# On the Significance of Digits in Interval Notation 

M.H. van Emden<br>Computer Science Dept, University of Victoria<br>Technical Report DCS-270-IR


#### Abstract

To analyse the significance of the digits used for interval bounds, we clarify the philosophical presuppositions of various interval notations. We use information theory to determine the information content of the last digit of the numeral used to denote the interval's bounds. This leads to the notion of efficiency of a decimal digit: the actual value as percentage of the maximal value of its information content. By taking this efficiency into account, many presentations of intervals can be made more readable at the expense of negligible loss of information.


## 1 Introduction

Once upon a time, it was a matter of professional ethics among computers never to write a meaningless decimal. Since then computers have become machines and thereby lost any form of ethics, professional or otherwise. The human computers of yore were helped in their ethical behaviour by the fact that it took effort to write spurious decimals. Now the situation is reversed: the lazy way is to use the default precision of the I/O library function. As a result, we are deluged with meaningless decimals.

Of course interval arithmetic is not guilty of such negligence. After all, the very raison d'être of the subject is to be explicit about the precision of computed results. Yet, even interval arithmetic is plagued by superfluous decimals, albeit in a more subtle way. In this note we first review the various interval notations. We argue in favour of a rarely used notation called "tail", or "factored", which has the advantage of avoiding the repetition of decimals that are necessarily the same. We analyse the information content of the remaining decimals.

## 2 Philosophical implications of an interval notation

Several papers [5, 6, 7] discussing interval notations have been published recently. The various notations have different implications, just as people have different reasons for being interested in interval arithmetic.

For some, intervals are a way of denoting a fuzzy, or perhaps probabilistic, quantity. Others use intervals to give an indication of the extent to which rounding has introduced error in a computation. Here we assume an interpretation of intervals that does not necessarily negate the above interpretations, but differs in the way it is made precise. We call it the set interpretation of interval arithmetic.

The set interpretation According to the set interpretation, variables range over the real numbers. These reals are represented in computer memory as sets of reals. The constraint is that if variable $x$ is represented by set $S$, we have $x \in S$. Thus the set interpretation differs from conventional numerical analysis in the absence of errors. It is either true or false that $x$ belongs to $S$.

The fact that $S$ contains more than one real is not an error. In conventional numerical analysis, an error arises when, for example, a real variable $x$ with value 0.1 is represented by a floating-point number $f$. An error arises because $x=f$ is false. On the other hand, representing $x$ by $S$ is not an error if $x \in S$.

Of course, the statement $x \in S$ provides only a limited amount of information about $x$. The larger $S$ is, the less information. In the set interpretation of interval arithmetic we distinguish error, which is avoidable, from the inescapable fact that the amount of information yielded by a finite machine is finite.

Consequences of the set interpretation Interval arithmetic is no exception to the rule that finite machines can only give a finite amount of information. In interval arithmetic the sets of reals are limited to those that are easily representable: closed, connected sets of reals that have finite floating-point numbers as bounds, if they have a bound at all. Unbounded closed connected sets of reals use the infinities of the floating-point standard in the obvious way. Each of this finite set of sets of reals can be represented by a pair of floating-point numbers. It is also the case that for every set of reals, there exists a unique least floating-point interval containing it.

This is the set interpretation of interval arithmetic. Its virtues include that it is familiar. In fact, many people are surprised to hear it given a name, as this is what they always thought intervals to be. Another virtue is that, if the set interpretation is followed up in all its consequences, it allows resolution of potential ambiguities in interval arithmetic, especially in interval division involving unbounded intervals, intervals containing zero, or intervals containing nothing but zero (3].

## 3 Interval notations

If one accepts the advantages of the set interpretation of interval arithmetic, then one prefers a notation for an interval that suggests a set. The traditional notation, exemplified by $[1.233,1.235]$ has this advantage. Although widely
used, it is not practical, as is apparent from the statement that an unknown real $x$ belongs to

$$
\begin{equation*}
[+0.6180339887498946804,+0.6180339887498950136] \tag{1}
\end{equation*}
$$

The problem with this ubiquitous notation is that it is hard to separate two important pieces of information: where the interval is, and how wide it is. To remedy this defect, Hyvönen (5] described a notation according to which one writes instead

$$
\begin{equation*}
+0.61803398874989[46804,50136] \tag{2}
\end{equation*}
$$

The situation is similar when we are annoyed by having to write

$$
0.61803398874989 x+0.61803398874989 y
$$

which we prefer to have in factored form: $0.61803398874989(x+y)$. Hence we propose to refer to (2) as factored notation for intervals ${ }^{2}$. The name is more than an analogy: in general, one factors with respect to a multiplicative infix operation, of which concatenation on strings is an example.

In the example the bounds are in normalized scientific notation and have the same exponent. In general, factored notation converts an interval [ $a \times 10^{p}, b \times$ $\left.10^{q}\right]$, with normalized numerals as bounds, first to $\left[a, b \times 10^{q-p}\right] \times 10^{p}$, where the upper bound is not necessarily normalized. When $p \neq q$, then this cannot be shortened by taking an initial string of common first decimals outside the brackets. It can only be shortened by limiting the precision of $a$ and $b$, a topic we address later in the paper.

Table 1 contains an overview of interval notations. Most of the table is adopted from Hyvönen [5]. In this overview we distinguish three categories: (a) those that suggest a set, (b) those that suggest a number degraded by an error, and (c) those that suggest a pure number. The Classic and Factored notations belong to category (a). Under category (b) we have added, in analogy with the Tilde notation, the Plus notation. This latter notation is useful in the improvement of the factored notation discussed later on in this paper. Category (c) is in the last line. Hyvönen used the name "Fortran notation". The notation is actually the "Single-number notation" for the Fortran implementation described in [6].

The virtue of the notations in category (b) is that they make explicit that a numeral is not to be interpreted according to mathematical notation, by which we mean that

$$
\begin{equation*}
d_{m} d_{m-1} \ldots d_{0} \cdot d_{-1} \ldots d_{-n} \tag{3}
\end{equation*}
$$

denotes the number $\sum_{i=-n}^{m} d_{i} 10^{i}$. Mathematical notation implies an infinite number of zeros after the last digit when $n>0$.

[^0]| Notation | Interval value | Name of notation |
| :--- | :--- | :--- |
| $[1.233,1.235]$ | $[1.233,1.235]$ | Classic |
| $1.23[3,5]$ | $[1.233,1.235]$ | Factored |
| $1.234 \pm 2$ | $[1.232,1.236]$ | Range |
| $1.234^{\sim}$ | $[1.2335,1.2345]$ | Tilde |
| $1.234+$ | $[1.234,1.235]$ | Plus |
| $1.234+[-1 e-3,2 e-3]$ | $[1.233,1.236]$ | Error |
| $1.234 *$ | $[1.233,1.235]$ | Star |
| 1.234 | $[1.233,1.235]$ | Single-Number |

Table 1: Overview of interval notations, adapted from Hyvönen [5].

Mathematical notation is not the only way to interpret (3). For a long time physicists, chemists, and engineers have used the convention that a numeral has as meaning any number that rounds to the number denoted by the numeral displayed. The coexistence of mathematical notation with the physics convention introduces an ambiguity that is often resolved by context. With intervals, the ambiguity becomes problematic, as we need numerals to denote the bounds of an interval in the classic notation. Are these to be interpreted according to mathematical notation, or according to the physics convention? It is implicit in most of the interval literature, and explicit in [5, 7], that the numerals in the bounds of an interval are to be interpreted according to the mathematical notation. In this paper we follow that rule.

We therefore propose to avoid category (c) and to give single-number an annotation to indicate that it does not have the usual mathematical meaning. This has been done by Hickey, who introduced [4] the Star notation of Table 11.

Difficulties of factored notation There are two problems with the classical notation. The first is the scanning problem: one needs to scan both bounds digit by digit to find the leftmost different digit. Only then does one have an idea of the width of the interval. The second problem, the problem of useless digits can also be found in (1]): the width of the interval is specified by no fewer than five digits. Restricting oneself to four digits for this purpose will give almost as much information about $x$ and that the difference is so small as not to be worth that fifth digit. As we will show below, the same holds almost always for all digits beyond the first two or three.

Factored notation solves the scanning problem; the problem of useless digits remains. To solve it also, we need to study quantitatively the information content of the statement that an unknown real $x$ is contained in an interval $[a, b]$.

## 4 Information theory

According to Shannon's theory of information (see for example, among many textbooks, [1]), observations can reduce the amount of uncertainty about the value of an unknown quantity. The amount of information yielded by an observation is the decrease (if any) in the amount of uncertainty. Shannon argues that the amount of uncertainty is appropriately measured by the entropy of the probability distribution over the possible values. For a uniform distribution on a finite number of values, this reduces to the logarithm of the number of possible values. It can be shown that the entropy for a distribution over $n$ outcomes is maximized by the uniform distribution over these outcomes.

When there are two equally probable possible values, and if one would like this logarithm to come out at unity, one takes 2 as base of the logarithms and one calls the unit of information bit, for binary unit of information. Thus, the binary digits carry at most one bit of information. Similarly, if one works with decimal digits, then it is convenient to use 10 as the basis of the logarithms.

Thus information theory determines for each number base the maximum amount of information that can be carried by a digit. Normally, if we don't know what a number is, and we are only given the first $k$ digits of a numeral denoting that number, we have no idea what the next digit should be. That is, all possibilities in $\{0,1,2,3,4,5,6,7,8,9\}$ are equally probable so that the uncertainty is $\log _{10} 10=1$. As a decimal digit can only distinguish between ten possibilities, the efficiency of the $(k+1)$ st digit is one.

In the set interpretation of interval arithmetic, we have information of the form that a real $x$ belongs to a set $S$. According to information theory, this represents an uncertainty equal to the entropy of the probability distribution over the elements of $S$. What distribution to assume? We are only interested in the large differences in information carried by the successive digits of factored notation. These are large compared to those due to the differences among plausible distributions.

The fact that we are only interested in sets that are bounded intervals, simplifies matters considerably. Plausible distributions for bounded intervals include the uniform and the beta distributions. From now on, if we know that $x$ is in an interval $I$, we assume that the probability of $x$ belonging to any subinterval of $I$ only depends on the width of that subinterval and not on where in $I$ this subinterval is located. This property is implied by the uniform distribution over $I$, and this is the distribution we assume for computation of the uncertainty in the statement $x \in I$. This uncertainty is equal to $-\log _{10} w$, in decimal units of information, where $w$ is the width of $I$.

## 5 Improvement of factored notation

Factored notation solves the scanning problem. In this section we solve the remaining problem that typically many of the digits inside the brackets are useless. We do this by applying the formula found in Section to determine the
information content of the digits in factored notation. As factored notation is just an abbreviation of it, this holds for classical notation as well.

We first consider a specific example in which we note a pattern of rapidly decreasing efficiency as more digits are added. We explain this phenomenon by a generally applicable formula, and use it to justify our recommendation to write no more than three decimal digits inside the brackets of factored notation.

For the example, we randomly selected an interval under the constraints that both bounds have 15 digits, that the first five be the same, and that the interval be nonempty. Thus we came to consider the interval $[a, b]$ that is, in factored notation,

$$
\begin{equation*}
0.389015[282749894,960538227] \tag{4}
\end{equation*}
$$

The information content is $-\log _{10}(b-a)$, which is about 6.169 decimal units. If we have to represent the information that a real is confined to this interval, but are only allowed to use two digits inside the brackets, then this interval has to be $0.389015[28,97]$. This interval has information content of about 6.161 . Thus we saved twice seven digits and lost an amount of information equal to 0.008 decimal units. Note that an optimally used pair of decimal digits in factored notation carries 1.000 decimal units of information.

This example suggests that two decimals inside the brackets already give almost all the information contained in the statement that $x$ is in (4). That only two decimal digits inside the brackets are enough could be a misleading feature of this particular example. To investigate this possibility, we analyse the information content remaining for all possible ways of shortening (4). From this we will see that a pattern emerges. We show that the pattern is not a peculiarity of the example. Because the pattern almost always occurs, we give it a name: Rule of One Tenth. Before investigating this rule, we first need to be more precise about shortening the representation of an interval.

Inflation Consider the statement that $x \in[a, b]$. Let $\left[a^{\prime}, b^{\prime}\right]$ properly contain $[a, b]$. Now it may be the case that $x \in\left[a^{\prime}, b^{\prime}\right]$ conveys almost as much information about $x$ as $x \in[a, b]$ and yet $\left[a^{\prime}, b^{\prime}\right]$ requires fewer digits to write. Then [ $\left.a^{\prime}, b^{\prime}\right]$ is a more efficient representation than $[a, b]$.

A more efficient representation such as $\left[a^{\prime}, b^{\prime}\right]$ may be obtained by one or more applications of an operation we refer to as "inflation".

Definition 1 Let I be the representation of an interval of which the bounds have a finite number of decimals. The operation of inflation has as result the representation of the smallest interval containing I where each bound has one less decimal than the corresponding bound in I.

In Table 2 we see some examples of inflation. Line 0 is a typical case. Line 1 illustrates that inflation may apply to intervals with an unequal number of decimals in the bounds. Line 2 is included to illustrate that inflation decreases the number of digits, so that the four-digit 0.9999 changes to the three-digit numeral 1.00.

| line number | before inflation | after inflation |
| ---: | :--- | :--- |
| 0 | $0.123[456,789]$ | $0.123[45,79]$ |
| 1 | $0.1[2345,34]$ | $0.1[234,4]$ |
| 2 | $0 .[1234,9999]$ | $[0.123,1.00]$ |
| 3 | $0.123[450,670]$ | $0.123[45,67]$ |
| 4 | $0.123[499,501]$ | $0.123[49,51]$ |

Table 2: Examples of inflation.

|  | left boundary $a$ | right boundary $b$ | $-\log _{10}(b-a)$ | information <br> loss |
| ---: | :--- | :--- | :--- | :--- |
| 0 | 0.389015282749894 | 0.389015960538227 | 6.168905911 |  |
| 1 | 0.38901528274989 | 0.38901596053823 | 6.168905907 | 0.000000005 |
| 2 | 0.3890152827498 | 0.3890159605383 | 6.168905804 | 0.000000103 |
| 3 | 0.389015282749 | 0.389015960539 | 6.168904843 | 0.000000961 |
| 4 | 0.38901528274 | 0.38901596054 | 6.168898435 | 0.000006407 |
| 5 | 0.3890152827 | 0.3890159606 | 6.168834366 | 0.000064069 |
| 6 | 0.389015282 | 0.389015961 | 6.168130226 | 0.000704140 |
| 7 | 0.38901528 | 0.38901597 | 6.161150909 | 0.006979316 |
| 8 | 0.3890152 | 0.3890160 | 6.096910013 | 0.064240896 |
| 9 | 0.389015 | 0.389016 | 6 | 0.096910013 |
| 10 | 0.38901 | 0.38902 | 5 | 1 |
| 11 | 0.3890 | 0.3891 | 4 | 1 |
| 12 | 0.389 | 0.390 | 3 | 1 |
| 13 | 0.38 | 0.39 | 2 | 1 |
| 14 | 0.3 | 0.4 | 1 | 1 |
| 15 | 0 | 1 | 0 | 1 |

Table 3: Intervals $[a, b]$ containing an unknown real $x$. Information loss as the result of successive inflations. Given that $x$ is in $[0,1]$, the information content of $x \in[a, b]$ is $-\log _{10}(b-a)$. The loss due to inflation is in the last column.

Let us now consider the change in interval width due to inflation. In line 3 of Table 2 we see that it can be as little as zero. Line 4 shows that the width can increase by a factor of 10 . In such a case, the digits saved by inflation carry as much information as is possible for a decimal digit.

In Table 3 we see in the top line the bounds of interval (4). Each next line shows the result of inflation applied to the previous line. Thus it is true that $x$ is contained in each interval of the table. In the fourth column we see the information content of the statement that $x$ belongs to the interval shown in that line. The last column shows the decrease in information compared to the line before. This decrease is to be compared to the information content of the omitted decimal, which is 1 . Thus, the last column contains the efficiency of showing the last decimal in each bound in the line before.

As one goes down the table, considering successively more succinct, yet
true statements about $x$, one sees an interesting transition about halfway. Of course something special has to happen at the point where factored notation is $0.38901[5,6]$. The next more succinct intervals are, successively, $0.3890[1,2]$, $0.389[0,1]$ and so on. In this range, the information decrease is 1 , exactly the information content of the decimal digit saved. That is, the digits that are saved here are fully efficient. Factored notation is not as useful here as it was higher up in the table. In fact, it is redundant, as there is always a pair of successive single decimals inside the brackets. An ad-hoc notation in the style of tilde notation has a considerable advantage here. I adopted the one proposed by Hickey (1) and called it "Plus" in Table 1.

Let us now consider the most important part of Table 3. Suppose one considers shortening the interval in the top line to $0.389015[28,97]$ and suppose one worries that too much information has been lost. The last column in line 7 shows that the additional digits contained in line 8 add only about one tenth of the amount information contained in the last digits of line 7 , which is already pretty low at around one tenth of those in the line above that. One can summarize the last column above line 8 by the Rule of One Tenth:

Each additional digit carries about one tenth of the information in the previous one.

The rule holds quite well from line 8 upwards. If it would be exact, the last column in line 1 would be $6 * 10^{-9}$ instead of the $5 * 10^{-9}$ actually observed. Is this rule a fortuitous feature of this particular example? In the following, we will argue that it is not.

The general case In Table 3 we see that the Rule of One Tenth only holds over many lines with considerable fluctuations from line to line. In fact, in Table 2 we saw that inflation can cause an increase in interval width of as little as a factor of one and as much as a factor of ten. These factors correspond to information losses of 0 and 1 , respectively. What can we say in general about interval widening due to inflation?

We consider for the general case the interval shown digit by digit as

$$
\begin{equation*}
0 . x_{1} \ldots x_{j-1}\left[y_{j} \ldots y_{j+k-1} p, z_{j} \ldots z_{j+k-1} q\right] \tag{5}
\end{equation*}
$$

where $y_{j}<z_{j}$ and $k \geq 2$. We ask whether the number of digits can be safely decreased by one application of the inflation operation.

If $p=q=0$, width does not increase, so inflation can be applied without any loss of information. The largest information loss occurs if $p=9$ and $q=1$, in which case the width increases by $18 \times 10^{-j-k}$. Let us take $10^{-j-k+1}$ as a typical width increase, as it is a convenient value near midway these extremes.

This increase should be compared with the width $w$ of (5). The comparison is obscured by the large variation of $w$. It may be as little as $10^{-j-k}$ (see last line of Table 2) and nearly as much as $10^{-j+1}$. In the case (5) is narrowest, inflation widens it typically by a factor ten. In that case $p$ and $q$ carry as
much information as is possible for a decimal digit. Perhaps all decimals should be kept. In the case (5) is widest, inflation widens it by a negligible amount. Inflation is advisable.

Apparently it does not help to consider the extreme values of $w$, as they lead to contradictory advice. So let us consider average values of $w$. We assume $k \geq 2$ (we retain at least two digits inside the brackets). If the average is in the order of $10^{-j}$, then inflation causes negligible information loss. If the average width is near $10^{-j-k}$, then inflation causes the full amount of information loss, so this is the worst case. To simplify matters, we make the worst case worse and assume that $w$ can range from 0 to $10^{-j+1}$. This is only a small change, as we are only interested in $k>2$, in which case the range from 0 to $10^{-j-k}$ is negligible compared to the range from 0 to $10^{-j+1}$.

It is simplest to assume that the probability distribution of $w$ is not far from uniform between 0 and $10^{-j+1}$. In that case, it will usually be the case that $w \in\left[10^{-j}, 10^{-j+1}\right]$.

But one may prefer not to make assumptions about the probability distribution of $w$. Then one may accept the assumption that the digits between the brackets in (5) are independent random variables with a uniform distribution on $\{0, \ldots, 9\}$ under the constraint that $y_{j}<z_{j}$. The average width of (5) can then be expressed as

$$
\begin{equation*}
w=\sum_{s=0}^{9} \sum_{t=s+1}^{9} p_{s t} w_{s t} \tag{6}
\end{equation*}
$$

where $p_{s t}$ is the probability of $y_{j}=s$ and $z_{j}=t$ and $w_{s t}$ is the average width under the constraint that $y_{j}=s$ and $z_{j}=t$. For $i$ between 0 and 8 , if $y_{j}=i$, then $z_{j}$ can be $i, \ldots, 9$. Under the assumption about the distributions of the digits involved, we have $p_{s t}=1 / \sum_{i=1}^{9} i=1 / 45$.

We are interested in a lower bound for $w_{s t}$. Each width is bounded below by $(t-s-1) * 10^{-j}$. Whatever the distribution, the average is also bounded below by $(t-s-1) * 10^{-j}$. Because this bound depends only on $t-s$, we rewrite (6) as

$$
w=\sum_{d=1}^{9} \sum_{a=0}^{9-d} p_{a, a+d} w_{a, a+d}
$$

Using $w_{a, a+d} \geq(d-1) * 10^{-j}$ and $p_{s t}=1 / 45$ we have

$$
\begin{aligned}
w & \geq(1 / 45) \sum_{d=1}^{9}(d-1) * 10^{-j} \\
& \geq(36 / 45) * 10^{-j}=(4 / 5) * 10^{-j}
\end{aligned}
$$

Moreover, $w$ is bounded above by $10^{-j+1}$. So it is reasonable to assume that $w$ is in the order of $10^{-j}$.

Hence inflation widens an interval with a width of about $10^{-j}$ to one that has a width of about $10^{-j}+10^{-j-k+1}=10^{-j}\left(1+10^{-k+1}\right)$. Thus, the uncertainty
decreased by the last digit is in the order of $\log _{10}\left(1+10^{-k+1}\right)$, which is about $10^{-k+1}$, neglecting a factor of $\ln 10$.

This is also the decrease in information gain for every additional digit inside the brackets in factored notation. This is also the Rule of Ten observed in Table 3 when averaging over many rows. We can expect that the third decimal in a factored notation only increases information by 0.01 of the potential information in a decimal digit, and is therefore of questionable value. We recommend factored notation with two decimals inside the brackets, while keeping in mind that the rule does not apply in rare cases such as line 4 in Table 2 .

## 6 Conclusions

Interval methods are coming of age. When interval software was experimental, it didn't matter whether interval output was easy to read. Now that the main technical challenges have been overcome, and we at least know how to ensure that the floating-point bounds include all reals that are possible values of the variable concerned, we need to turn our attention to small, mundane matters, which include taking care of the convenience of users. Factored notation is an advance in this respect. However, without some attention to the number of digits inside the brackets, one runs the risk of specifying in maximum accuracy not the number under consideration, but the unavoidable lack of information about this number.

## 7 Acknowledgements

I owe a debt of gratitude to the anonymous referees for their valuable suggestions. Many thanks to Frédéric Goualard for helpful comments on a draft of this paper. We acknowledge generous support by the University of Victoria, the Natural Science and Engineering Research Council NSERC, the Centrum voor Wiskunde en Informatica CWI, and the Nederlandse Organisatie voor Wetenschappelijk Onderzoek NWO.

## References

[1] R.B. Ash. Information Theory. Interscience, 1965.
[2] Pascal Van Hentenryck, Laurent Michel, and Yves Deville. Numerica: A Modeling Language for Global Optimization. MIT Press, 1997.
[3] T. Hickey, Q. Ju, and M.H. van Emden. Interval arithmetic: from principles to implementation. Journal of the ACM, 2001. To appear.
[4] Timothy J. Hickey. CLIP: A CLP(intervals) dialect for metalevel constraint solving. In PADL2000, pages 200-214. Springer-Verlag, 2000. Lecture Notes in Computer Science 1753.
[5] E. Hyvönen. Interval input and output. In W. Kramer and J.W. von Gudenberg, editors, Scientific Computing, Validated Numerics, Interval Methods, pages 41-52. Kluwer, 2001.
[6] Michael Schulte, Vitaly Zelov, G. William Walster, and Dmitri Chiriaev. Single-number interval I/O. In Developments in Reliable Computing, pages 141,148. Kluwer Academic Publishers, 1999.
[7] M.H. van Emden. Factored notation for interval I/O. Technical Report DCS-264-IR, Department of Computer Science, University of Victoria. Paper cs.NA/0102023 in Computing Research Repository (CoRR), February 2001.
[8] M.H. van Emden. Algorithmic power from declarative use of redundant constraints. Constraints, pages 363-381, 1999.


[^0]:    ${ }^{1}$ I'm not making this up; see page 122 of [2].
    2 The notation has been occasionally used without comment in the literature; see for example [8]. Credit goes to Hyvönen, whose paper [5] was the first to appear in print that drew attention to it and named it. Independently 1 did so in 7. Hyvönen called it "tail notation".

