Queueing Systems 47, 45–52, 2004

A Note on the $GI/GI/\infty$ System with Identical Service and Interarrival-Time Distributions

E.A. VAN DOORN and A.A. JAGERS {e.a.vandoorn;a.a.jagers}@utwente.nl
Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

Received 10 December 2002; Revised 19 June 2003

Dedicated to Frits Göbel on the occasion of his 70th birthday

Abstract. We study the stationary distribution of the number of busy servers in a $GI/GI/\infty$ system in which the service-time distribution is identical to the interarrival-time distribution, and obtain several representations for the variance. As a result we can verify an expression for the variance, conjectured by Rajaratnam and Takawira (IEEE Trans. Vehicular Technol. 50 (2001) 954–970), when the common distribution of interarrival and service times is a gamma distribution.

Keywords: infinite-server system, busy-server distribution, gamma distribution, Mellin transform

AMS subject classification: 60K25, 44A15, 68M20, 90B22

1. Introduction

We are interested in the steady-state distribution – and in particular the first two moments – of the number of busy servers in a $GI/GI/\infty$ system in which the service time distribution is identical to the interarrival-time distribution. This particular model comes about when an infinite pool of customers has to be served by a tandem service system consisting of a finite group of servers (of size N, say) followed by an infinite-server group. If service times at all servers are independent and identically distributed random variables, then the arrival process at the second group is simply a superposition of N independent renewal processes, each with an interarrival-time distribution equal to the service-time distribution. The number of busy servers in the infinite-server system is therefore the sum of N independent and identically distributed random variables, each representing the number of busy servers in the infinite-server group when $N = 1$. It is the stationary distribution of the latter random variable which is our concern in this note.

The model of a tandem system consisting of a finite-server group followed by an infinite-server group, but with a Poisson arrival process to the first group, has been proposed by Rajaratnam and Takawira in [6] and earlier papers (see the references in [6]) in the performance analysis of cellular mobile networks. In this setting it is of interest to know the stationary distribution – and in particular the mean and variance – of the number of busy servers in the infinite-server group, but this distribution seems difficult to obtain. Rajaratnam and Takawira [6] therefore resort to an approximate analysis which involves the model studied in this note.

We reveal in the next section a remarkably simple expression for $M(t)$, the *time-dependent* mean number of busy servers in the $GI/GI/\infty$ system, when interarrival and service times are identically distributed. This result enables us to find V, the variance of the *stationary* number of busy servers, by exploiting a classic result of Takács [7]. Some special cases are considered in section 3. In particular, we will verify an expression for V which was conjectured by Rajaratnam and Takawira [6] in the case that interarrival and service times have identical gamma distributions.

We finally note that an interesting representation for V has been given by Yamazaki et al. [8], but it does not seem to lead to the explicit expression of section 2.

2. The number of busy servers

We start off with some general notation and terminology. Let us consider a $GI/GI/\infty$ system in which the interarrival times have a common distribution function F, and the service times have a common distribution function H. Both F and H are supposed to have a finite first moment, so that the arrival and service rates

$$\lambda \equiv \left(\int_0^\infty t\,\mathrm{d}F(t)\right)^{-1} \quad \text{and} \quad \mu \equiv \left(\int_0^\infty t\,\mathrm{d}H(t)\right)^{-1},$$

respectively, are positive. We assume that the system starts empty at time 0 and that the time until the first arrival has distribution function G. By m we denote the renewal function associated with F, that is,

$$m(t) \equiv \sum_{n=1}^{\infty} F^{n*}(t), \quad t \geqslant 0,$$

where F^{n*} stands for the n-fold convolution of F. By $X(t)$ and X we denote the number of busy servers at time t and in steady state, respectively, and we let $B_n(t)$ and B_n be their respective nth binomial moments, that is,

$$B_n(t) \equiv \sum_{k=n}^{\infty} \binom{k}{n} \Pr\bigl\{X(t) = k\bigr\} \quad \text{and} \quad B_n \equiv \sum_{k=n}^{\infty} \binom{k}{n} \Pr\{X = k\}, \qquad n \geqslant 1.$$

The next theorem summarizes some classic results of Takács' [7] on these binomial moments.

Theorem 1. (i) The binomial moments $B_n(t)$ exist for all n and $t \geqslant 0$, and, if $G = F$, satisfy the recurrence relation

$$B_n(t) = \int_0^t B_{n-1}(t-u)\bigl(1 - H(t-u)\bigr)\,\mathrm{d}m(u), \quad t \geqslant 0,\ n = 1, 2, \ldots, \tag{1}$$

where $B_0(t) = 1$.

(ii) There exists a unique steady-state distribution with binomial moments B_n satisfying

$$B_n = \lambda \int_0^\infty B_{n-1}(t)\big(1 - H(t)\big)\,\mathrm{d}t, \quad n = 1, 2, \ldots, \tag{2}$$

where $B_n(t),\ n = 0, 1, \ldots,$ are the time-dependent binomial moments in the case $G = F$.

Takács requires F to be non-lattice for (2), but, as observed in [8], this condition can be dropped. For generalizations of Takács' findings and related results we refer to Pakes and Kaplan [5], Kaplan [3], Liu et al. [4] and Ayhan et al. [2], and the references there.

As announced, we wish to obtain the mean M and variance V of the stationary number of busy servers in the case that $F = H$. To this end we first observe a surprisingly simple corollary to theorem 1 concerning $M(t)$, the mean number of busy servers at time t.

Corollary 2. If $G = F = H$ then $M(t) = H(t)$ for all $t \geqslant 0$.

Proof. From theorem 1(i) we see that

$$M(t) = B_1(t) = \int_0^t \big(1 - H(t-u)\big)\,\mathrm{d}m(u) = m(t) - H * m(t).$$

But since $m(t) \equiv \sum_{n=1}^\infty F^{n*}(t) = \sum_{n=1}^\infty H^{n*}(t)$, the result follows immediately. □

Remark. A more direct (and perhaps more appealing) argument leads to a generalization of corollary 2 in which we do not require $G = F$. Indeed, let T_i be the arrival time of the ith customer, and S_i his service time. Then we can write

$$X(t) = \sum_{i=1}^\infty I_{[T_i, T_i+S_i)}(t), \quad t \geqslant 0,$$

where I_A denotes the indicator function of a set A. Taking expectations on both sides we get

$$M(t) = \sum_{i=1}^\infty \Pr\{T_i \leqslant t < T_i + S_i\} = \sum_{i=1}^\infty \Pr\{T_i \leqslant t < T_{i+1}\} = \Pr\{T_1 \leqslant t\},$$

since $T_{i+1} - T_i$ and S_i are independent and identically distributed, and also independent of T_i. So we actually have $M(t) = G(t)$ for all $t \geqslant 0$.

If $F = H$ then theorem 1(ii) (or Little's formula) tells us $M = B_1 = 1$, since $\lambda^{-1} = \mu^{-1} = \int_0^\infty (1 - H(t))\,\mathrm{d}t$. Moreover, corollary 2 tells us that $B_1(t) = M(t) = H(t)$, which upon substitution in (2) gives us B_2 and hence $V = 2B_2 + M - M^2 = 2B_2$. Summarizing we have the following.

Theorem 3. If $F = H$ then the mean M and variance V of the stationary number of busy servers in the system $GI/GI/\infty$ are given by

$$M = 1 \quad \text{and} \quad V = 2\mu \int_0^\infty H(t)\bigl(1 - H(t)\bigr)\,\mathrm{d}t. \tag{3}$$

We easily see that changing the unit of time does not affect the value of V (which is obvious also on physical grounds). We also note that

$$0 < V < 2\mu \int_0^\infty \bigl(1 - H(t)\bigr)\,\mathrm{d}t = 2.$$

The special case of a gamma distribution (discussed in section 3.2) may be used to show that both lower and upper bound can be approached arbitrarily close by choosing the parameter c in (8) sufficiently large and small, respectively.

It is interesting to observe that V can be represented as

$$V = \mu \int_0^\infty \int_0^\infty |t_1 - t_2|\,\mathrm{d}H(t_1)\,\mathrm{d}H(t_2), \tag{4}$$

so that V/μ may be interpreted as the expected absolute value of the difference of two service times.

If H has a continuous density h on $(0, \infty)$, with $h(t) = \mathcal{O}(t^{a-1})$ for some $a > 0$ and $t \downarrow 0$, and $h(t) = \mathcal{O}(t^{-b-2})$ for some $b > 0$ and $t \to \infty$, we may also express V in terms of the Mellin transform of h, given by

$$M(h, z) \equiv \int_0^\infty t^{z-1} h(t)\,\mathrm{d}t, \quad 1 - a < \Re(z) < 2 + b.$$

To do so we use the general Parseval relation for Mellin transforms (and integration by parts), and end up with the contour-integral representation

$$V = \frac{\mu}{\pi \mathrm{i}} \int_{\kappa - \mathrm{i}\infty}^{\kappa + \mathrm{i}\infty} \frac{M(h, z+1) M(h, 2-z)}{z(z-1)}\,\mathrm{d}z, \quad -\min(a, b) < \kappa < 0. \tag{5}$$

3. Special cases

In this section we will look more closely at two special cases, namely H is a mixture of a degenerate and an exponential distribution, and H is a gamma distribution. Since V is independent of the unit of time it is no restriction of generality to assume $\mu = 1$ in the remainder of this section.

3.1. *Mixtures of degenerate and exponential distributions*

Let us assume that the interarrival and service times in the $GI/GI/\infty$ system are all drawn from a mixture of a degenerate and an exponential distribution with means 1, that is, for some p, $0 \leqslant p < 1$,

$$H(t) = F(t) = pI_{[1,\infty)}(t) + (1-p)\big(1 - \mathrm{e}^{-t}\big), \quad t \geqslant 0. \tag{6}$$

Substitution of this distribution function in (3) readily yields

$$V \equiv V(p) = (1-p)\left(1 + \left(\frac{4}{\mathrm{e}} - 1\right)p\right). \tag{7}$$

Rajaratnam and Takawira [6] observed that $V(0) = 1$ and $V(1) = 0$, and found via simulation that $V(1/2) \approx 5/8 = 0.6250$ (in reality, $V(1/2) = 1/4 + 1/\mathrm{e} \approx 0.6179$). Hence, they proposed the quadratic interpolation formula

$$V(p) \approx (1-p)\left(1 + \frac{p}{2}\right),$$

which is pretty close to (7) since $4/\mathrm{e} - 1 \approx 0.4715$.

3.2. *Gamma distributions*

Now suppose that the interarrival and service times in the $GI/GI/\infty$ system are drawn from a common gamma distribution with mean 1, that is,

$$F(t) = H(t) = \frac{1}{\Gamma(c)} \int_0^t c(cu)^{c-1} \mathrm{e}^{-cu}\, \mathrm{d}u, \quad t \geqslant 0, \tag{8}$$

where c is some positive constant and Γ is the gamma function

$$\Gamma(a) \equiv \int_0^\infty u^{a-1} \mathrm{e}^{-u}\, \mathrm{d}u, \quad a > 0.$$

Analysing the $E_c/E_c/\infty$ queue by standard Markovian techniques, Rajaratnam and Takawira [6] found that for small integral values of c the variance V of the stationary number of busy servers is given by

$$V \equiv V(c) = 2^{1-2c} \frac{\Gamma(2c+1)}{\Gamma(c+1)^2}, \tag{9}$$

and they conjectured the validity of this expression for arbitrary $c > 0$. Before proving this in two different ways, we note that by the duplication formula for the gamma function (see [1], equation (6.1.18)) $V(c)$ may also be written as

$$V(c) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(c + 1/2)}{\Gamma(c+1)}. \tag{10}$$

As a consequence, by [1], equation (6.1.47),

$$V(c) \sim \frac{2}{\sqrt{\pi c}}\left\{1 - \frac{1}{8}c^{-1} + \cdots\right\} \quad \text{as } c \to \infty.$$

Moreover, $V(c)$ is a decreasing function of c by the log-convexity of the gamma function on $(0, \infty)$. In fact, letting $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ (the *digamma function*), we have

$$\frac{\mathrm{d}}{\mathrm{d}c} \log V(c) = \psi\left(c + \frac{1}{2}\right) - \psi(c+1) < 0,$$

since, by [1], equation (6.4.10),

$$\psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2} > 0, \quad z \notin \{0, -1, -2, \ldots\}.$$

Theorem 4. The variance of the stationary number of busy servers in the system $GI/GI/\infty$ when interarrival and service times have a common distribution function (8) is given by (9), or (10), for all $c > 0$.

Proof. Elementary substitution of (8) in (3), followed by appropriate changes of variables and a change in the order of integration, leads to

$$\begin{aligned}
V(c) &= \frac{2}{(\Gamma(c))^2} \int_0^\infty \left(\int_0^t c(cu)^{c-1} \mathrm{e}^{-cu}\, \mathrm{d}u\right)\left(\int_t^\infty c(cv)^{c-1}\mathrm{e}^{-cv}\, \mathrm{d}v\right) \mathrm{d}t \\
&= \frac{2}{(\Gamma(c))^2} \int_0^\infty \left(\int_0^1 (ct)^c x^{c-1} \mathrm{e}^{-cxt}\, \mathrm{d}x\right)\left(\int_1^\infty (ct)^c y^{c-1} \mathrm{e}^{-cyt}\, \mathrm{d}y\right) \mathrm{d}t \\
&= \frac{2}{(\Gamma(c))^2} \int_0^1 \int_1^\infty (xy)^{c-1} \int_0^\infty (ct)^{2c} \mathrm{e}^{-c(x+y)t}\, \mathrm{d}t\, \mathrm{d}y\, \mathrm{d}x \\
&= \frac{2}{c(\Gamma(c))^2} \int_0^1 \int_1^\infty \frac{(xy)^{c-1}}{(x+y)^{2c+1}} \int_0^\infty u^{2c} \mathrm{e}^{-u}\, \mathrm{d}u\, \mathrm{d}y\, \mathrm{d}x \\
&= \frac{2\Gamma(2c+1)}{(\Gamma(c+1))^2} \int_0^1 \int_1^\infty \frac{c(xy)^{c-1}}{(x+y)^{2c+1}}\, \mathrm{d}y\, \mathrm{d}x,
\end{aligned}$$

where we have used $\Gamma(c+1) = c\Gamma(c)$. Finally, substitution of $y = u/v$ and $x = 1/v$ and another change in the order of integration gives us

$$\begin{aligned}
\int_0^1 \int_1^\infty \frac{c(xy)^{c-1}}{(x+y)^{2c+1}}\, \mathrm{d}y\, \mathrm{d}x &= \int_1^\infty \int_v^\infty \frac{c\, u^{c-1}}{(1+u)^{2c+1}}\, \mathrm{d}u\, \mathrm{d}v \\
&= \int_1^\infty \int_1^u \frac{c\, u^{c-1}}{(1+u)^{2c+1}}\, \mathrm{d}v\, \mathrm{d}u
\end{aligned}$$

$$= \int_1^{\infty} \frac{c\, u^{c-1}(u-1)}{(1+u)^{2c+1}}\, \mathrm{d}u$$
$$= \int_1^{\infty} \mathrm{d}\left(\frac{-u^c}{(1+u)^{2c}}\right) = \frac{1}{2^{2c}}, \tag{11}$$

as required. □

Remark. The integral in (11) is closely related to a particular value of a hypergeometric function. In fact, by [1], equations (15.3.1) and (15.1.21), we have

$$\int_1^{\infty} \frac{c\, u^{c-1}(u-1)}{(1+u)^{2c+1}}\, \mathrm{d}u = \int_0^1 \frac{ct^{c-1}(t-1)}{(1+t)^{2c+1}}\, \mathrm{d}t$$
$$= \frac{\Gamma(c+1)\Gamma(2)}{\Gamma(c+2)} F(2c+1, c; c+2; -1) = 2^{-2c},$$

as before.

Second proof of theorem 4. We apply (5) with $h(t) = c^c t^{c-1} \mathrm{e}^{-ct}/\Gamma(c)$ and $b = \infty$ to obtain

$$M(h, z) = \frac{c^c}{\Gamma(c)} \int_0^{\infty} t^{z+c-2} \mathrm{e}^{-ct}\, \mathrm{d}t = c^{1-z} \frac{\Gamma(z+c-1)}{\Gamma(c)}, \quad \Re(z) > 1-c,$$

and

$$V(c) = \frac{c^{-1}}{\Gamma(c)^2 \pi \mathrm{i}} \int_{\kappa - \mathrm{i}\infty}^{\kappa + \mathrm{i}\infty} \frac{\Gamma(z+c)\Gamma(1-z+c)}{z(z-1)}\, \mathrm{d}z, \quad -c < \kappa < 0.$$

By replacing $z(z-1)$ with $\Gamma(2-z)/\Gamma(-z)$, followed by a substitution of $z = -c - s$, we get (up to a factor) a Mellin–Barnes integral for the hypergeometric function value mentioned in the above remark. Actually, by [1], equations (15.3.2) and (15.1.21), it follows that

$$V(c) = \frac{c^{-1}}{\Gamma(c)^2 \pi \mathrm{i}} \int_{\kappa - \mathrm{i}\infty}^{\kappa + \mathrm{i}\infty} \frac{\Gamma(s+2c+1)\Gamma(s+c)\Gamma(-s)}{\Gamma(s+c+2)}\, \mathrm{d}s$$
$$= 2 \frac{\Gamma(2c+1)}{\Gamma(c+2)\Gamma(c+1)} F(2c+1, c; c+2; -1) = 2^{1-2c} \frac{\Gamma(2c+1)}{\Gamma(c+1)^2},$$

as before. □

We finally note that for integral values of c a third proof of (9) may be based on the interpretation (4) of $V(c)$ as the expected absolute value of the difference of two service times. Namely, imagine two service times S_1 and S_2, each consisting of c exponentially distributed phases of mean $1/c$, starting at time 0, and a counter going up (or down) one unit each time a phase of S_1 (or S_2) elapses, until one of the service times ends. If we denote the state of the counter after the nth count by X_n, and let

$$N \equiv \min\{n\colon |X_{c+n}| = c - n\},$$

then, for $n = 1, 2, \ldots, N$, X_n is distributed as $\sum_{k=1}^{n} Y_k$, where $Y_1, Y_2, \ldots$ are independent random variables taking the values $+1$ and -1 with equal probabilities. Moreover, at the time of the Nth count either S_1 or S_2 has ended, and the remaining part of the surviving service time consists of $c - N$ exponentially distributed phases with means $1/c$. Consequently,

$$V(c) = \mathbb{E}|S_1 - S_2| = \frac{c - \mathbb{E}N}{c}. \tag{12}$$

A combinatorial argument shows that N has a truncated negative binomial distribution

$$\Pr\{N = n\} = 2\binom{c+n-1}{n}\left(\frac{1}{2}\right)^{c+n}, \quad n = 0, 1, \ldots, c-1,$$

with first moment

$$\mathbb{E}N = c - c\binom{2c}{c}\left(\frac{1}{2}\right)^{2c-1},$$

which, together with (12), gives us (9) again.

Acknowledgement

The authors thank B. Sanders of Vodafone Group Research and Development in Maastricht for drawing their attention to reference [6].

References

[1] M. Abramowitz and I.A. Stegun (eds.), *Handbook of Mathematical Functions* (Dover, New York, 1972).

[2] H. Ayhan, J. Limon-Robles and M.A. Wortman, On the time-dependent occupancy and backlog distributions for the $GI/G/\infty$ queue, J. Appl. Probab. 36 (1999) 558–569.

[3] N. Kaplan, Limit theorems for a $GI/G/\infty$ queue, Ann. Probab. 3 (1975) 780–789.

[4] L. Liu, B.R.K. Kashyap and J.G.C. Templeton, On the $GI^X/G/\infty$ system, J. Appl. Probab. 27 (1990) 671–683.

[5] A.G. Pakes and N. Kaplan, On the subcritical Bellman–Harris process with immigration, J. Appl. Probab. 11 (1974) 652–668.

[6] M. Rajaratnam and F. Takawira, Handoff traffic characterization in cellular networks under nonclassical arrivals and service time distributions, IEEE Trans. Vehicular Technol. 50 (2001) 954–970.

[7] L. Takács, On a coincidence problem in telephone traffic, Acta Math. Acad. Sci. Hung. 9 (1958) 45–80.

[8] G. Yamazaki, K. Sigman and M. Miyazawa, Moments in infinite channel queues, Comput. Math. Appl. 24 (1992) 1–6.