI and for studying its stability consists in introducing the so-called "gap functions", while stability analysis of an equilibrium point of dynamical systems can be made by means of Lyapunov functions. In this paper we show strict relationships between gap functions and Lyapunov functions.



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1 Variational inequalities and globally projected dynamical systems

Definition 1.1. For a closed convex set $K \subseteq \mathbb{R}^n$ and vector function $F : K \to \mathbb{R}^n$, the variational inequality VI(F,K) is to determine a vector $x^* \in K$, such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall\, x \in K,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

We focus our attention on the so-called globally projected dynamical systems (see Friesz et al (1994), Pappalardo and Passacantando (2002), Xia and Wang (2000)). Given a closed convex set $K \subset \mathbb{R}^n$, we denote P_K the usual projection operator:

$$P_K(x) = \arg\min_{y \in K} \|x - y\|.$$

Definition 1.2. We define the globally projected dynamical system GPDS(F,K,α) as the ordinary differential equation

$$\dot{x} = P_K(x - \alpha\, F(x)) - x,$$

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where α is a positive constant, $K \subseteq \mathbb{R}^n$ is a closed convex set, and F is a continuous vector field defined on K.

We remark that the right-hand side of GPDS(F,K,α) is continuous on K and it can be different from $-F(x)$ even if x is an interior point to K.

The equilibrium points of GPDS(F,K,α) are naturally defined as the vectors $x^* \in K$ such that

$$P_K(x^* - \alpha\, F(x^*)) - x^* = 0.$$

It is well known that $x^* \in K$ is solution to VI(F,K) if and only if for any $\alpha > 0$ one has

$$x^* = P_K(x^* - \alpha\, F(x^*)).$$

Therefore for any $\alpha > 0$ the equilibrium points of the GPDS(F,K,α) coincide with the solutions of VI(F,K).

Recall now a definition about the stability of an equilibrium point; we will use $B(x, r)$ to denote the open ball with center x and radius r.

Definition 1.3.

1. An equilibrium point x^* of GPDS(F,K,α) is called *stable* if for any $\varepsilon > 0$ there is a $\delta > 0$ such that, for every $x \in B(x^*, \delta)$ the solution $x(t)$ of GPDS(F,K,α) with $x(0) = x$ is defined and $x(t) \in B(x^*, \varepsilon)$ for all $t > 0$.

2. A stable equilibrium point x^* is called *asymptotically stable* if there is a $\delta > 0$ such that for every solution $x(t)$ with $x(0) \in B(x^*, \delta)$ one has $\lim_{t\to+\infty} x(t) = x^*$.

2 Lyapunov functions and gap functions

Stability analysis of an equilibrium point of a classical (i.e. not projected) dynamical systems can be made by means of Lyapunov function method (see Hirsch and Smale (1974), La Salle and Lefschetz (1961)). We show that this tool can be used also for a globally projected dynamical system.

Definition 2.1. Let x^* an equilibrium point of GPDS(F,K,α). If $V : U \to \mathbb{R}$ is a continuous function defined on a neighborhood U of x^*, differentiable on $U \backslash \{x^*\}$, such that

1. $V(x^*) = 0$ and $V(x) > 0$ if $x \neq x^*$;

2. for all $x \in U$, if $x(t)$ is solution to GPDS(F,K,α) with $x(0) = x$, then

$$\frac{d}{dt} V(x(t))|_{t=0} \leq 0;$$

then V is called *Lyapunov function* for x^*;

if for all $x \neq x^*$ the solution $x(t)$ to GPDS(F,K,α) passing through x when $t = 0$ is such that

$$\frac{d}{dt} V(x(t))|_{t=0} < 0,$$

then V is called *strict Lyapunov function* for x^*.

A Lyapunov function for x^* has a local minimum at x^* and it is nonincreasing along the solutions of GPDS(F,K,α). The existence of a Lyapunov function for an equilibrium point guarantees stability properties for it.

Proposition 2.1. Let x^* be an equilibrium point of GPDS(F,K,α). If there exists a Lyapunov function V for x^* then x^* is stable; furthermore if V is a strict Lyapunov function, then x^* is asymptotically stable.

Proof. Let $\varepsilon > 0$ be so small that the closed ball around x^* of radius ε lies entirely in U. Let $k > 0$ be the minimum value of V on the boundary of $B(x^*, \varepsilon)$, that is $\partial B(x^*, \varepsilon)$. If we consider the neighborhood of x^*

$$Q = \{x \in B(x^*, \varepsilon) : \ V(x) < k\},$$

then no solution starting in Q can meet $\partial B(x^*, \varepsilon)$ since V is nonincreasing on solution curves. Hence every solution starting in Q never leaves $B(x^*, \varepsilon)$, thus x^* is stable.

Now we suppose that V is a strict Lyapunov function. Let $x(t)$ an arbitrary solution starting in Q, since V is decreasing along $x(t)$, we have $\lim_{t \to +\infty} V(x(t)) = l \geq 0$. If $l > 0$ then there is $\delta < \varepsilon$ such that

$$0 \leq V(x) < l \qquad \forall\, x \in B(x^*, \delta),$$

hence $x(t)$ lies in the closed annulus $C(\delta, \varepsilon)$ for all $t \geq 0$. Since the continuous function

$$g(x) = \langle F(x), \nabla V(x) \rangle < 0 \qquad \forall\, x \in U,$$

g has a maximum value $M < 0$ on $C(\delta, \varepsilon)$, then

$$\frac{d}{dt} V(x(t)) = g(x(t)) \leq M < 0 \qquad \forall\, t \geq 0,$$

and thus $\lim_{t \to +\infty} V(x(t)) = -\infty$. This is impossible, hence $l = 0$, that is $x(t) \to x^*$ and the proof is complete. $\square$

A gap (or merit) function for a variational inequality provides an equivalent optimization formulation for it, which can be solved by minimizing the gap function. The gap function introduced in Fukushima (1992):

$$g_\beta(x) = \sup_{y \in K} [\langle F(x), x - y \rangle - \beta \, \|y - x\|^2], \qquad \beta > 0,$$

is such that:

- $g_\beta(x) \geq 0$ for all $x \in K$;
- $g_\beta(x^*) = 0$ if and only if x^* is solution to VI(F,K);

i.e. the VI(F,K) is equivalent to the optimization problem

$$\text{minimize} \quad g_\beta(x) \quad \text{subject to} \ \ x \in K$$

Moreover, if the operator F is differentiable on $\mathbb{R}^n$, then g_β is differentiable too and

$$\nabla g_\beta(x) = F(x) - [JF(x) - \beta\, I](P_K(x - \beta^{-1} F(x)) - x) \qquad \forall\, x \in \mathbb{R}^n.$$

We now show the main result of this paper. Under mild conditions on jacobian matrix JF, this gap function for VI(F,K) is a Lyapunov function for a suitable choice of the parameter for an isolated equilibrium point of GPDS(F,K,α).

Theorem 2.1. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathcal{C}^1$ vector field and x^* is an isolated equilibrium point of GPDS(F,K,α). If JF(x) is positive semidefinite (definite) for all x in a neighborhood of x^*, then the gap function $g_{\frac{1}{\alpha}}$ is a (strict) Lyapunov function for x^*.

Proof. Since x^* is isolated then there exists a neighborhood $N(x^*)$ of x^* in K such that $g_{\frac{1}{\alpha}}(x) > 0$ for all $x \in N(x^*) \backslash \{x^*\}$ and $g_{\frac{1}{\alpha}}(x^*) = 0$. Moreover let $x \in N(x^*)$ be arbitrary and $x(t)$ the solution of

GPDS(F,K,α) with $x(0) = x$, then

$$\begin{aligned}
&\frac{d}{dt} g_{\frac{1}{\alpha}}(x(t))|_{t=0} = \langle \nabla g_{\frac{1}{\alpha}}(x), \dot{x}(0)\rangle = \\
&= \frac{1}{\alpha} \langle P_K(x - \alpha F(x)) - (x - \alpha F(x)), P_K(x - \alpha F(x)) - x\rangle - \\
&-\langle JF(x)(P_K(x - \alpha F(x)) - x), (P_K(x - \alpha F(x)) - x)\rangle \leq \\
&\leq -\langle JF(x)(P_K(x - \alpha F(x)) - x), (P_K(x - \alpha F(x)) - x)\rangle \leq 0.
\end{aligned}$$

Therefore if JF(x) is positive semidefinite (definite) then $g_{\frac{1}{\alpha}}$ is a (strict) Lyapunov function for x^*. □

3 Gradient systems

A gradient system, denoted by DS(∇V), is a classical (i.e. not projected) dynamical system of the form

$$\dot{x} = -\nabla V(x),$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^2$ function.

Using the implicit function theorem, we find that at regular points of V ($\nabla V(x) \neq 0$) the vector field $-\nabla V(x)$ is perpendicular to the level surfaces of V. Since the trajectories of the gradient system are tangent to $-\nabla V(x)$, at regular points of V the trajectories cross level surfaces orthogonally. Moreover equilibrium points of DS(∇V) are stationary points of V.

We observe that in a gradient system, Lyapunov functions are defined naturally as $V(x)+$ constant; indeed if $x(t)$ is solution of DS(∇V) then

$$\frac{d}{dt} V(x(t)) = -\|\nabla V(x(t))\|^2 \leq 0.$$

The second result of this paper is the following: if we consider an isolated equilibrium point of a gradient system DS(∇V), then it is stable if and only if it is a local minimum of function V, moreover stability is equivalent to asymptotic stability, as the theorem shows.

Theorem 3.1. Let x^* be an isolated equilibrium point of DS(∇V). Then the following statements are equivalent:

1. x^* is local minimum of V;
2. x^* is stable for DS(∇V);
3. x^* is asymptotically stable for DS(∇V);.

Proof.

$1 \Longrightarrow 3$. Since x^* is an isolated local minimum of V, then the function $x \to V(x) - V(x^*)$ is a strict Lyapunov function for x^*, in some neighborhood of x^*. By proposition 2.1 x^* is asymptotically stable.

$3 \Longrightarrow 2$. By definition.

$2 \Longrightarrow 1$. Since x^* is isolated then there is $\varepsilon > 0$ such that x^* is the only equilibrium point in $B(x^*, \varepsilon)$. Since x^* is stable, there is $\delta < \varepsilon$ such that every solution starting in $B(x^*, \delta)$ never leaves $B(x^*, \varepsilon)$. We assert that $V(x) \geq V(x^*)$ for all $x \in B(x^*\delta)$.

We suppose by contradiction that there is $y \in B(x^*, \delta)$ such that $V(y) < V(x^*)$, then we take account of the solution $y(t)$ passing through y, and we pose $h(t) = V(y(t))$. For all $t \geq 0$ we have

$$h'(t) = \langle \nabla V(y(t)), \dot{y}(t)\rangle = -\|\nabla V(y(t))\|^2 < 0,$$

namely h is decreasing. We call $\xi = V(x^*) - V(y)$, by continuity there is η such that $0 < \eta < \delta$ and

$$V(x^*) - \frac{\xi}{2} \leq V(x) \leq V(x^*) + \frac{\xi}{2} \qquad \forall\, x \in B(x^*, \eta).$$

Moreover by stability we have $y(t) \in B(x^*, \varepsilon)$ for all $t \geq 0$, and also $y(t) \notin B(x^*, \eta)$ for all $t \geq 0$ since

$$V(y(t)) < V(y) = V(x^*) - \xi < V(x^*) - \frac{\xi}{2},$$

thus $y(t)$ belongs to the closed annulus $C(\eta, \varepsilon)$ for all $t \geq 0$. Since $C(\eta, \varepsilon)$ is compact we have $\min\limits_{x \in C(\eta,\varepsilon)} \|\nabla V(x)\| = m > 0$, then

$$h'(t) = -\|\nabla V(y(t))\|^2 \leq -m^2 < 0 \qquad \forall\, t \geq 0,$$

thus

$$\lim_{t \to +\infty} h(t) = -\infty,$$

but this is impossible because V is lower bounded on $B(x^*, \varepsilon)$. □

Corollary 3.1. If V is a convex function, then every isolated equilibrium point of DS(∇V) is asymptotically stable.

Now, we consider GPDS(∇V,K,α). Its equilibrium points coincide with solutions of VI(∇V,K), i.e. the stationary points of V on K.

We can generalize theorem 3.1 for GPDS(∇V,K,α).

Theorem 3.2. Let x^* be an isolated equilibrium point of GPDS(∇V,K,α). The following statements are equivalent:

1. x^* is local minimum of V on K;
2. x^* is stable for GPDS(∇V,K,α);
3. x^* is asymptotically stable for GPDS(∇V,K,α).

Proof.

$1 \Longrightarrow 3$ and $3 \Longrightarrow 2$. As in the theorem 3.1.

$2 \Longrightarrow 1$. The proof scheme is the same as theorem 3.1, the only difference is the derivative of $h(t) = V(y(t))$:

$$\begin{aligned} h'(t) &= \langle \nabla V(y(t)), P_K(y(t) - \alpha\, \nabla V(y(t))) - y(t) \rangle \leq \\ &\leq -\frac{1}{\alpha} \|P_K(y(t) - \alpha\, \nabla V(y(t))) - y(t)\|^2 < 0. \end{aligned}$$

Thus $y(t)$ belongs to some closed annulus $C(\eta, \varepsilon)$ for all $t \geq 0$ and $\min\limits_{x \in C(\eta,\varepsilon)} \|P_K(x - \alpha\, \nabla V(x)) - x\| > 0$, thus

$$\lim_{t \to +\infty} h(t) = -\infty,$$

but this is impossible because V is lower bounded on $B(x^*, \varepsilon)$. □

We summarize the relations among GPDS($\nabla V, K, \alpha$), the function V and VI($\nabla V, K$) in the following table:

GPDS($\nabla V, K, \alpha$)		Function V		VI($\nabla V, K$)
equilibrium point	$\Longleftrightarrow$	stationary point	$\Longleftrightarrow$	solution
$\Uparrow$		$\Uparrow$		
stable equilibrium point	$\Longleftrightarrow$	local minimum		
$\Updownarrow$				
asymptotically stable equilibrium point				

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