Stochastic Schrödinger equations as limit of discrete filtering

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Abstract

We consider an open model possessing a Markovian quantum stochastic limit and derive the limit stochastic Schrödinger equations for the wave function conditioned on indirect observations using only the von Neumann projection postulate. We show that the diffusion (Gaussian) situation is universal as a result of the central limit theorem with the quantum jump (Poissonian) situation being an exceptional case. It is shown that, starting from the correponding limiting open systems dynamics, the theory of quantum filtering leads to the same equations, therefore establishing consistency of the quantum stochastic approach for limiting Markovian models.

1 Introduction

The problem of describing the evolution of a quantum system undergoing continual measurement has been examined from a variety of different physical and mathematical viewpoints however a consensus is that the generic forms of the stochastic Schrödinger equation (SSE) governing the state $\psi_t(\omega)$, conditioned on recorded output ω, takes on of the two forms below:

$$|d\psi_t\rangle = (L-\lambda_t)|\psi_t\rangle\, d\hat{q}_t + \left(-iH - \frac{1}{2}\left(L^\dagger L - 2\lambda_t L + \lambda_t^2\right)\right)|\psi_t\rangle\, dt, \tag{1}$$

$$|d\psi_t\rangle = \left(\frac{L-\sqrt{\nu_t}}{\sqrt{\nu_t}}\right)|\psi_t\rangle\, d\hat{n}_t + \left(-iH - \frac{1}{2}L^\dagger L - \frac{1}{2}\nu_t + \sqrt{\nu_t}L\right)|\psi_t\rangle\, dt \tag{2}$$

Here H is the system's Hamiltonian and L a fixed operator of the system which is somehow involved with the coupling of the system to the apparatus. In (1) we have $\lambda_t(\omega) = \frac{1}{2}\left\langle \psi_t(\omega) \,|\, \left(L^\dagger + L\right)\psi_t(\omega)\right\rangle$ and $\{\hat{q}_t : t \geq 0\}$ is a Gaussian martingale process (in fact a Wiener process). In (2), $\nu_t(\omega) = \left\langle \psi_t(\omega) \,|\, L^\dagger L \psi_t(\omega)\right\rangle$ and

$\{\hat{n}_t : t \geq 0\}$ is a Poissonian martingale process. The former describing quantum diffusions [1-7], the latter quantum jumps [8-10].

(By the term martingale, we mean a bounded stochastic process whose current value agrees with the conditional expectation of any future value based on observations up to the present time. They are used mathematically to model noise and, in both cases above, they are to come from continually de-trending the observed output process.)

There is a general impression that the continuous time SSEs require additional postulates beyond the standard formalism of quantum mechanics and the von Neumann projection postulate. We shall argue that this is not so. Our aim is derive the SSEs above as continuous limits of a straightforward quantum dynamics with discrete time measurements. The model we look at is a generalization of one studied by Kist *et al.* [12] where the environment consists of two-level atoms which sequentially interact with the system and are subsequently monitored. The generalization occurs in considering the most general form of the coupling of the two level atoms to the system that will lead to a well defined Markovian limit dynamics. The procedure for conditioning the quantum state, based on discrete measurements is given by von Neumann's projection postulate. Recording a value of an observable with corresponding eigenspace-projection Π will result in the change of vector state $\psi \mapsto p^{-1/2}\Pi\psi$ where $p = \langle\psi|\Pi\psi\rangle$ is assumed non-zero. Let us suppose that at discrete times $t = \tau, 2\tau, 3\tau, \ldots$ the system comes in contact with an apparatus and that an indirect measurement is made. Based on the measurement output, which must be viewed as random, we get a time series $(\psi_j)_j$ of system vector states. The question is then whether such a time series might converge in the continuous time limit ($\tau \to 0$) and whether it will lead to the standard stochastic Schrödinger equations. We apply a procedure pioneered by Smolianov and Truman [13] to obtain the limit SSEs for the various choices of monitored variable: a key feature of this procedure approach is that only standard quantum mechanics and the projection postulate are needed!

The second point of the analysis is to square our results up with the theory of continuous-time quantum filtering [4],[14],[15],[16]. This applies to unitary, Markovian evolutions of quantum open systems (that is, joint system and Markovian environment) described by quantum stochastic methods [17],[18],[19]. Our model was specifically chosen to lead to a Markovian limit. Here we show that filtering theory applied to the limit dynamics leads to exactly the same limit SSEs we derive earlier. Needless to say, the standard forms (1) and (2) occur as generic forms.

We show that if the two level atoms are prepared in their ground states then we obtain jump equations (2) whenever we try to monitor if the post-interaction atom is still in its ground state. In all other cases we are lead to a diffusion equation which we show to universally have the form (1).

2 Limit of Continuous Measurements

Models of the type we consider here have been treated in the continuous time limit by [20],[21], and [22]. In this section we recall the notations and results of [22] detailing how a discrete-time repeated interaction-measurement can, in the continuous time limit, be described as an open quantum dynamics driven by quantum Wiener and Poisson Processes.

Let $\mathcal{H}_S$ be has state space of our system and at times $t = \tau, 2\tau, 3\tau, \ldots$ it interacts with an apparatus. We denote by $\mathcal{H}_{E,k}$ the state space describing the apparatus used at time $t = k\tau$ - this will be a copy of a fixed Hilbert space $\mathcal{H}_E$. We are interested in the Hilbert spaces

$$\mathcal{F}_E^{t]} = \bigotimes_{k=1}^{\lfloor t/\tau \rfloor} \mathcal{H}_{E,k}, \qquad \mathcal{F}^{(\tau)} = \bigotimes_{k=\lfloor t/\tau \rfloor + \tau}^{\infty} \mathcal{H}_{E,k}, \qquad \mathcal{F}^{(\tau)} = \mathcal{F}_{t]}^{(\tau)} \otimes \mathcal{F}_{(t}^{(\tau)} \tag{3}$$

where $\lfloor x \rfloor$ means the integer part of x. (We fix a vector $e_0 \in \mathcal{H}_E$ and use this to stabilize the infinite direct product.) We shall refer to $\mathcal{F}_{t]}^{(\tau)}$ and $\mathcal{F}_{(t}^{(\tau)}$ as the *past* and *future environment spaces* respectively.

We are interested only in the evolution between the discrete times $t = \tau, 2\tau, 3\tau, \ldots$ and to this end we require, for each $k > 0$, a unitary (Floquet) operator, V_k, to be applied at time $t = k\tau$: its action will be on the joint space $\mathcal{H}_S \otimes \mathcal{F}^{(\tau)}$ but it will have non-trivial action only on the factors $\mathcal{H}_S$ and $\mathcal{H}_{E,k}$. The V_k's will be copies of a fixed unitary V acting on the representative space $\mathcal{H}_S \otimes \mathcal{H}_E$. The unitary operator $U_t^{(\tau)}$ describing the evolution from initial time zero to time t is then

$$U_t^{(\tau)} = V_{\lfloor t/\tau \rfloor} \cdots V_2 V_1 \tag{4}$$

It acts on $\mathcal{H}_S \otimes \mathcal{F}_{t]}^{(\tau)}$ but, of course, has trivial action on the future environment space. The same is true of the discrete time dynamical evolution of observables $X \in \mathfrak{B}(\mathcal{H}_S)$, the space of bounded operators on $\mathcal{H}_S$, given by

$$J_t^{(\tau)}(X) = U_t^{(\tau)\dagger} (X \otimes 1_\tau) U_t^{(\tau)}, \tag{5}$$

where 1_τ is the identity on $\mathcal{F}^{(\tau)}$. The discrete time evolution satisfies the difference equation

$$\frac{1}{\tau}\left(U_{\lfloor t/\tau \rfloor + \tau}^{(\tau)} - U_{\lfloor t/\tau \rfloor}^{(\tau)} \right) = \left(V_{\lfloor t/\tau \rfloor + \tau} - 1 \right) U_{\lfloor t/\tau \rfloor}^{(\tau)}. \tag{6}$$

The state for the environment is chosen to be the vector $\Psi^{(\tau)}$ on $\mathcal{F}^{(\tau)}$ given by

$$\Psi^{(\tau)} = e_0 \otimes e_0 \otimes e_0 \otimes e_0 \cdots$$

and, since e_0 will typically be identified as the ground state on $\mathcal{H}_E$, we shall call $\Psi^{(\tau)}$ the *vacuum vector for the environment.*

2.1 Spin-$\frac{1}{2}$ Apparatus

For simplicity, we take $\mathcal{H}_E \equiv \mathbb{C}^2$. We may think of the apparatus as comprising of a two-level atom (qubit) with ground state e_0 and excited state e_1. We introduce the transition operators

$$\sigma^+ = |e_1\rangle \langle e_0| \quad \sigma^- = |e_0\rangle \langle e_1|$$

The copies of these operators for the $k-$th atom will be denoted σ_k^+ and σ_k^-. The operators $\sigma_k^\pm$ are Fermionic variables and satisfy the anti-commutation relation

$$\{\sigma_k^\pm, \sigma_k^\pm\} = 0, \quad \{\sigma_k^-, \sigma_k^+\} = 1 \tag{7}$$

while commuting for different atoms.

2.2 Collective Operators

We define the *collective operators* $A^\pm(t;\tau), \Lambda(t;\tau)$ to be

$$A^\pm(t;\tau) := \sqrt{\tau} \sum_{k=1}^{\lfloor t/\tau \rfloor} \sigma_k^\pm; \qquad \Lambda(t;\tau) := \sum_{k=1}^{\lfloor t/\tau \rfloor} \sigma_k^+ \sigma_k^-. \tag{8}$$

For times $t, s > 0$, we have the commutation relations

$$\begin{aligned} \left[A^-(t;\tau), A^+(s;\tau)\right] &= \tau \left\lfloor \left(\frac{t \wedge s}{\tau} \right) \right\rfloor - 2\tau \Lambda(t \wedge s, \tau), \\ \left[\Lambda(t;\tau), A^\pm(s;\tau)\right] &= \pm A^\pm(t \wedge s;\tau), \end{aligned}$$

where $s \wedge t$ denotes the minimum of s and t. In the limit where τ goes to zero while s and t are held fixed, we have the approximation

$$\left[A^-(t;\tau), A^+(s;\tau)\right] \approx t \wedge s. \tag{9}$$

The collective fields $A^\pm(t;\tau)$ converge to a Bosonic quantum Wiener processes $A_t^\pm$ as $\tau \to 0$, while $\Lambda(t;\tau)$ converges to the Bosonic conservation process Λ_t [17]. This is an example of a general class of well-known quantum stochastic limits [27],[29]. Intuitively, we may use the following rule of thumb for $t = j\tau$:

$$\begin{array}{rr} \tau \simeq dt, & \sqrt{\tau}\sigma_j^- \simeq dA_t^-, \\ \sqrt{\tau}\sigma_j^+ \simeq dA_t^+, & \sigma_j^+\sigma_j^- \simeq d\Lambda_t. \end{array} \tag{10}$$

These replacements are usually correct when we go from a finite difference equation involving the discrete spins to a quantum stochastic differential equation involving differential processes.

The limit processes are denoted as $A_t^{10} = A_t^+$, $A_t^{01} = A_t^-$ and $A_t^{11} = \Lambda_t$ respectively and we emphasize that they are not considered to live on the same Hilbert space as the discrete system but on a Bose Fock space $\Gamma_+\left(L^2(\mathbb{R}^+, dt)\right)$. (See the appendix for details.) We also set $A_t^{00} = t$.

2.3 The Interaction

The $k-$th Floquet operator is then taken to be

$$V_k = \exp\left\{-i\tau H_k^{(\tau)}\right\} \tag{11}$$

where

$$H_k^{(\tau)} := \frac{1}{\tau} H_{11} \otimes \sigma_k^+ \sigma_k^- + \frac{1}{\sqrt{\tau}} H_{10} \otimes \sigma_k^+ + \frac{1}{\sqrt{\tau}} H_{01} \otimes \sigma_k^- + H_{00}. \tag{12}$$

Here we must take H_{11} and H_{00} to be self-adjoint and require that $(H_{01})^\dagger = H_{10}$. We may identify H_{00} with the free system Hamiltonian H_S. We shall assume that the operators $H_{\alpha\beta}$ are bounded on $\mathcal{H}_S$ with H_{11} also bounded away from zero.

The scaling of the spins $\sigma_k^\pm$ by $\tau^{-1/2}$ is necessary if we want to obtain a quantum diffusion associated with the H_{10} and H_{01} terms and a zero-intensity Poisson process associated with H_{11} in the $\tau \to 0$ limit.

We shall also employ the following summation convention: whenever a repeated raised and lowered Greek index appears we sum the index over the values zero and one. With this convention,

$$H_k^{(\tau)} \equiv H_{\alpha\beta} \otimes \left[\frac{\sigma_k^+}{\sqrt{\tau}}\right]^\alpha \left[\frac{\sigma_k^-}{\sqrt{\tau}}\right]^\beta \tag{13}$$

were we interpret the raised index as a power: that is, $[x]^0 = 1$, $[x]^1 = x$.

2.4 Continuous Time Limit Dynamics

We consider the Floquet unitary on $\mathcal{H}_S \otimes \mathcal{H}_E$ given by

$$\begin{aligned} V &= \exp\left\{-i\tau\, H_{\alpha\beta} \otimes \left[\frac{\sigma^+}{\sqrt{\tau}}\right]^\alpha \left[\frac{\sigma^-}{\sqrt{\tau}}\right]^\beta\right\} \\ &\simeq 1 + \tau L_{\alpha\beta} \otimes \left[\frac{\sigma^+}{\sqrt{\tau}}\right]^\alpha \left[\frac{\sigma^-}{\sqrt{\tau}}\right]^\beta \end{aligned}$$

where $\simeq$ means that we drop terms that do not contribute in our $\tau \to 0$ limit. Here the so-called *Itô coefficients* $L_{\alpha\beta}$ are given by

$$L_{\alpha\beta} = -iH_{\alpha\beta} + \sum_{n\geq 2} \frac{(-i)^n}{n!} H_{\alpha 1} (H_{11})^{n-2} H_{1\beta}, \tag{14}$$

that is,

$$\begin{aligned} L_{11} &= e^{-iH_{11}} - 1; & L_{10} &= \frac{e^{-iH_{11}} - 1}{H_{11}} H_{10}; \\ L_{01} &= H_{01} \frac{e^{-iH_{11}} - 1}{H_{11}}; & L_{00} &= -iH_{00} + H_{01} \frac{e^{-iH_{11}} - 1 + iH_{11}}{(H_{11})^2} H_{10}. \end{aligned}$$

The relationship between the Hamiltonian coefficients $H_{\alpha\beta}$ and the Itô coefficients was first given in [23]. We remark that they satisfy the relations

$$L_{\alpha\beta} + L_{\beta\alpha}^{\dagger} + L_{1\alpha}^{\dagger} L_{1\beta} = 0. \tag{15}$$

guaranteeing unitarity [17]. Consequently we have

$$L_{11} = W - 1; \quad L_{10} = L; \quad L_{01} = -L^{\dagger} W; \quad L_{00} = -iH - \frac{1}{2} L^{\dagger} L \tag{16}$$

with $W = \exp\{-iH_{11}\}$ unitary, $H = H_{00} - H_{01} \frac{H_{11} - \sin(H_{11})}{(H_{11})^2} H_{10}$ self-adjoint and L is bounded but otherwise arbitrary. (Note that $\frac{x - \sin x}{x^2} > 0$ for $x > 0$.)

2.4.1 Convergence of Unitary Evolution

In the above notations, the discrete time family $\left\{ U_t^{(\tau)} : t \geq 0 \right\}$ converges to quantum stochastic process $\{U_t : t \geq 0\}$ on $\mathcal{H}_S \otimes \Gamma_+ \left(L^2 \left(\mathbb{R}^+, dt \right) \right)$ in the sense that, for all $u, v \in \mathcal{H}_S$, integers n, m and for all $\phi_j, \psi_j \in L^2 \left(\mathbb{R}^+, dt \right)$ Riemann integrable, we have the uniform convergence ($\tau \to 0$)

$$\begin{aligned} &\left\langle A^+ \left(\phi_m, \tau \right) \cdots A^+ \left(\phi_1, \tau \right) u \otimes \Psi^{(\tau)} \right| U_t^{(\tau)} A^+ \left(\psi_n, \tau \right) \cdots A^+ \left(\psi_1, \tau \right) v \otimes \Psi^{(\tau)} \Big\rangle \\ &\qquad \to \left\langle A^+ \left(\phi_m \right) \cdots A^+ \left(\phi_1 \right) u \otimes \Psi \right| U_t \, A^+ \left(\psi_n \right) \cdots A^+ \left(\psi_1 \right) v \otimes \Psi \big\rangle \end{aligned} \tag{17}$$

The process U_t is moreover unitary, adapted and satisfies the quantum stochastic differential equation (QSDE, see appendix)

$$dU_t = L_{\alpha\beta} \otimes dA_t^{\alpha\beta} \, U_t, \quad U_0 = 1. \tag{18}$$

2.4.2 Convergence of Heisenberg Dynamics

Likewise, for X a bounded observable on $\mathcal{H}_S$ the discrete time family $\left\{ J_t^{(\tau)} \left(X \right) \right\}$ converges to the quantum stochastic process $J_t \left(X \right) = U_t^{\dagger} \left(X \otimes 1 \right) U_t$ on $\mathcal{H}_S \otimes \Gamma_+ \left(L^2 \left(\mathbb{R}^+, dt \right) \right)$ in the same weak sense as in (17). The limit Heisenberg equations of motion are

$$dJ_t \left(X \right) = J_t \left(\mathcal{L}_{\alpha\beta} X \right) \otimes dA_t^{\alpha\beta}, \quad J_0 \left(X \right) = X \otimes 1 \tag{19}$$

where

$$\mathcal{L}_{\alpha\beta} \left(X \right) := L_{\beta\alpha}^{\dagger} X + X L_{\alpha\beta} + L_{1\alpha}^{\dagger} X L_{1\beta}. \tag{20}$$

We remark that $\mathcal{L}_{\alpha\beta} \left(1 \right) = 0$, by the unitarity conditions (15). A completely positive semigroup $\{\Xi_t : t \geq 0\}$ is then defined by $\langle u | \, \Xi_t \left(X \right) \, v \rangle := \langle u \otimes \Psi | \, J_t \left(X \right) \, v \otimes \Psi \rangle$ and we have $\Xi_t = \exp\{t \mathcal{L}_{00}\}$ where the Lindblad generator is $\mathcal{L}_{00} \left(X \right) = \frac{1}{2} \left[L^{\dagger}, X \right] L + \frac{1}{2} L^{\dagger} \left[X, L \right] - i \left[X, H \right]$ with $L = \frac{e^{-iH_{11}} - 1}{H_{11}} H_{10}$ and $H = H_{00} - H_{01} \frac{H_{11} - \sin(H_{11})}{(H_{11})^2} H_{10}$.

We remark that such QSDEs occur naturally in Markovian limits for quantum field reservoirs [24], [25], see also [16].

3 Conditioning on Measurements

We now consider how the measurement of an apparatus indirectly leads to a conditioning of the system state. For clarity we investigate the situation of a single apparatus to begin with. Initially the apparatus is prepared in state e_0 and after a time τ we have the evolution

$$\phi \otimes e_0 \to V\left(\phi \otimes e_0\right) \simeq \left(1+\tau L_{00}\right) \phi \otimes e_0 + \sqrt{\tau} L_{10} \phi \otimes e_1 . \tag{21}$$

We shall measure the σ^x-observable. This can be written as

$$\sigma^x = \sigma^+ + \sigma^- = \left|e_+\right\rangle \left\langle e_+\right| - \left|e_-\right\rangle \left\langle e_-\right|$$

where $\left|e_+\right\rangle = \frac{1}{\sqrt{2}}\left|e_1\right\rangle + \frac{1}{\sqrt{2}}\left|e_0\right\rangle$ and $\left|e_-\right\rangle = \frac{1}{\sqrt{2}}\left|e_1\right\rangle - \frac{1}{\sqrt{2}}\left|e_0\right\rangle$. (Actually, the main result of this section will remain true provided we measure an observable with eigenstates $\left|e_\pm\right\rangle$ different to $\left|e_0\right\rangle$ and $\left|e_1\right\rangle$. We will show this universality later.)

Taking the initial joint state to be $\phi \otimes e_0$, we find that the probabilities to get the eigenvalues ± 1 are

$$p_\pm = \left\langle \phi \otimes e_0 \right| V^\dagger \left(1_S \otimes \Pi_\pm\right) V \ \phi \otimes e_0 \left.\right\rangle \simeq \frac{1}{\sqrt{2}}\left[1 \pm 2\lambda\sqrt{\tau}\right] + O\left(\tau^{3/2}\right)$$

where we introduce the expectation

$$\lambda = \frac{1}{2}\left\langle \phi \right| \left(L + L^\dagger\right) \phi \left.\right\rangle .$$

A pair of linear maps, $\mathcal{V}_\pm$ on $\mathfrak{h}_S$ are defined by

$$\left(\mathcal{V}_\pm \phi\right) \otimes e_\pm \equiv \left(1_S \otimes \Pi_\pm\right) V \ \left(\phi \otimes e_0\right) \tag{22}$$

and to leading order we have

$$\mathcal{V}_\pm \simeq \frac{1}{\sqrt{2}}\left[1 \pm L\sqrt{\tau} + \left(-iH - \frac{1}{2}L^\dagger L\right) \tau\right] .$$

The wave function $\mathcal{W}_\pm \phi$, conditioned on a $\pm$-measurement, is therefore

$$\mathcal{W}_\pm \phi := \frac{\mathcal{V}_\pm \phi}{\sqrt{p_\pm}} \tag{23}$$

$\mathcal{W}_\pm$ will be non-linear as the probabilities $p_\pm$ clearly depend on the state ϕ. We then have the development

$$\mathcal{W}_\pm \phi \simeq \phi \pm \sqrt{\tau}\ \left(L - \lambda\right) \phi + \tau \left[-iH - \frac{1}{2}L^\dagger L + \lambda\left(\frac{3}{2}\lambda - L\right)\right] \phi . \tag{24}$$

We now introduce a random variable η which takes the two possible values ± 1 with probabilities $p_\pm$. We call η the *discrete output variable*. Then

$$\begin{aligned} \mathbb{E}\left[\eta\right] &= p_+ + p_- = 2\lambda\sqrt{\tau} + O\left(\tau\right) &(25)\\ \mathbb{E}\left[\eta^2\right] &= p_+ + p_- = 1 + O\left(\tau\right) . &(26)\end{aligned}$$

Now let us deal with repeated measurements. We shall record an output sequence of ± 1 and we write η_j as the random variable describing the jth output. Statistically, the η_j are not independent: each η_j will depend on $\eta_1, \cdots, \eta_{j-1}$.

We set, for $j = \lfloor t/\tau \rfloor$ and fixed $\phi \in \mathfrak{h}_S$,

$$\phi_t^{(\tau)} = \mathcal{V}_{\eta_n} \cdots \mathcal{V}_{\eta_1} \phi, \quad \psi_t^{(\tau)} = \mathcal{W}_{\eta_n} \cdots \mathcal{W}_{\eta_1} \phi = \frac{1}{\left\| \phi_t^{(\tau)} \right\|} \phi_t^{(\tau)}. \tag{27}$$

We shall have $\Pr \left\{ \eta_{j+1} = \pm 1 \right\} \simeq \frac{1}{2} \left[1 \pm \sqrt{\tau} \lambda_j^{(\tau)} \right]$, where $\lambda_j^{(\tau)} = \frac{1}{2} \left\langle \psi_j^{(\tau)} \right| \left(L + L^\dagger \right) \left. \psi_j^{(\tau)} \right\rangle$, and

$$\begin{aligned} \mathbb{E}_j^\tau \left[\eta_{j+1} \right] &= 2\lambda_j^{(\tau)} \sqrt{\tau} + O(\tau) & (28) \\ \mathbb{E}_j^\tau \left[\left(\eta_{j+1} \right)^2 \right] &= 1 + O(\tau) & (29) \end{aligned}$$

where $\mathbb{E}_j^\tau$ is conditional expectation wrt. the variables $\left(\eta_1, \cdots, \eta_j \right)$. The state $\psi_{(j+1)\tau}^{(\tau)}$ after the $(j+1)$-st measurement depends on the state $\psi_{j\tau}^{(\tau)}$ and η_{j+1} and we have, to relevant order, the finite difference equation

$$\psi_{(j+1)\tau}^{(\tau)} - \psi_{j\tau}^{(\tau)} \simeq \eta_{j+1} \sqrt{\tau} \left(L - \lambda_j^{(\tau)} \right) \psi_{j\tau}^{(\tau)} + \tau \left[-iH - \frac{1}{2} L^\dagger L + \lambda_j^{(\tau)} \left(\frac{3}{2} \lambda_j^{(\tau)} - L \right) \right] \psi_{j\tau}^{(\tau)}$$

We wish to consider the process

$$q^{(\tau)}(t) = \sqrt{\tau} \sum_{j=1}^{\lfloor t/\tau \rfloor} \eta_j,$$

however, it has a non-zero expectation and so is not suitable as a noise term. Instead, we introduce new random variables $\zeta_j := \eta_j - \sqrt{\tau}\, 2\lambda_{j-1}^{(\tau)}$ and consider $\hat{q}^{(\tau)}(t) = \sqrt{\tau} \sum_{j=1}^{\lfloor t/\tau \rfloor} \zeta_j$. We now use a simple trick to show that it is mean-zero to required order. First of all, observe that $\mathbb{E}_{j-1}^\tau \left[\zeta_j \right] = 0$ and so $\mathbb{E} \left[\zeta_j^\tau \right] = \mathbb{E} \left[\mathbb{E}_{j-1}^\tau \left[\zeta_j^\tau \right] \right] = O(\tau)$. Similarly $\mathbb{E} \left[\zeta_j^2 \right] = 1 + O(\sqrt{\tau})$. It follows that $\left\{ \hat{q}^{(\tau)}(t) : t \geq 0 \right\}$ converges in distribution to a mean-zero martingale process, which we denote as $\{ \hat{q}_t : t \geq 0 \}$, with correlation $\mathbb{E} \left[\hat{q}_t \hat{q}_s \right] = t \wedge s$. We may therefore take $\hat{q}_t$ to be a Wiener process. Likewise, $\left\{ q^{(\tau)}(t) : t \geq 0 \right\}$ should converge to a stochastic process $\{ q_t : t \geq 0 \}$ which is adapted wrt. $\hat{q}$: in other words, q_t should be determined as a function of the $\hat{q}$-process for times $s \leq t$ for each time $t > 0$. In particular,

$$q_t = \hat{q}_t + 2 \int_0^s \lambda_s ds$$

where $\lambda_t = \dfrac{1}{2} \left\langle \psi_t \right| \left(L + L^\dagger \right) \left. \psi_t \right\rangle$.

In the limit $\tau \to 0$ we obtain the limit stochastic differential

$$\left| d\psi_t \right\rangle = \left(L - \lambda_l \right) \left| \psi_t \right\rangle dq_t + \left[-iH - \frac{1}{2} L^\dagger L + \lambda_j \left(\frac{3}{2} \lambda_t - L \right) \right] \left| \psi_t \right\rangle dt$$

with initial condition $|\psi_0\rangle = |\phi\rangle$. In terms of the Wiener process $\hat{q}$ we have

$$|d\psi_t\rangle = (L - \lambda_t)\,|\psi_t\rangle\, d\hat{q}_t + \left[-iH - \frac{1}{2}\left(L^\dagger L - 2\lambda_t L + \lambda_t^2\right)\right]|\psi_t\rangle\, dt. \tag{30}$$

This is, of course, the standard form of the diffusive Stochastic Schrödinger equation (1).

3.1 Universality of Gaussian SSE

We consider measurements on an observable of the form

$$X = x_+\,|e_+\rangle\langle e_+| + x_-\,|e_-\rangle\langle e_-| \tag{31}$$

where x_+ and x_- are real eigenvalues, while e_+ and e_- are any orthonormal eigenvectors in $\mathcal{H}_E$ not the same as e_0 and e_1. Generally speaking we will have

$$e_+ = \sqrt{q}e_0 + e^{i\theta}\sqrt{1-q}e_1, \quad e_- = \sqrt{1-q}e_0 - \sqrt{q}e^{i\theta}e_1 \tag{32}$$

with $0 < q < 1$. For convenience we set $q_+ = q$ and $q_- = 1 - q$. The phase $\theta \in [0, 2\pi)$ will actually play no role in what follows and can always be removed by the "gauge" transformations $e_0 \hookrightarrow e_0$, $e^{i\theta}e_1 \hookrightarrow e_1$ which is trivial from our point of view since it leaves the ground state invariant. We therefore set $\theta = 0$.

The probability that we measure X to be $x_\pm$ after the interaction will be

$$\begin{aligned} p_\pm &= \left\langle \phi\otimes e_0\right| V^\dagger\left(1_S \otimes \Pi_\pm\right) V\ \phi\otimes e_0\rangle \qquad (33)\\ &\simeq q_\pm\left[1 + \eta_\pm 2\lambda\sqrt{\tau} - \nu\left(1-\eta_\pm^2\right)\tau\right] \qquad (34)\end{aligned}$$

where we introduce the expectations

$$\nu = \left\langle \phi | L^\dagger L\,\phi\right\rangle = \|L\phi\|^2$$

and the weighting

$$\eta_+ = \sqrt{\frac{q_-}{q_+}}, \qquad \eta_- = -\sqrt{\frac{q_+}{q_-}}. \tag{35}$$

We may now introduce a random variable η taking the values $\eta_\pm$ with probabilities $p_\pm$. We remark that

$$\begin{aligned} \mathbb{E}[\eta] &= p_+\eta_+ + p_-\eta_- = 2\lambda\sqrt{\tau} + O(\tau), \qquad (36)\\ \mathbb{E}\left[\eta^2\right] &= p_+\eta_+^2 + p_-\eta_-^2 = 1 - 2\lambda\frac{q_+^2 - q_-^2}{\sqrt{q_+q_-}}\sqrt{\tau} + O(\tau). \qquad (37)\end{aligned}$$

We then have the finite difference equation

$$\begin{aligned} \psi_{j+1}^{(\tau)} \simeq \psi_j^{(\tau)} + \sqrt{\tau}\eta_{j+1}^{(\tau)}\left(L - \lambda_j^{(\tau)}\right)\psi_j^{(\tau)} \\ +\tau\left[-iH - \frac{1}{2}L^\dagger L + \frac{1}{2}\left(1 - \eta_{(j+1)}^{(\tau)2}\right)\nu_{(j)}^{(\tau)} + \eta_{(j+1)}^{(\tau)2}\left(\frac{3}{2}\lambda_j^{(\tau)2} - \lambda_j^{(\tau)}L\right)\right]\psi_j^{(\tau)} \qquad (38)\end{aligned}$$

which is the same as before except for the new term involving $\nu_{(j)}^{(\tau)} = \left\langle \psi_j^{(\tau)} \right| L^\dagger L \, \psi_j^{(\tau)} \Big\rangle$. We may replace the η^2-terms by their averaged value of unity when transferring to the limit $\tau \to 0$: in particular the term with $\nu_j^{(\tau)}$ is negligible. Defining the processes $q_t^{(\tau)}$ and $\hat{q}_t^{(\tau)}$ as before, we are lead to the same SSE as (30).

3.2 Poissonian Noise

Now let us consider what happens if we take the input observable to be $\sigma^+\sigma^-$. (This corresponds to the choice $q_+ = 1$, $q_- = 0$.) We now record a result of either zero or one with probabilities $p_\varepsilon = \left\langle \phi \otimes e_0 | V^\dagger \left(1 \otimes \Pi_\varepsilon\right) V \phi \otimes e_0 \right\rangle$ where $\varepsilon = 0, 1$ and $\Pi_\varepsilon = |e_\varepsilon\rangle \langle e_\varepsilon|$. Explicitly we have

$$p_0 \simeq 1 - \nu\tau, \quad p_1 \simeq \nu\tau.$$

As before, we define a conditional operator $\mathcal{V}_\varepsilon$ on $\mathcal{H}_S$ by $(\mathcal{V}_\varepsilon \phi) \otimes e_\varepsilon = \left(1 \otimes \Pi_\varepsilon\right) V \left(\phi \otimes e_0\right)$ and a conditional map $\mathcal{W}_\varepsilon = \left(p_\varepsilon\right)^{-1/2} \mathcal{V}_\varepsilon$. Here we will have

$$\mathcal{W}_0 \phi \simeq \left[1 + \tau \left(-iH - \frac{1}{2} L^\dagger L + \frac{1}{2}\nu\right)\right] \phi, \quad \mathcal{W}_1 \phi \simeq \frac{1}{\sqrt{\nu}} L\phi.$$

Iterating this in a repeated measurement strategy, we record an output series $\left(\varepsilon_1^{(\tau)}, \varepsilon_2^{(\tau)}, \cdots\right)$ of zeroes and ones with difference equation for the conditioned states given by

$$\begin{aligned} \psi_{j+1}^{(\tau)} \;\; &\simeq \;\; \psi_j^{(\tau)} + \varepsilon_{j+1}^{(\tau)} \left(\frac{L - \sqrt{\nu_j^{(\tau)}}}{\sqrt{\nu_j^{(\tau)}}}\right) \psi_j^{(\tau)} \\ &\quad + \tau \left(1 - \varepsilon_{j+1}^{(\tau)}\right) \left(-iH - \frac{1}{2} L^\dagger L + \frac{1}{2} \nu_j^{(\tau)}\right) \psi_j^{(\tau)} \end{aligned} \tag{39}$$

where $\nu_j^{(\tau)} := \left\langle \psi_j^{(\tau)} | L^\dagger L \psi_j^{(\tau)} \right\rangle$. The $\varepsilon_j^{(\tau)}$'s may be viewed as dependent Bernoulli variables. In particular let $\mathbb{E}_j\left[\cdot\right]$ denote conditional expectation with respect to the first j of these variables, then $\mathbb{E}_j \left[\exp\left(iu\varepsilon_{j+1}^{(\tau)}\right)\right] \simeq 1 - \tau\nu_j^{(\tau)} \left(ie^u - 1\right)$. We now define a stochastic process $n_t^{(\tau)}$ by

$$n_t^{(\tau)} := \sum_{j=1}^{\lfloor t/\tau \rfloor} \varepsilon_j^{(\tau)}$$

and in the limit $\tau \to 0$ this converges to a non-homogeneous compound Poisson process $\{n_t : t \geq 0\}$. Specifically, if f is a smooth test function, then

$$\lim_{\tau \to 0} \mathbb{E}\left[\exp\left\{i \sum_{j=1}^{\lfloor t/\tau \rfloor} f\left(j\tau\right) \varepsilon_j^{(\tau)}\right\}\right] = \mathbb{E}\left[\exp\left\{\int_0^t ds \, \nu_s \left(e^{if(s)} - 1\right)\right\}\right]$$

with mean square limit $\nu_t := \lim_{\tau\to 0} \nu^{(\tau)}_{\lfloor t/\tau \rfloor}$. The Itô table for n_t is $(dn_t)^2 = dn_t$, and we have $\mathbb{E}_{t]}[dn_t] = \nu_t dt$ where $\mathbb{E}_{t]}[\cdot]$ is conditional expectation with respect to n_t.

The limit stochastic Schrödinger equation is then

$$|d\psi_t\rangle = \frac{\left(L - \sqrt{\nu_t}\right)}{\sqrt{\nu_t}} |\psi_t\rangle \, dn_t + \left(-iH - \frac{1}{2}\left(L^\dagger L - \nu_t\right)\right) |\psi_t\rangle \, dt \tag{40}$$

and one readily shows that the normalization $\|\psi_t\| = 1$ is preserved. Replacing n_t by the compensated Poisson process $\hat{n}_t = n_t - \int_0^t \nu_s ds$, we find

$$|d\psi_t\rangle = \frac{\left(L - \sqrt{\nu_t}\right)}{\sqrt{\nu_t}} |\psi_t\rangle \, d\hat{n}_t + \left(-iH - \frac{1}{2}\left(L^\dagger L + \nu_t\right) + \nu_t L\right) |\psi_t\rangle \, dt \tag{41}$$

This is the standard jump-type SSE (2).

3.3 Discrete Input / Output Processes

In quantum theory the probabilistic notion of events is replaced by orthogonal projections. For the measurements of σ_x, the relevant projectors are $\Pi^{(j)}_{\pm} = \frac{1}{2}\left[1 \pm \sigma^{(j)}_x\right]$ and so far we have worked in the Schrödinger representation. We note the property that, for $j \neq k$,

$$\left[\Pi^{(j)}_{\pm}, V_k\right] = 0.$$

In the Heisenberg picture we are interested alternatively in the projectors

$$\tilde{\Pi}^{(j)}_{\pm} := U^{(\tau)\dagger}_{j\tau} \, \Pi^{(j)}_{\pm} \, U^{(\tau)}_{j\tau}. \tag{42}$$

Note that $\left[\tilde{\Pi}^{(j)}_{\pm}, \Pi^{(k)}_{\pm}\right] = 0$ for $j < k$.

The family $\left\{\Pi^{(j)}_{\pm} : j = 1, 2, \cdots\right\}$ is auto-commuting: a property that is sometimes referred to as leading to a *consistent history* of measurement outputs. The family $\left\{\tilde{\Pi}^{(j)}_{\pm} : j = 1, 2, \cdots\right\}$ is likewise also auto-commuting. To see this, we note that for $n > j$,

$$\begin{aligned} U^{(\tau)\dagger}_{n\tau} \, \Pi^{(j)}_{\pm} \, U^{(\tau)}_{n\tau} &= V_1^\dagger \cdots V_n^\dagger \, \Pi^{(j)}_{\pm} \, V_n \cdots V_1 \\ &= V_1^\dagger \cdots V_j^\dagger \, \Pi^{(j)}_{\pm} \, V_j \cdots V_1 \equiv \tilde{\Pi}^{(j)}_{\pm} \end{aligned}$$

and so, for any j and k, $\left[\tilde{\Pi}^{(j)}_{\pm}, \tilde{\Pi}^{(k)}_{\pm}\right] = U^{(\tau)\dagger}_{n\tau} \left[\Pi^{(j)}_{\pm}, \Pi^{(k)}_{\pm}\right] U^{(\tau)}_{n\tau} = 0$ where we need only take n greater than both j and k.

For a given random output sequence $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots)$, we have for $n = \lfloor t/\tau \rfloor$

$$\left(\phi^{(\tau)}_t(\boldsymbol{\eta})\right) \otimes e_{\eta_1} \otimes \cdots \otimes e_{\eta_n} \otimes \Phi^{(\tau)}_{(t} = \left(\Pi^{(n)}_{\eta_n} V_n\right) \cdots \left(\Pi^{(1)}_{\eta_1} V_1\right) \, \phi \otimes \Phi^{(\tau)}$$

and the right hand side can be written alternatively as

$$\Pi_{\eta_n}^{(n)} \cdots \Pi_{\eta_1}^{(1)} U_t^{(\tau)} \phi \otimes \Phi^{(\tau)} \text{ or } U_t^{(\tau)} \tilde{\Pi}_{\eta_n}^{(n)} \cdots \tilde{\Pi}_{\eta_1}^{(1)} \phi \otimes \Phi^{(\tau)}.$$

The probability of a given output sequence $(\eta_1, \cdots, \eta_n)$ is then

$$\begin{aligned} \left\| \phi_t^{(\tau)} (\boldsymbol{\eta}) \right\|^2 &= \left\langle \chi_t^{(\tau)} \right| \Pi_{\eta_n}^{(n)} \cdots \Pi_{\eta_1}^{(1)} \ \chi_t^{(\tau)} \right\rangle \\ &= \left\langle \phi \otimes \Phi^{(\tau)} \right| \tilde{\Pi}_{\eta_n}^{(n)} \cdots \tilde{\Pi}_{\eta_1}^{(1)} \ \phi \otimes \Phi^{(\tau)} \right\rangle \end{aligned}$$

where $\chi_t^{(\tau)} := U_t^{(\tau)} \phi \otimes \Phi^{(\tau)}$.

Likewise, we have $\psi_t^{(\tau)} (\boldsymbol{\eta}) = \left\| \phi_t^{(\tau)} (\boldsymbol{\eta}) \right\|^{-1} \phi_t^{(\tau)} (\boldsymbol{\eta})$ and for any observable G of the system we have the random expectation

$$\begin{aligned} \left\langle \psi_t^{(\tau)} \right| G \ \psi_t^{(\tau)} \right\rangle &= \left\| \phi_t^{(\tau)} \right\|^{-2} \left\langle \chi_t^{(\tau)} \right| \left(G \otimes 1_E^{(\tau)} \right) \Pi_{\eta_n}^{(n)} \cdots \Pi_{\eta_1}^{(1)} \ \chi_t^{(\tau)} \right\rangle \\ &= \left\| \phi_t^{(\tau)} \right\|^{-2} \left\langle \phi \otimes \Phi^{(\tau)} \right| J_t^{(\tau)} (G) \ \tilde{\Pi}_{\eta_n}^{(n)} \cdots \tilde{\Pi}_{\eta_1}^{(1)} \ \phi \otimes \Phi^{(\tau)} \right\rangle . \end{aligned}$$

Let us introduce new spin variables $\tilde{\sigma}_j^{\pm}$ defined by

$$\tilde{\sigma}_j^{\pm} = U_{j\tau}^{(\tau)\dagger} \left(\sigma_j^{\pm} \right) U_{j\tau}^{(\tau)}$$

In particular, let $\tilde{\sigma}_j^x = \tilde{\sigma}_j^+ + \tilde{\sigma}_j^-$ then $\tilde{\Pi}_{\pm}^{(j)} = \frac{1}{2} \left[1 \pm \tilde{\sigma}_j^x \right]$. We may the construct the following adapted processes

$$\tilde{A}^{\pm} (t, \tau) := \sqrt{\tau} \sum_{j=1}^{\lfloor t/\tau \rfloor} \tilde{\sigma}_j^{\pm}; \qquad \tilde{\Lambda} (t; \tau) := \sum_{j=1}^{\lfloor t/\tau \rfloor} \tilde{\sigma}_j^{+} \tilde{\sigma}_j^{-}.$$

The current situation can be described as follows. Either we work in the Schrödinger picture, in which case we are dealing in the quantum stochastic process $Q(t;\tau) = A^+(t;\tau) + A^-(t;\tau)$, or in the Heisenberg picture, in which case we are dealing with $\tilde{Q}(t;\tau) = U_t^{(\tau)\dagger} X(t;\tau) U_t^{(\tau)} = \tilde{A}^+(t;\tau) + \tilde{A}^-(t;\tau)$. Adopting the terminology due to Gardiner, the Q-process is called the *input process* while the $\tilde{Q}$ is called the *output process*. We may likewise refer to the $\Pi_{\pm}^{(j)}$'s as *input events* and the $\tilde{\Pi}_{\pm}^{(j)}$'s as *output events*.

It is useful to know that, to relevant order, the output spin variables are

$$\begin{aligned} \sqrt{\tau} \tilde{\sigma}_j^- &\simeq U_{(j-1)\tau}^{(\tau)\dagger} \left(\sqrt{\tau} W \sigma_j^- + \tau L \right) U_{(j-1)\tau}^{(\tau)} \\ &= J_{(j-1)\tau}^{(\tau)} (W) \sqrt{\tau} \sigma_j^- + J_{(j-1)\tau}^{(\tau)} (L) \ \tau \end{aligned} \tag{43}$$

We shall study the continuous-time versions of these processes next.

4 The Stochastic Schrödinger Equation

We now review the simple theory of quantum filtering. Let $Y = \{Y_t : t \geq 0\}$ be an adapted, self-adjoint quantum stochastic process having trivial action on the system. In particular we suppose that it arises as a quantum stochastic integral

$$Y_t = y_0 + \sum_{\alpha,\beta} \int_0^t Y_{\alpha\beta}(t)\, dA_t^{\alpha\beta}$$

where the $Y_{\alpha\beta}(t) = (Y_{\beta\alpha}(t))^\dagger$ are again adapted processes and $y_0 \in \mathbb{C}$. We shall assume that the process is self-commuting:

$$[Y_t, Y_s] = 0, \text{ for all } t, s > 0.$$

This requires the consistency conditions $[Y_{\alpha\beta}(t), Y_s] = 0$ whenever $t > s$.

The process Y can then be represented as a classical stochastic process $\{y_t : t \geq 0\}$ with canonical (that is to say, minimal) probability space $(\Omega, \Sigma, \mathbb{Q})$ and associated filtration $\left\{\Sigma_{t]} : t \geq 0\right\}$ of sigma-algebras. For each $t > 0$, we define the Dyson-ordered exponential

$$\vec{T} \exp\left\{\int_0^t f(u)\, dY_u\right\} := \sum_{n\geq 0} \frac{1}{n!} \int_{\Delta_n(t)} dY_{t_n} \cdots dY_{t_1}\, f(t_n) \cdots f(t_1)$$

where $\Delta_n(t)$ is the ordered simplex $\{(t_n, \cdots, t_1) : t > t_n > \cdots > t_1 > 0\}$. The algebra generated by such Dyson-ordered exponentials will be denotes as $\mathfrak{C}_{t]}^Y$. Essentially, this is the spectral algebra of process up to time t and it can be understood (at least when the Y's are bounded) as the von Neumann algebra $\mathfrak{C}_{t]}^Y = \{Y_s : 0 \leq s \leq t\}''$ where we take the commutants in $\mathfrak{B}\left(\mathcal{H}_S \otimes \mathcal{F}_{t]}\right)$. In the following we shall assume that $\mathfrak{C}_{t]}^Y$ is a maximal commuting von Neumann sub-algebra of $\mathfrak{B}\left(\mathcal{F}_{t]}\right)$. In other words, there are no effects generated by the environmental noise other than those that can be accounted for by the observed process. The commutant of $\mathfrak{C}_{t]}^Y$ will be denoted as

$$\mathfrak{A}_{t]}^Y = \left(\mathfrak{C}_{t]}^Y\right)' = \left\{A \in \mathfrak{B}(\mathcal{H}_S \otimes \mathcal{F}) : [Z, A] = 0, \forall Z \in \mathfrak{C}_{t]}^Y\right\}$$

and this is often referred to as the algebra of observables that are not demolished by the observed process up to time t. We note the isotonic property $\mathfrak{C}_{t]}^Y \subset \mathfrak{C}_{s]}^Y$ whenever $t < s$ and it is natural to introduce the inductive limit algebra $\mathfrak{C}^Y := \lim_{t\to 0} \mathfrak{C}_{t]}^Y$.

From our assumption of maximality, we have that

$$\mathfrak{A}_{t]}^Y \equiv \mathfrak{B}(\mathcal{H}_S) \otimes \mathfrak{C}_{t]}^Y \otimes \mathfrak{B}\left(\mathcal{F}_{(t}\right).$$

A less simple theory would allow for effects of unobserved noises and one would have $\mathfrak{C}_{t]}^Y$ as the centre of $\mathfrak{A}_{t]}^Y$. One is then interested in conditional expectations from $\mathfrak{A}_{t]}^Y$ into $\mathfrak{C}_{t]}^Y$. Here we are interested in the Hilbert space aspects and so we take advantage of the simple setup that arises when $\mathfrak{C}_{t]}^{Y}$ is assumed maximal. (For the more general case where $\mathfrak{C}_{t]}^{Y}$ is not maximal, see [15].)

It is convenient to introduce the Hilbert spaces $\mathcal{H}_{t]}^{Y}$ and $\mathcal{G}_{t]}^{Y}$ defined though

$$\overline{\mathfrak{A}_{t]}^{Y}\left(\mathcal{H}_{S}\otimes\Psi_{t]}\right)}\equiv\mathcal{H}_{t]}^{Y}\otimes\left\{\mathbb{C}\Psi_{(t}\right\},\quad\overline{\mathfrak{C}_{t]}^{Y}\left(\Psi_{t]}\right)}\equiv\mathcal{G}_{t]}^{Y}$$

where we understand $\mathcal{H}_{t]}^{Y}$ as a subspace of $\mathcal{H}_{S}\otimes\mathcal{F}_{t]}$ and $\mathcal{G}_{t]}^{Y}$ as a subspace of $\mathcal{F}_{t]}$. From our maximality condition we shall have

$$\mathcal{H}_{t]}^{Y}=\mathcal{H}_{S}\otimes\mathcal{G}_{t]}^{Y}.$$

(Otherwise $\mathcal{H}_{S}\otimes\mathcal{G}_{t]}^{Y}$ would be only a subset of $\mathcal{H}_{t]}^{Y}$.) A Hilbert space isomorphism $\mathfrak{I}_{t}$ from $\mathcal{H}_{S}\otimes\mathcal{G}_{t]}^{Y}$ to $\mathcal{H}_{S}\otimes L^{2}\left(\Omega,\Sigma,\mathbb{Q}\right)$ is then defined by linear extension of the map

$$\phi\otimes\vec{T}\exp\left\{\int_{0}^{t}f\left(u\right)dY_{u}\right\}\mapsto\phi\vec{T}\exp\left\{\int_{0}^{t}f\left(u\right)dy_{u}\right\}$$

where the Dyson-ordered exponential on the right hand side has the same meaning as for its operator-valued counterpart.

We remark that, in particular, we have the following isomorphism between commutative von Neumann algebras:

$$\mathfrak{I}_{t}\,\mathfrak{C}_{t]}^{Y}\,\mathfrak{I}_{t}^{-1}=L^{\infty}\left(\Omega,\Sigma_{t},\mathbb{Q}\right).$$

By extension, $\mathfrak{C}^{Y}$ can be understood as being isomorphic to $L^{\infty}\left(\Omega,\Sigma,\mathbb{Q}\right)$. Now fix a unit vector ϕ in $\mathcal{H}_{S}$ and consider the evolved vector

$$\chi_{t}=U_{t}\ \left(\phi\otimes\Psi\right)$$

which will lie in $\mathcal{H}_{S}\otimes\mathcal{G}_{t]}^{Y}$. In particular, we have a Σ-measurable function $\phi_{t}\left(\cdot\right)$ corresponding to $\mathfrak{I}_{t}\,\chi_{t}$. Here ϕ_{t} is a $\mathcal{H}_{S}$-valued random variable on $\left(\Omega,\Sigma,\mathbb{Q}\right)$ which is adapted to the filtration $\left\{\Sigma_{t]}:t\geq0\right\}$. We shall have the normalization condition

$$\int_{\Omega}\left\|\phi_{t}\left(\omega\right)\right\|_{S}^{2}\,\mathbb{Q}\left[d\omega\right]=1.$$

In general, $\left\|\phi_{t}\left(\omega\right)\right\|_{S}^{2}$ is not unity, however, as it is positive and normalized, we may introduce a second measure $\mathbb{P}$ on $\left(\Omega,\Sigma\right)$ defined by

$$\mathbb{P}\left[A\right]:=\int_{A}\left\|\phi_{t}\left(\omega\right)\right\|_{S}^{2}\,\mathbb{Q}\left[d\omega\right]$$

whenever $A\in\Sigma_{t}$. We remark that, for $B\in\mathfrak{C}_{t]}^{Y}$, we have

$$\left\langle\chi_{t}\right|B\chi_{t}\rangle=\int_{\Omega}\mathfrak{I}_{t}\,B\,\mathfrak{I}_{t}^{-1}\,\mathbb{P}\left[d\omega\right].$$

It is convenient to introduce a normalized $\mathcal{H}_S$-valued variable ψ_t defined almost everywhere by

$$\psi_t(\omega) := \|\phi_t(\omega)\|_S^{-1} \ \phi_t(\omega).$$

We now define a conditional expectation $\mathcal{E}_{t]}^Y$ from $\mathfrak{A}_{t]}^Y$ to the von Neumann sub-algebra $\mathfrak{C}_{t]}^Y$ by the following identification almost everywhere

$$\mathfrak{I}_t \, \mathcal{E}_{t]}^Y [A] \ \mathfrak{I}_t^{-1} := \langle \psi_t | \, \mathfrak{I}_t \, A \, \mathfrak{I}_t^{-1} \, \psi_t \rangle .$$

This expectation leaves the state determined by χ_t invariant:

$$\begin{aligned}
\left\langle \chi_t \right| \mathcal{E}_{t]}^Y [A] \ \chi_t \Big\rangle &= \int_\Omega \langle \psi_t | \, \mathfrak{I}_t \, A \, \mathfrak{I}_t^{-1} \, \psi_t \rangle \ \mathbb{P}[d\omega] \\
&= \int_\Omega \langle \phi_t | \, \mathfrak{I}_t \, A \, \mathfrak{I}_t^{-1} \, \phi_t \rangle \ \mathbb{Q}[d\omega] \\
&= \langle \chi_t | \, A \, \chi_t \rangle .
\end{aligned}$$

This property then uniquely fixes the conditional expectation. If we consider the action of $\mathcal{E}_{t]}^Y$ from $\mathfrak{C}^Y$, only, to $\mathfrak{C}_{t]}^Y$ then this must play the role of a classical conditional expectation $\mathbb{E}_{t]}^y$ from Σ-measurable to Σ_t-measurable functions, again uniquely determined by the fact that it leaves a probability measure, in this case $\mathbb{P}$, invariant. We have the usual property that $\mathbb{E}_{t]}^y \circ \mathbb{E}_{s]}^y = \mathbb{E}_{t \wedge s]}^y$. We shall denote by $\mathbb{E}^y = \mathbb{E}_{0]}^y$ the expectation wrt. $\mathbb{P}$.

Let us first remark that the classical process $\{y_t : t \geq 0\}$ introduced above is not necessarily a martingale on $(\Omega, \Sigma, \mathbb{P})$ wrt. the filtration $\left\{ \Sigma_{t]} : t \geq 0 \right\}$. Indeed, we have

$$\begin{aligned}
\mathbb{E}^y [dy_t] &= \langle \chi_t | \, dY_t \, \chi_t \rangle \\
&= \left\langle \phi \otimes \Psi \right| d\tilde{Y}_t \, \phi \otimes \Psi \Big\rangle
\end{aligned}$$

where we define the output process $\tilde{Y}$ by

$$\tilde{Y}_t := U_t^\dagger \, Y_t \, U_t$$

From the quantum Itô calculus, we obtain

$$\begin{aligned}
d\tilde{Y}_t &= U_t^\dagger \, dY_t \, U_t + dU_t^\dagger \, Y_t \, U_t + U_t^\dagger \, Y_t \, dU_t + dU_t^\dagger \, Y_t \, dU_t \\
&\quad + dU_t^\dagger \, dY_t \, U_t + U_t^\dagger \, dY_t \, dU_t + dU_t^\dagger \, dY_t \, dU_t \\
&= U_t^\dagger \, Y_{\alpha\beta}(t) \ U_t \, dA_t^{\alpha\beta} + U_t^\dagger \, \mathcal{L}_{\alpha\beta}(1) \, Y_t \, U_t \, dA_t^{\alpha\beta} \\
&\quad + U_t^\dagger \ \left(L_{1\alpha}^\dagger Y_{1\beta} + Y_{\alpha 1} L_{1\beta} + L_{1\alpha}^\dagger Y_{11} L_{1\beta} \right) U_t \, dA_t^{\alpha\beta}.
\end{aligned}$$

Noting that $\mathcal{L}_{\alpha\beta}(1) = 0$, we see that

$$d\tilde{Y}_t = U_t^\dagger \left(Y_{\alpha\beta}(t) + L_{1\alpha}^\dagger Y_{1\beta}(y) + Y_{\alpha 1}(t) L_{1\beta} + L_{1\alpha}^\dagger Y_{11}(t) L_{1\beta} \right) U_t \, dA_t^{\alpha\beta}.$$

In particular, we define $\tilde{A}_t^{\alpha\beta} := U_t^\dagger A_t^{\alpha\beta} U_t$ and they are explicitly

$$\begin{aligned} d\tilde{\Lambda}_t &= d\tilde{A}_t^{11} = d\Lambda_t + J_t\left(W^\dagger L\right) dA_t^+ + J_t\left(L^\dagger W\right) dA_t^- + J_t\left(L^\dagger L\right) dt \\ d\tilde{A}_t^+ &= d\tilde{A}_t^{10} = J_t\left(W^\dagger\right) dA_t^+ + J_t\left(L^\dagger\right) dt \\ d\tilde{A}_t^- &= d\tilde{A}_t^{01} = J_t(W)\, dA_t^- + J_t(L)\, dt \end{aligned}$$

with $\tilde{A}_t^{00} = t$.

We remark that $\mathbb{E}^y[dy_t] = \bar{y}_t dt$ where

$$\begin{aligned} \bar{y}_t &= \left\langle \chi_t \right| \left(Y_{00}(t) + L_{10}^\dagger Y_{10}(t) + Y_{01}(t) L_{10} + L_{10}^\dagger Y_{11}(t) L_{10} \right) \chi_t \Big\rangle \\ &= \int_\Omega \mathbb{P}[d\omega] \left\langle \psi_t(\omega) \right| \left[L^\dagger\right]^\alpha [L]^\beta \, \psi_t(\omega) \Big\rangle \, y_{\alpha\beta}(t;\omega) \end{aligned}$$

and we use the notations $y_{\alpha\beta}(t;\cdot) = \mathfrak{I}_t Y_{\alpha\beta}(t) \mathfrak{I}_t^{-1}$. Therefore, a martingale on $(\Omega, \Sigma, \mathbb{P})$ wrt. the filtration $\left\{\Sigma_{t]} : t \geq 0\right\}$ is given by the process $\{\hat{y}_t : t \geq 0\}$ defined as

$$d\tilde{y}_t(\omega) = dy_t(\omega) - \left\langle \psi_t(\omega) \right| \left[L^\dagger\right]^\alpha [L]^\beta \, \psi_t(\omega) \Big\rangle \, y_{\alpha\beta}(t;\omega) \; dt.$$

4.1 Filtering based on observations of $Q_t = A_t^+ + A_t^-$

Let us choose, for our monitored observables, the process $Q_t = A_t^+ + A_t^-$. Here the output process will be $\tilde{Q}_t$ with differentials

$$d\tilde{Q}_t = J_t\left(W^\dagger\right) dA_t^+ + J_t(W)\, dA_t^- + J_t\left(L^\dagger + L\right) dt.$$

By the previous arguments, we construct a classical process $y = q$ giving the distribution of Q in the vacuum state: as is well-known, this is a Wiener process and $(\Omega, \Sigma, \mathbb{Q})$ will be the canonical Wiener space. (In fact, $\mathfrak{I}_t$ is then the Wiener-Itô-Segal isomorphism [18].) The corresponding martingale process will then be $\hat{q}$ defined through

$$d\hat{q}_t = dq_t - 2\lambda_t dt, \quad \hat{q}_0 = 0$$

where

$$\lambda_t(\omega) := \frac{1}{2} \left\langle \psi_t(\omega) \right| \left(L + L^\dagger\right) \psi_t(\omega) \right\rangle.$$

A differential equation for ψ_t can be obtained as follows. The state $\chi_t = U_t\, \phi \otimes \Psi$ will satisfy the vector-process QSDE

$$d\chi_t = L_{\alpha\beta} dA_t^{\alpha\beta} \chi_t = L_{\alpha 0} dA_t^{\alpha 0} \chi_t \tag{44}$$

since we have $dA_t^{\alpha 1} \chi_t = U_t dA_t^{\alpha 1} \phi \otimes \Psi = 0$ - that is the Itô differentials $d\Lambda_t$ and dA_t^- commute with U_t annihilate the Fock vacuum. It is convenient to restore the annihilation differential, this time as $L_{10} dA_t^- \phi_t = 0$, in which case we obtain the equivalent QSDE

$$d\chi_t = -\left(iH + \frac{1}{2}L^\dagger L\right)\chi_t\, dt + L\, dQ_t\, \chi_t.$$

It should be immediately obvious that the process $\phi_t(\cdot)$ will satisfy the sde $|d\phi_t\rangle = L|\phi_t\rangle\, dq_t - \left(iH + \frac{1}{2}L^\dagger L\right)|\phi_t\rangle\, dt$. Here which we shall write $\phi_t(\cdot)$ as $|\phi_t(\cdot)\rangle$ to emphasize the fact that it in $\mathcal{H}_S$-valued process. From the Itô rule $(dq_t)^2 = dt$, we find that

$$d\|\phi_t\|^2 = \langle d\phi_t|\phi_t\rangle + \langle \phi_t|d\phi_t\rangle + \langle d\phi_t|d\phi_t\rangle = \left\langle \phi_t\right|\left(L^\dagger + L\right)\phi_t\big\rangle\, dq_t.$$

The derivative rule is

$$\begin{aligned} d\|\phi_t\|^{-1} &= \left(\|\phi_t\|^2 + d\|\phi_t\|^2\right)^{-1/2} - \left(\|\phi_t\|^2\right)^{-1/2} \\ &= \|\phi_t\|^{-1} \sum_{k\geq 1} \binom{-1/2}{k} \|\phi_t\|^{-2k} \left(d\|\phi_t\|^2\right)^k \end{aligned} \tag{45}$$

and here we must use the Itô rule $d\|\phi_t\|^2 = 2\lambda_t dt$ where here $\lambda_t := \frac{1}{2}\left\langle\psi_t\right|\left(L^\dagger + L\right)\psi_t\big\rangle$. This leads to

$$d\|\phi_t\|^{-1} = -\frac{1}{2}\|\phi_t\|^{-1}\lambda_t\, dq_t + \frac{3}{8}\|\phi_t\|^{-1}\lambda_t^2\, dt.$$

This leads to the SDE for $|\psi_t\rangle$: $|d\psi_t\rangle = \|\phi_t\|^{-1}|d\phi_t\rangle + d\left(\|\phi_t\|^{-1}\right)|\phi_t\rangle + d\left(\|\phi_t\|^{-1}\right)|d\phi_t\rangle$ and this is explicitly

$$|d\psi_t\rangle = (L - \lambda_t)|\psi_t\rangle\, dq_t + \left(-iH - \frac{1}{2}L^\dagger L - \lambda_t L + \frac{3}{2}\lambda_t^2\right)|\psi_t\rangle\, dt. \tag{46}$$

Finally, substituting in for the martingale process $\hat{q}$ we obtain

$$|d\psi_t\rangle = (L - \lambda_t)|\psi_t\rangle\, d\hat{q}_t + \left(-iH - \frac{1}{2}\left(L^\dagger L - 2\lambda_t L + \lambda_t^2\right)\right)|\psi_t\rangle\, dt. \tag{47}$$

4.2 Filtering based on observations of Λ_t

Let us now choose, for our monitored observables, the gauge process Λ_t. Unfortunately, we hit on a snag: the gauge is trivially zero in the vacuum state, that is, it is a Poisson process of zero intensity. A trick to deal with this is to replace the gauge process with a unitarily equivalent process Λ_t^f given by

$$\Lambda_t^f := e^{A^-(f) - A^+(f)}\,\Lambda_t\, e^{A^+(f) - A^-(f)}$$

for $f \in L^2(\mathbb{R}^+, dt)$ a real-valued function with $f(t) > 0$ for all times $t > 0$. The process is defined alternatively by

$$d\Lambda_t^f = d\Lambda_t + f(t)\, dA_t^+ + f(t)\, dA_t^- + f(t)^2\, dt, \quad \Lambda_0^f = 0.$$

It satisfies the Itô rule $d\Lambda_t^f \, d\Lambda_t^f = d\Lambda_t^f$ and we have that $\left\langle \Psi | \, d\Lambda_t^f \, \Psi \right\rangle = f(t)^2 \, dt$. We see that Λ_t^f corresponds to a classical process $y = n^f$ which is a non-homogeneous Poisson process with intensity density f^2 and we shall denote by $(\Omega, \Sigma, \mathbb{Q})$ the canonical probability space.

Now, from (44), we find $d\chi_t = -\left(iH + \frac{1}{2}L^\dagger L + fL\right)\chi_t \, dt + f^{-1} L \, dn_t^f \, \chi_t$ and the corresponding $\mathcal{H}_S$-valued process satisfies

$$
|d\phi_t\rangle = -\left(iH + \frac{1}{2}L^\dagger L + fL\right)|\phi_t\rangle \; dt + f^{-1}L \; |\phi_t\rangle \; dn_t^f
$$

from which we find that

$$
d \left\| \phi_t \right\|^2 = \frac{1}{f(t)^2} \left\langle \phi_t \right| \left(L^\dagger + f\right)(L + f)\, \phi_t \rangle \; \left(dn_t^f - f(t)^2 \, dt \right).
$$

Substituting into (45) we find after some re-summing

$$
\begin{aligned}
d \left\| \phi_t \right\|^{-1} \quad &= \quad \left\| \phi_t \right\|^{-1} \left(\frac{f(t)}{\sqrt{\nu_t + 2f(t)\lambda_t + f(t)^2}} - 1 \right) dn_t^f \\
&\quad + \frac{1}{2} \left\| \phi_t \right\|^{-1} \left(\nu_t + 2f(t)\lambda_t \right) dt
\end{aligned}
$$

where $\nu_t(\omega) := \left\langle \psi_t(\omega) \, | \, L^\dagger L \, \psi_t(\omega) \right\rangle$ and $\lambda_t(\omega)$ is as defined above. Note that $\nu_t + 2f(t)\lambda_t + f(t)^2 = \left\langle \psi_t(\omega) \, | \, \left(L^\dagger + f(t)\right)\left(L + f(t)\right) \, \psi_t(\omega) \right\rangle$. The resulting sde for the normalized state ψ_t is then

$$
\begin{aligned}
|d\psi_t\rangle \quad &= \quad \left(\frac{L + f(t) - \sqrt{\nu_t + 2f(t)\lambda_t + f(t)^2}}{\sqrt{\nu_t + 2f(t)\lambda_t + f(t)^2}} \right) |\psi_t\rangle \; dn_t^f \\
&\quad + \left(-iH - \frac{1}{2}L^\dagger L - f(t)\,L + \frac{1}{2}\left[\nu_t + 2f(t)\lambda_t\right] \right) |\psi_t\rangle \; dt.
\end{aligned}
$$

Now n^f is decomposed into martingale and deterministic part according to

$$
dn^f = d\hat{n}^f + \left(\nu_t + 2f(t)\lambda_t + f(t)^2 \right) dt
$$

and so we have

$$
\begin{aligned}
|d\psi_t\rangle \quad &= \quad \left(\frac{L + f(t) - \sqrt{\nu_t + 2f(t)\lambda_t + f(t)^2}}{\sqrt{\nu_t + 2f(t)\lambda_t + f(t)^2}} \right) |\psi_t\rangle \; d\hat{n}_t^f \\
&\quad + \Bigg[-iH - \frac{1}{2}L^\dagger L - f(t)\,L + \frac{1}{2}\left[\nu_t + 2f(t)\lambda_t\right] \\
&\quad + \left(L + f(t) - \sqrt{\nu_t + 2f(t)\lambda_t + f(t)^2} \right) \sqrt{\left(\nu_t + 2f(t)\lambda_t + f(t)^2 \right)} \Bigg] |\psi_t\rangle \; dt.
\end{aligned}
$$

We now take the limit $f \to 0$ to obtain the result we want and this leaves us with the sde

$$
\begin{aligned}
|d\psi_t\rangle &= \left(\frac{L - \sqrt{\nu_t}}{\sqrt{\nu_t}}\right)|\psi_t\rangle \, d\hat{n}_t + \left(-iH - \frac{1}{2}L^\dagger L - \frac{1}{2}\nu_t + \sqrt{\nu_t}L\right)|\psi_t\rangle \, dt \\
&= \left(\frac{L - \sqrt{\nu_t}}{\sqrt{\nu_t}}\right)|\psi_t\rangle \, dn_t + \left(-iH - \frac{1}{2}L^\dagger L + \frac{1}{2}\nu_t\right)|\psi_t\rangle \, dt.
\end{aligned}
$$

Here n_t will be a non-homogeneous Poisson process with intensity ν_t.

5 Appendix

5.1 Bosonic Noise

Let $\mathcal{H}$ be a fixed Hilbert space. The n-particle Bose states take the basic form $\phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_n = \sum_{\sigma \in \mathfrak{S}_n} \phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(n)}$ where we sum over the permutation group $\mathfrak{S}_n$. The n-particle state space is denoted $\mathcal{H}^{\hat{\otimes} n}$ and the Bose Fock space, with one particle space $\mathcal{H}$, is then $\Gamma_+(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\hat{\otimes} n}$ with vacuum space $\mathcal{H}^{\hat{\otimes} 0}$ spanned by a single vector Ψ.

The Bosonic creator, annihilator and differential second quantization fields are, respectively, the following operators on Fock space

$$
\begin{aligned}
A^+(\psi) \; \phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_n &= \sqrt{n+1}\, \psi \hat{\otimes} \phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_n \\
A^-(\psi) \; \phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_n &= \frac{1}{\sqrt{n}} \sum_j \langle \psi | \phi \rangle \, \hat{\otimes} \phi_1 \hat{\otimes} \cdots \hat{\otimes} \widehat{\phi_j} \hat{\otimes} \cdots \hat{\otimes} \phi_n \\
d\Gamma(T) \; \phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_n &= \sum_j \phi_1 \hat{\otimes} \cdots \hat{\otimes} \left(T\phi_j\right) \hat{\otimes} \cdots \hat{\otimes} \phi_n
\end{aligned}
$$

where $\psi \in \mathcal{H}$ and $T \in \mathfrak{B}(\mathcal{H})$.

Now choose $\mathcal{H} = L^2(\mathbb{R}^+, dt)$ and on the Fock space $\mathcal{F} = \Gamma_+\left(L^2(\mathbb{R}^+, dt)\right)$ set

$$
A_t^{\pm} := A^{\pm}\left(1_{[0,t]}\right); \quad \Lambda_t := d\Gamma\left(\tilde{1}_{[0,t]}\right) \tag{48}
$$

where $1_{[0,t]}$ is the characteristic function for the interval $[0,t]$ and $\tilde{1}_{[0,t]}$ is the operator on $L^2(\mathbb{R}^+, dt)$ corresponding to multiplication by $1_{[0,t]}$.

An integral calculus can be built up around the processes $A_t^{\pm}, \Lambda_t$ and t and is known as (Bosonic) quantum stochastic calculus. This allows us to consider quantum stochastic integrals of the type $\int_0^T \{F_{10}(t) \otimes dA_t^+ + F_{01}(t) \otimes dA_t^- + F_{11}(t) \otimes d\Lambda_t + F_{00}(t) \otimes dt\}$ on $\mathcal{H}_0 \otimes \Gamma_+\left(L^2(\mathbb{R}^+, dt)\right)$ where $\mathcal{H}_0$ is some fixed Hilbert space (termed the initial space).

We note the natural isomorphism $\Gamma_+\left(L^2(\mathbb{R}^+, dt)\right) \cong \mathcal{F}_{t]} \otimes \mathcal{F}_{(t}$ where $\mathcal{F}_{t]} = \Gamma_+\left(L^2([0,t], dt)\right)$ and $\mathcal{F}_{(t} = \Gamma_+\left(L^2((t,\infty), dt)\right)$. A family $(F_t)_t$ of operators on $\mathcal{H}_0 \otimes \Gamma_+\left(L^2(\mathbb{R}^+, dt)\right)$ is said to be adapted if F_t acts trivially on the future space $\mathcal{H}_{(t}$ for each t.

The Leibniz rule however breaks down for this theory since products of stochastic integrals must be put to Wick order before they can be re-expressed again as stochastic integrals. The new situation is summarized by the quantum Itô rule $d\left(FG\right)=\left(dF\right)G+F\left(dG\right)+\left(dF\right)\left(dG\right)$ and the quantum Itô table

$\times$	dA^{+}	$d\Lambda$	dA^{-}	dt
dA^{+}	0	0	0	0
$d\Lambda$	dA^{+}	$d\Lambda$	0	0
dA^{-}	dt	dA^{-}	0	0
dt	0	0	0	0

It is convenient to denote the four basic processes as follows:

$$A_t^{\alpha\beta}=\left\{\begin{array}{cc} \Lambda_t, & (1,1)\,; \\ A_t^{+}, & (1,0)\,; \\ A_t^{-}, & (0,1)\,; \\ t, & (0,0)\,. \end{array}\right.$$

The Itô table then simplifies to $dA_t^{\alpha\beta}dA_t^{\mu\nu}=0$ except for the cases

$$dA_t^{\alpha 1}dA_t^{1\beta}=dA_t^{\alpha\beta}. \tag{49}$$

The fundamental result [17] is that there exists an unique solution U_t to the quantum stochastic differential equation (QSDE)

$$dU_t=L_{\alpha\beta}\otimes dA_t^{\alpha\beta},\qquad U_0=1$$

whenever the coefficients $L_{\alpha\beta}$ are in $\mathfrak{B}\left(\mathcal{H}_0\right)$. The solution is automatically adapted and, moreover, will be unitary provided that the coefficients take the form (16).

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