# Interval Computations and their Categorification 

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#### Abstract

By the example of the proof of Minkowski's conjecture on critical determinant we give a category theory framework for interval computation.


## 1 Introduction

The purpose of this paper is to describe interval computations under the proof of Minkowski's conjecture on the critical determinant of the region $|x|^{p}+|y|^{p}<$ $1, p>1$ in terms of (certain kind of) categories and functors. Let $D_{p} \subset \mathbf{R}^{2}=$ $(x, y), p>1$ be the 2 -dimensional region:

$$
|x|^{p}+|y|^{p}<1
$$

The well known Minkowski conjecture [1, 3, 4, 5, 6, 7] about critical determinant of the region $D_{p}$ can be reformulated as the problem of minimization on moduli space $\mathcal{M}$ of admissible lattices of the region $D_{p}$ [10]. The moduli space is defined by the equation

$$
\begin{equation*}
\Delta(p, \sigma)=(\tau+\sigma)\left(1+\tau^{p}\right)^{-\frac{1}{p}}\left(1+\sigma^{p}\right)^{-\frac{1}{p}} \tag{1}
\end{equation*}
$$

in the domain

$$
D_{p}: \infty>p>1,1 \leq \sigma \leq \sigma_{p}=\left(2^{p}-1\right)^{\frac{1}{p}}
$$

of the $\{p, \sigma\}$ plane, where $\sigma$ is some real parameter; here $\tau=\tau(p, \sigma)$ is the function uniquely determined by the conditions

$$
A^{p}+B^{p}=1,0 \leq \tau \leq \tau_{p}
$$

where
$A=A(p, \sigma)=\left(1+\tau^{p}\right)^{-\frac{1}{p}}-\left(1+\sigma^{p}\right)^{-\frac{1}{p}}, B=B(p, \sigma)=\sigma\left(1+\sigma^{p}\right)^{-\frac{1}{p}}+\tau\left(1+\tau^{p}\right)^{-\frac{1}{p}}$,
$\tau_{p}$ is defined by the equation $2\left(1-\tau_{p}\right)^{p}=1+\tau_{p}^{p}, 0 \leq \tau_{p}<1$.
Minkowski's analytic conjecture:
For any real $p$ and $\tau$ with conditions $p>1, p \neq 2,0<\tau<\tau_{p}$

$$
\begin{equation*}
\Delta(p, \sigma)>\min \left(\Delta(p, 1), \Delta\left(p, \sigma_{p}\right)\right) \tag{2}
\end{equation*}
$$

A.V. Malyshev and I have proved by interval computation

Theorem. For all $p \geq 1.01, p \neq 2$ and for all $1<\sigma<\sigma_{p}$ with the exception of $\{p, \sigma\}$ from the domain $P_{2, \sqrt{(3)}}=\left\{2 \leq p \leq 2.000003, d \leq \sigma \leq \sigma_{p}\right\} d=$ 1.7320503 the inequality (2) takes place.

Nondifferential interval method of computation of implicitly defined functions is one of the ingredients of this proof. It can be interpreted as a contracting map on the interval space. The map is defined by some paths of the corresponding program.
The category theory view on manifolds in mathematics and on programs in computer science as well as interval computations on Minkowski moduli space led us to the necessity of the introduction of interval manifolds, presheaves on them and functors from path categories of programs to the interval categories. Using these and some other notions, we present a category theory framework for interval computation. Proofs are omitted.

## 2 Lattices, Admissible Lattices and Critical determinants

In this section we recall notions of lattices, admissible lattices and critical determinants and some of their properties [2].
Let $a_{1}, \ldots, a_{n}$ be the independent points (a basis) of $\mathbf{R}^{n}$. The set $\Lambda$ of points

$$
x=u_{1} a_{1}+u_{2} a_{2}+\ldots+u_{n} a_{n},\left(u_{1}, \ldots, u_{n} \text { integers }\right)
$$

is called a lattice. The system of points $a_{1}, \ldots, a_{n}$ is called a basis of $\Lambda$.
If $\Lambda$ is a lattice and $A$ is a basis of $\Lambda$, then $|\operatorname{det} A|$ is called the determinant of $\Lambda$. It is denoted by $d(\Lambda)$.

Let $M$ be an arbitrary set in $\mathbf{R}^{n}, O=(0,0) \in \mathbf{R}^{n}$. A lattice $\Lambda$ is called admissible for $M$, or $M$-admissible, if it has no points $\neq \mathrm{O}$ in the interior of $M$. It is called strictly admissible for $M$ if it does not contain a point $\neq \mathrm{O}$ of $M$.

The critical determinant of a set $M$ is the quantity $\Delta(M)$ given by

$$
\Delta(M)=\inf \{d(\Lambda): \Lambda \text { strictly admissible for } M\}
$$

with the understanding that $\Delta(M)=\infty$ if there are no strictly admissible lattices. The set $M$ is said to be of the finite or the infinity type according to whether $\Delta(M)$ is finite or infinite.

Of course, we may also consider the greatest lower bound of $d(\Delta)$ on the collection of all $M$-admissible lattices. We put

$$
\Delta_{0}(M)=\inf \{d(\Lambda): \Lambda \text { admissible for } M\}
$$

Then clearly, $\Delta_{0}(M) \leq \Delta(M)$. Here, for large classes of sets, the equality sing hold. For the class of open sets, this is trivial. As we consider open sets, we will denote the critical determinants by $\Delta(M)$.

## 3 Interval Cellular Covering

For any $n$ and any $j, 0 \leq j \leq n$, an $j$-dimensional interval cell, or $j-I$-cell, in $\mathbf{R}^{n}$ is a subset $I c$ of $\mathbf{R}^{n}$ such that (possibly, after permutation of variables) it has the form
$I c=\left\{x \in \mathbf{R}^{n}: \underline{a}_{i}, \bar{a}_{i}, r_{k} \in \mathbf{R}:\right.$
$\underline{a}_{i} \leq x_{i} \leq \bar{a}_{i}, 1 \leq i \leq j$,
$\left.x_{j+1}=r_{1}, \cdots, x_{n}=r_{n-j}\right\}$. Here $\underline{a}_{i} \leq \bar{a}_{i}$.

If $j=n$ then we have an $n$-dimensional interval vector. Let $\mathcal{P}$ be the hyperplane that contains $I c$. These is the well known fact:

Lemma 1 The dimension of Ic is equal to the minimal dimension of hyperplanes that contain Ic.

Let $\mathcal{P}$ be the such hyperplane, Int $I c$ the set of interior points of $I c$ in $\mathcal{P}, B d I c=$ $I c \backslash I n t I c$. For $m$-dimensional I-cell $I c$ let $d_{i}$ be an $(m-1)$-dimensional I-cell from $B d I c$. Then $d_{i}$ is called an $(m-1)$-dimensional face of the I-cell $I c$.

Definition 1 Let $D$ be a bounded set in $\mathbf{R}^{n}$. By interval cellular covering Cov we will understand any finite set of $n$-dimensional I-cells such that their union contains $D$ and adjacent I-cells are intersected by their faces only. By $|\operatorname{Cov}|$ we will denote the union of all I-cells from Cov.

Let Cov be the interval covering. By its subdivision we will understand an interval covering Cov such that $|\operatorname{Cov}|=\left|\operatorname{Cov}^{*}\right|$ and each I-cell from Cov is contained in an I-cell from Cov. In the paper we will consider mainly bounded horizontal and vertical strips in $\mathbf{R}^{2}$, their interval coverings and subdivisions. Let $\mathbf{X}=\left(\mathbf{x}_{1}, \cdots, \mathbf{x}_{n}\right)=\left(\left[\underline{x}_{1}, \bar{x}_{1}\right], \cdots,\left[\underline{x}_{n}, \bar{x}_{n}\right]\right.$ be the n-dimensional real interval vector with $\underline{x}_{i} \leq x_{i} \leq \bar{x}_{i}$ ("rectangle" or "box"). Let $f$ be a real continuous function of $n$ variables that is defined on $\mathbf{X}, O f-$ its optimal interval evaluation 8 on $\mathbf{X}$. The pair $(\mathbf{X}, O f)$ is called the interval functional element. If $E f$ is an interval that contains $O f$ then we will call the pair $(\mathbf{X}, E f)$ the extension
of $(\mathbf{X}, O f)$ or eif-element.
Let $f$ be the constant signs function on $\mathbf{X}$. If $f>0$ (respectively $f<0$ ) on $\mathbf{X}$ and $O f>0$ (respectively $O f<0$ ) then we will call $(\mathbf{X}, O f)$ the correct interval functional element (shortly $c$-element). More generally we will call the correct interval functional element an extension $(\mathbf{X}, E f)$ of $(\mathbf{X}, O f)$ that has the same sign as $O f$.

## 4 Some Categories and Functors of Interval Mathematics

A set of intervals with inclusion relation forms a category $\mathcal{C I} \mathcal{P}$ of preorder [12].

Definition 2 contravariant functor from $\mathcal{C I P}$ to the category of sets is called the interval presheaf.

For a finite set $F S=\left\{\mathbf{X}_{i}\right\}$ of $m$-dimensional intervals in $\mathbf{R}^{n}, m \leq n$, the union $V$ of the intervals forms a piecewise-linear manifold in $\mathbf{R}^{n}$. Let $G$ be the graph of the adjacency relation of intervals from $F S$. The manifold $V$ is connected if $G$ is a connected graph. In the paper we are considering connected manifolds. Let $f$ be a constant signs function on $\mathbf{X} \in F S$. The set $\left\{\left(\mathbf{X}_{j}, O f\right)\right\}$ of $c-$ elements (if exists) is called a constant signs continuation of $f$ on $\left\{\mathbf{X}_{j}\right\}$. If $\left\{\mathbf{X}_{j}\right\}$ is the maximal subset of $F S$ relatively a constant signs function $f$ then $\left\{\left(\mathbf{X}_{j}, O f\right)\right\}$ is called the constant signs continuation of $f$ on $F S$.

## 5 On Interval Operads

Operads was introduced by J. May [13. Operadic language is useful for investigation of many problems in mathematics and physics. Some recent applications of operads in physics is given by J. Morava [14. Here we give a short description of interval operad. The space of continuous interval functions of $j$ variables forms the topological space $I \mathcal{C}(j)$. Its points are operations $I^{j} \mathbf{R} \rightarrow I \mathbf{R}$ of arity $j$. $I \mathcal{C}(0)$ is a single point $*$. The class of interval spaces $I^{n} \mathbf{R}, n \geq 0$ forms the category [12]. We will consider the spaces with base points and denote the category of those spaces by $I \mathcal{U}$. Let $X \in I \mathcal{U}$ and for $k \geq 0$ let $I \mathcal{E}(k)$ be the space of maps $M\left(X^{k}, X\right)$. There is the action (by permuting the inputs) of the symmetric group $S_{k}$ on $I \mathcal{E}(k)$. The identity element $1 \in I \mathcal{E}(1)$ is the identity map of $X$.

Proposition-Definition 1 In the above mentioned conventions let $k \geq 0$ and $j_{1}, \ldots, j_{k} \geq 0$ be integers. Let for each choice of $k$ and $j_{1}, \ldots, j_{k}$ there is a map

$$
\gamma: I \mathcal{E}(k) \times I \mathcal{E}\left(j_{1}\right) \ldots \times I \mathcal{E}\left(j_{k}\right) \rightarrow I \mathcal{E}\left(j_{1}+\ldots+j_{k}\right)
$$

given by multivariable composition. If maps $\gamma$ satisfy associativity, equivalence and unital properties then $I \mathcal{E}$ is the endomorphism interval operad $I \mathcal{E}_{X}$ of $X$.

## 6 Interval Programs as Functors

Let $A=I \mathbf{R}$ be the interval algebra [8 with interval arithmetic operations. In many cases extra interval operations are required. So we have to extend the notion of interval algebra. Let us define the "operator domain" $\Omega$ of interval computations as sequence of sets $\Omega_{0}$ (interval constants and variables), $\Omega_{1}$ (unary interval operations), $\Omega_{2}$ (binary interval operations). .... In these notations the set $T_{\Omega}$ of all "non-branching" programs in $\Omega$ is defined as the least subset of $\left(\bigcup_{n=0}^{\infty} \Omega_{n} \bigcup\{()\}\right)^{*}$ such that following axioms are satisfied:
(t) $\Omega_{0} \subseteq T_{\Omega}$;
(tt) for $n \geq 1, \omega \in \Omega_{n}$ and $t_{1}, \ldots, t_{n} \in T_{\Omega}, \omega\left(t_{1}, \ldots, t_{n}\right) \in T_{\Omega}$.
$\Omega$-interval algebra is constructed from the interval algebra $A$ and functions $\omega_{A}: A^{n} \rightarrow A, \omega \in \Omega_{n}$. Below in the section our results follows Gougen 15] who discussed the non-interval case.
Proposition $1 T_{\Omega}$ is an initial object in the category of $\Omega$ - interval algebras.
Let $I P$ be the interval program that implements an interval computation, $G=$ $G(I P)$ the graph of the flow diagram of $I P, \mathcal{G}^{\otimes}$ the category of all paths in $G$. Let $\mathcal{I P F}$ be the category of interval sets with partial interval functions.
Proposition 2 Interval program IP defines a functor $\overline{I P}: \mathcal{G}^{\otimes} \rightarrow \mathcal{I P} \mathcal{F}$.

## Some Problems

Let $E$ be an interval expression with a tree $T$. Let $w(E)$ be the diameter of $E$ on a given interval data.
Problem 1 How to transform $E$ to the interval expression of the minimal complexity?

Problem 2 How to transform $E$ to an expression with min $w(E)$ ?
Let $S$ be a sufficiently smooth bounded surface with boundary over $x y$-plane in $\mathbf{R}^{3}$.
Let $S_{x y}=\operatorname{Proj}_{x y} S$ be the projection of $S$ on the $x y-$ plane, $S_{x}=\operatorname{Proj}_{x} S_{x y}, S_{y}=$ $\operatorname{Proj}_{y} S_{x y}$.
Problem 3 Compute the set of points that minimize for each fixed $x$ the distance

$$
z_{\min }=\left\{\min z(x, y) \mid x \in S_{x}, y \in S_{y}\right\} .
$$

I know a solution of the problem 3 for Minkowski-Cohn moduli space (1) and for some another surfaces.

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