# Joint Multicast Beamforming and User Scheduling in Large-scale Antenna Systems

Longfei Zhou, Student Member, IEEE, Zi Xu, Wei Jiang, Member, IEEE, and Wu Luo, Member, IEEE

Abstract—This paper studies the joint multicast beamforming and user scheduling problem, with the objective of minimizing total transmitting power across multiple channels by jointly assigning each user to appropriate channel and designing multicast beamformer for each channel. The problem of interest is formulated in two different optimization problems, a mixed binary quadratically constrained quadratic program and a highlystructured nonsmooth program. Two different algorithms, based on convex relaxation and convex restriction, respectively, are proposed to solve the problem. The performance ratio between the approximate solution provided by the convex-relaxationbased algorithm and optimal solution is proved to be upper bounded by a constant independent of problem data. The convex-restriction-based algorithm is guaranteed to converge to a critical point to the nonsmooth formulation problem. Finally, extensive simulation results verify the theoretical analysis and demonstrate the advantage of the proposed co-design scheme over conventional fixed scheduling and random scheduling in terms of power consumption.

*Index Terms*—Multicast beamforming, user scheduling, semidefinite relaxation, approximation ratios, sequential convex approximation, dual fast gradient projection

#### I. INTRODUCTION

Demands for high-rate wireless services, such as Internet TV, on-line gaming, and multimedia downloading, continue to grow explosively in the worldwide. Wireless multicast is regarded as one of key enabling technologies in future cellular systems to boost the capacity of wireless networks and cater to the customer demands. When combined with large-scale antenna arrays at base station (BS), wireless multicast is able to take full advantage of available channel state information at transmitter (CSIT) to provide enhanced data rates and relatively high transmission reliability [1]. Wireless multicast has always been an important part of the evolution of multimedia broadcast multicast service (MBMS) in wireless communication standards such as UMTS, LTE and LTE-Advanced [2].

Lots of downlink multicast beamforming problems have been discussed for different scenarios. Single-group multicast beamforming for single-cell system was first investigated in [3], and then extended to multi-group multicast in [4]. Multicast beamforming with per-antenna power constraints was further discussed in [5] [6]. Furthermore, coordinated multicast beamforming under per-BS power constraints for multi-cell system was considered in [7] [8]. Some other issues, such as energy efficient design, user selection and real-time implementation, were also studied in [9] [10] [11].

A commonly-used formulation of above studies is transmitting power minimization under quality of service (QoS) constraints. A key difficulty with such formulation is that the problem may be infeasible, especially when the number of users is much larger than the number of antennas. In such a situation, part of users should be removed out (admission control) or scheduled in orthogonal resource dimensions, such as time, frequency, and code slots, which is crucial for practical applications. The former leads to a variety of joint beamforming and admission control problems.

In [12], the authors addressed the joint multicast beamforming and admission control problem based on semidefinite programming relaxation (SDR) and greedy membership deflation. The basic idea is sequentially dropping a weakest user, then solving the relaxed problems and finally checking whether the suboptimal rank-one solution satisfies all QoS constraints. Recently, network energy efficient design and sparse optimization of the joint multicast beamforming and admission control for green Cloud-RAN was further discussed in [13]. For the particular satellite communication systems, system sum rate optimization of the multi-group multicast precoding and user scheduling under per-antenna power constraints and underlying framing structure constraints was considered in [14]. In [15], a closed-form asymptotically optimal solution was proposed for the joint multi-group multicast beamforming and user grouping in massive MIMO systems.

This paper studies the joint multicast beamforming and user scheduling in large-scale antenna systems with a massive number of users. We assume that each user takes interest in multiple information symbols but is assigned to receive one of its interested information symbols. Each information symbol is transmitted over an orthogonal channel, such as, time slot and frequency subcarrier. The problem of interest is minimizing the total transmitting power across all orthogonal channels by jointly assigning each user to appropriate channel and designing multicast beamforming vector for each information symbol such that each user should successfully decode at least one information symbol. Since channel quality of each user at all channels should be taken into consideration, this problem is much different from the admission control in [12], where only channel quality of all users at a fixed channel is considered.

Our main contributions are summarized as follows. First, the problem of interest is cleverly formulated in two different op-

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L. Zhou, W. Jiang, and W. Luo are with the State Key Laboratory of Advanced Optical Communication Systems and Networks, Peking University, 100871, China. E-mail: {zhoulongfei,jiangwei, luow}@pku.edu.cn.

Z. Xu are with Department of Mathematics, College of Sciences, Shanghai University, Shanghai 200444, People's Republic of China. E-mail: xuzi@i.shu.edu.cn

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timization problems, a mixed binary quadratically constrained quadratic program and a highly-structured nonsmooth program. Second, a polynomial-time SDR algorithm is proposed to address the problem. The worst-case approximation ratio of SDR is proved to be O(QK) for general channel scenario, and  $O(K^{1/Q})$  for the special case of homogeneous channel scenario, where Q and K is the number of orthogonal channels and the number of users, respectively. Our result is an important improvement and generalization upon those in [16] [17] [18]. Third, a sequential convex approximation (SCA) scheme is proposed for the nonsmooth formulation problem and an efficient dual fast gradient projection (DFGP) algorithm is devised for the subproblems. The overall algorithm is matrixfree, i.e., based solely on matrix-vector multiplications and comparison operations, and guaranteed to converge to a critical point to the nonsmooth formulation problem. Finally, extensive simulation results are provided to verify the theoretical analysis and demonstrate the advantage of the proposed codesign scheme over conventional fixed scheduling and random scheduling in terms of transmitting power consumption.

The remainder of paper is outlined as follows. Section II describes the system model and the two problem formulations. Section III presents the SDR algorithm and theoretical performance analysis. Section IV details the SCA-DFGP algorithm and its convergence result and computational complexity. Section V and Section VI provides comprehensive simulation results to assess the performance of the proposed algorithms and concludes the paper, respectively.

*Notation*: In the rest of this paper, boldface italic lowercase and uppercase characters denote column vectors and matrices, respectively. The operators  $(\cdot)^{T}, (\cdot)^{H}, |\cdot|, Tr(\cdot), ||\cdot||_{2}$ , and  $||\cdot||_{F}$ , correspond to the transpose, the conjugate transpose, the absolute value, the trace and the Euclidean norm and the Frobenius norm operations, while Re( $\cdot$ ) and Im( $\cdot$ ) denotes the real part and imaginary part of complex number, respectively.

#### **II. SYSTEM MODEL AND PROBLEM FORMULATION**

#### A. System Model

We consider a downlink multicast scenario consisting of a BS with M antennas and K single-antenna users. Assume that there are Q orthogonal channels  $C_q$  ( $q \in Q, Q = \{1, 2, ..., Q\}$ ) between the BS and each user, such as nonoverlapping time slots or orthogonal subcarriers. Let  $\tilde{h}_{k,q} \in \mathbb{C}^M$  denote the complex channel vector between the BS and the k-th user for channel  $C_q$ . Note that for each user these Q channel vectors could be identical if the coherence bandwidth or coherence time is sufficiently large. Such a special case will be referred to as homogeneous channel scenario. The BS uses an  $M \times 1$ beamforming vector  $w_q$  to send a zero-mean and unit-variance common information symbol  $x_q$  to the interested users over channel  $C_q$ . The signal received from  $C_q$  by the k-th user is

$$y_{k,q} = \tilde{h}_{k,q}^{\mathrm{H}} w_q x_q + n_{k,q} \,\forall q \in Q \,\forall k \in \mathcal{K},$$
(1)

where  $\mathcal{K} = \{1, 2, ..., K\}$  is the user index set, and  $n_{k,q}$  is the zero-mean circularly-symmetric complex Gaussian random noise with variance  $\sigma_{k,q}^2$ , which is independent of  $x_q$  and

 $\hat{h}_{k,q}$ . The signal-to-noise ratio (SNR) at the *k*-th user can be expressed as

$$\gamma_{k,q} = \frac{\left|\tilde{\boldsymbol{h}}_{k,q}^{\mathrm{H}} \boldsymbol{w}_{q}\right|^{2}}{\sigma_{k,q}^{2}} \, \forall q \in \boldsymbol{Q} \, \forall k \in \mathcal{K}.$$
(2)

The QoS requirement for the *k*-th user to successfully decode information symbol  $x_q$  can be expressed as  $\gamma_{k,q} \ge \bar{\gamma}_q$ . Let  $\mathbf{h}_{k,q} = \tilde{\mathbf{h}}_{k,q}/(\sigma_{k,q}\sqrt{\bar{\gamma}_q})$  be the *k*-th user's normalized channel vector for  $C_q$  ( $q \in Q$ ). The QoS requirement can be rewritten as

$$\left|\boldsymbol{h}_{k,q}^{\mathrm{H}}\boldsymbol{w}_{q}\right|^{2} \ge 1.$$
(3)

We assume that each user takes interest in multiple information symbols but is assigned to receive one of its interested information symbols. When there are a large number of users in the system, it is impractical or inefficient to serve all users within a single channel. Therefore, properly scheduling all users to multiple channels is important to boost the system capacity.

# B. MBQCQP Formulation

A commonly-used disjunctive modelling technique is using binary variable  $b_{k,q} \in \{0, 1\}$  as scheduling indicator, i.e.,  $b_{k,q} = 1$  indicates that the *k*-th user is scheduled in channel  $C_q$ . Hence, the problem of interest can be formulated as the following mixed binary quadratically constrained quadratic program (MBQCQP)

$$\min_{\{\boldsymbol{w}_q\},\{b_{k,q}\}} \quad \sum_{q=1}^{Q} \|\boldsymbol{w}_q\|_2^2 \tag{4a}$$

s.t. 
$$\left| \boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{w}_{q} \right|^{2} \ge b_{k,q} \; \forall q \in \boldsymbol{Q} \; \forall k \in \mathcal{K},$$
 (4b)

$$\sum_{q=1}^{\infty} b_{k,q} = 1 \; \forall k \in \mathcal{K}, \tag{4c}$$

$$b_{k,q} \in \{0,1\} \ \forall q \in \mathcal{Q} \ \forall k \in \mathcal{K}.$$
(4d)

In the above problem, the objective function are quadratic in the continuous variables and the disjunctive constraints contain both continuous and binary variables. This class of MBQCQP problem is extremely difficult partly as they are nonconvex even with the binary variables being fixed [17] [18]. In the special case of  $b_{k,1} = 1$  for all  $k \in \mathcal{K}$ , the problem reduces to the single-group multicast beamforming problem, which is a continuous QCQP and NP-hard in general [3].

# C. Nonsmooth Reformulation

For each user, ensuring the QoS requirement (3) in at least one channel is equivalent to making the QoS requirement in the best channel be satisfied. Hence, the feasible set of continuous variables  $\{w_q\}$  in (4) can be equivalently described by the following nonsmooth constraints

$$\max_{q \in Q} \left\{ \left| \boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{w}_{q} \right|^{2} \right\} \ge 1 \; \forall k \in \mathcal{K}.$$
(5)

Denote  $W = [w_1, w_2, \dots, w_Q] \in \mathbb{C}^{M \times Q}$  and

$$f_k(\boldsymbol{W}) = \max_{q \in Q} \left\{ \left| \boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{w}_q \right|^2 \right\}.$$
(6)

Observing that the binary variables in (4) is absent from the objective function, we obtain a nonsmooth reformulation of (4) as follows

$$\min_{\boldsymbol{W} \in \mathbb{C}^{M \times Q}} \|\boldsymbol{W}\|_F^2 \tag{7a}$$

s.t. 
$$f_k(W) \ge 1 \ \forall k \in \mathcal{K}.$$
 (7b)

In this equivalent reformulation, all binary variables are removed out at the expense of a small number of nonsmooth constraints. The main obstacle in (7) is, of course, the nonsmoothness and nonconvexity of constraints. However, each constraint function  $f_k(W)$  is highly structured, and making use of the available structure in an appropriate way will give efficient algorithms to solve (7). After solving (7), we can properly assign each user to the channel in which the user attains the best QoS among all channels.

We will propose a semidefinite relaxation (SDR) approach in section III to solve (4), and an efficient nonsmooth optimization approach in section IV to solve (7).

# III. SEMIDEFINITE RELAXATION APPROACH

In this section, a SDR technique with performance guarantee is developed for solving the MBQCQP formulation (4). The main idea is to simultaneously use the continuous relaxation for the binary variables and the SDR for the continuous variables. After solving the SDR problem, a randomization procedure is used to generate approximate solutions to the original MBQCQP formulation from an optimal solution of the SDR problem. Furthermore, we analyze the bound on the approximation ratio between the optimal value of the MBQCQP problem and that of the associated SDR.

Upon changing the optimization variables to  $W_q = w_q w_q^H$ and then doing the SDP relaxation for  $W_q$  and the continuous relaxation for  $\{b_{k,q}\}$  in (4), we obtain the following problem

$$\min_{\{\mathbf{W}_q\},\{b_{k,q}\}} \quad \sum_{q=1}^{Q} \operatorname{Tr}(\mathbf{W}_q)$$
(8a)

s.t. 
$$\boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{W}_{q} \boldsymbol{h}_{k,q} \ge b_{k,q} \; \forall q \in \boldsymbol{Q} \; \forall k \in \mathcal{K},$$
 (8b)

$$\sum_{q=1}^{Q} b_{k,q} = 1 \; \forall k \in \mathcal{K}, \tag{8c}$$

$$0 \le b_{k,q} \le 1 \,\forall q \in \mathbf{Q} \,\forall k \in \mathcal{K},\tag{8d}$$

$$W_a \ge 0 \ \forall q \in Q. \tag{8e}$$

We observe that these continuous variables  $\{b_{k,q}\}$  in (8) can be eliminated out from the problem without loss of optimality. An equivalent problem is obtained as follows

$$\min_{\{\mathbf{W}_q\}} \quad \sum_{q=1}^{Q} \operatorname{Tr}(\mathbf{W}_q) \tag{9a}$$

s.t. 
$$\sum_{q=1}^{Q} \boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{W}_{q} \boldsymbol{h}_{k,q} \ge 1 \; \forall k \in \mathcal{K},$$
(9b)

$$W_q \ge 0 \ \forall q \in Q. \tag{9c}$$

One can verify that each feasible solution to (9) is also feasible to (8) and vice versa. Moreover, the same formulation is obtained if similar convex relaxation is applied to the nonsmooth problem (7).

For the special case of homogeneous channel scenario that the Q channel vectors for each user are identical, i.e.,  $\tilde{h}_{k,1} = \tilde{h}_{k,2} = \cdots = \tilde{h}_{k,Q} \stackrel{\Delta}{=} \tilde{h}_k \forall q \in Q$ , problem (9) is symmetric with respect to the arguments  $\{W_q\}$  and could be further reduced to

$$\min_{\mathbf{W}_{1}} \quad \mathrm{Tr}(\mathbf{W}_{1}) \tag{10a}$$

s.t. 
$$\tilde{\boldsymbol{h}}_{k}^{\mathrm{H}} \boldsymbol{W}_{1} \tilde{\boldsymbol{h}}_{k} \ge 1 \ \forall k \in \mathcal{K},$$
 (10b)

$$\geq 0.$$
 (10c)

Surprisingly, besides much simpler formulation, it will been shown in next subsection that (10) provides better performance guarantee for homogeneous channel scenario than for general channel scenario.

 $W_1$ 

Problem (9) and (10) are both convex, which can be efficiently solved using off-the-shelf interior point solvers such as SDPT3 and SeDuMi. Once an optimal solution  $\{W_q^*\}$  to (9) is obtained [for (10),  $W_1^* = W_2^* = \cdots = W_Q^*$ ], the Gaussian randomization method could be used to generate the candidate beamformers. The *l*-th candidate beamformer for channel block  $C_q$  is generated as  $\mathbf{x}_q^{(l)} = U_q \boldsymbol{\Sigma}_q^{1/2} \mathbf{v}_l$ , where  $U_q, \boldsymbol{\Sigma}_q$  are the eigen-decomposition factors of  $W_q^*$ , i.e.,  $W_q^* = U_q \boldsymbol{\Sigma}_q U_q^H$ , and  $\mathbf{v}_l \sim C \mathcal{N}(\mathbf{0}, \mathbf{I}_M)$ . It's easy to show that  $\mathbf{x}_q \sim C \mathcal{N}(\mathbf{0}, W_q^*)$ . Given such candidate beamformers { $\mathbf{x}_q$ } $_{q=1}^Q$ , we still need to determine the corresponding transmitting power. Let  $c_{k,q} = |\mathbf{h}_{k,q}^H \mathbf{x}_q|^2$ . Substituting  $\mathbf{w}_q = \sqrt{p_q} \mathbf{x}_q$  into (7), we have

$$\min_{\boldsymbol{p} \ge 0} \quad \sum_{q=1}^{Q} p_q \left\| \boldsymbol{x}_q \right\|^2 \tag{11a}$$

s.t. 
$$\max_{q \in Q} \left\{ p_q c_{k,q} \right\} \ge 1 \ \forall k \in \mathcal{K}.$$
(11b)

Albeit nonconvex, the problem (11) is a special monotonic optimization problem. Many kinds of outer approximation algorithms could be applied[19]. For the sake of analysis convenience, a simple scaling procedure is used to obtain high-quality approximate solution. The feasible approximate solution to (11) is given by  $p_q = p(\{x_q\}) \forall q \in Q$ , where

$$p(\lbrace \boldsymbol{x}_q \rbrace) = \frac{1}{\min_{k \in \mathcal{K}} \max_{q \in Q} \lbrace c_{k,q} \rbrace}.$$
 (12)

For completeness, the overall SDR-G algorithm for (4) or (7) is summarized in Algorithm 1.

#### A. Approximation Ratio

In this subsection, we will analyze the performance of proposed SDR-G algorithm. Denote the optimal value of SDR problem (9) by  $v_{SDR-LB}^*$ , the optimal value of MBQCQP problem (4) by  $v^*$ , and the objective value of the approximate solution to (7) by  $v_{SDR-G}^*$ . Obviously, we have

$$v_{\text{SDR-LB}} \le v^* \le v_{\text{SDR-G}}^* = \min_{l=1,\dots,L} p^{(l)}.$$
 (13)

Algorithm 1 SDR-G algorithm for (4) or (7)

Initialization Solve an optimal solution  $\{W_q^*\}$  to (9). for l = 1, ..., L do 1) Sample  $\mathbf{x}_q^{(l)} \sim C\mathcal{N}(\mathbf{0}, W_q^*)$  ( $\forall q \in Q$ ). 2) Calculate  $p^{(l)} = p(\{\mathbf{x}_q^{(l)}\}) \sum_{q \in Q} \|\mathbf{x}_q^{(l)}\|^2$  using (12). end for Output Let  $l^* = \operatorname{argmin}_{l=1,...,L} p^{(l)}$ . Select  $\{\sqrt{p^{(l^*)}}\mathbf{x}_q^{(l^*)}\}$  as the approximate solution to (7).

We will show that there exists a constant  $\theta > 0$  only depending on the number of orthogonal channels Q and the number of users K, such that

$$v_{\text{SDR-G}}^* \le \theta v_{\text{SDR-LB}} \tag{14}$$

holds true with overwhelming probability. Such a constant  $\theta$  is generally referred to as approximation ratio in computational complexity theory. It implies that the power loss due to the SDR approximation is at most  $10 \log_{10} \theta$  dB away from the optimal transmitting power  $v^*$  according to (13) and (14). The main results about the upper bound on the worst-case approximation ratio  $\theta$  are given in the following theorem.

Theorem 1: (1) For general channel scenario,

$$\theta \le 5QK \tag{15}$$

holds with probability at least  $1 - 0.9^{L}$ .

(2) For the special case of homogeneous channel scenario,

$$\theta \le 5K^{1/Q} \tag{16}$$

holds with probability at least  $1 - 0.9^{L}$ .

Please refer to Appendix for the proof of Theorem 1. Let's give some physical meaning explanations about why the worstcase approximation ratio are different between two scenarios. For general channel scenario, when Q - 1 channels are very poor simultaneously for all users, then scheduling all users into the rest channel are optimal. This degenerated problem is nothing but single-group multicast problem, for which the worst-case performance bound of O(K) provided by SDR is in fact tight up to a constant factor [20]. For homogeneous channel scenario, the bound of  $O(K^{1/Q})$  can be regarded as the result of a kind of user selection diversity according to the proof. For average-case general channel scenario, one could expect such diversity, which, however, vanishes in the worstcase scenario. For the special case of homogeneous channel scenario with Q = 2, a bound of Q(K) is shown in [18]. Moreover, by using a rank-two transmit beamformed Alamouti space-time code scheme for single-group multicast, a bound of  $O(\sqrt{K})$  is obtained in [16]. Our result is an interesting improvement and generalization upon above results.

#### IV. NONSMOOTH OPTIMIZATION APPROACH

Although SDR is a valuable benchmark for the problem, the computational burden of SDR is not well scalable to largescale antenna system. Moreover, the worst-case results imply that the performance of SDR-G may deteriorate considerably when there are a massive number of users in the system. Hence, we also provide an efficient algorithm to handle with such case. MBQCQP formulation (4) is difficult to solve due to a great number of binary variables and disjunctive constraints. We turn to highly-structured nonsmooth problem (7). Specifically, we devise a sequential approximation scheme to yield a series of smooth convex subproblems, and present a dual fast gradient projection algorithm to solve each subproblem. Finally, convergence and computational complexity of the overall algorithm is analyzed.

# A. Sequential Convex Approximations

Since nondifferential constraint function  $f_k(W)$  in (6) is the maximum of a finite number of convex quadratic functions,  $f_k(W)$  is convex as well. Therefore, we have the following subgradient inequality,

$$f_k(W) \ge f_k(V) + \langle G_k(V), W - V \rangle \,\forall W, \tag{17}$$

where  $G_k(V)$  is a subgradient of  $f_k(W)$  at V and  $\langle G, W \rangle = \text{Re}(\text{Tr}(G^H W))$  is the inner product of two complex matrices G and W.

At differentiable points, there is a unique subgradient of  $f_k(W)$ , i.e., the gradient, while at nondifferentiable points, there is an infinite set of subgradients. All subgradients  $G_k(W)$  of  $f_k(W)$  satisfying (17) form a convex set called subdifferential. According to subdifferential calculus of convex functions [21], we can write the subdifferential of  $f_k(W)$  as

$$\partial f_{k}(\mathbf{W}) = \left\{ \begin{bmatrix} \boldsymbol{d}_{1}, \boldsymbol{d}_{2}, \dots, \boldsymbol{d}_{Q} \end{bmatrix} : \\ \boldsymbol{d}_{q} = 2\alpha_{q}\boldsymbol{h}_{k,q}\boldsymbol{h}_{k,q}^{\mathrm{H}}\boldsymbol{w}_{q} \forall q \in \boldsymbol{Q}, \\ \alpha_{q} = 0 \forall q \notin I_{k}(\boldsymbol{W}), \\ \sum_{q \in I_{k}(\boldsymbol{W})} \alpha_{q} = 1, \alpha_{q} \ge 0 \right\},$$
(18)  
$$I_{k}(\mathbf{W}) = \left\{ q : \left| \boldsymbol{h}_{k,q}^{\mathrm{H}}\boldsymbol{w}_{q} \right|^{2} = f_{k}(\boldsymbol{W}) \right\}.$$

Our choice of subgradient is

$$G_k(\mathbf{W}) = \frac{2}{|I_k(\mathbf{W})|} \sum_{q \in I_k(\mathbf{W})} \mathbf{h}_{k,q} \mathbf{h}_{k,q}^{\mathrm{H}} \mathbf{w}_q, \qquad (19)$$

where  $|I_k(W)|$  is the cardinality of set  $I_k(W)$ . The idea behind such choice is ensuring equal probability of scheduling each user into the active channels.

By iteratively linearizing  $f_k(W)$  at  $W^{(n)}$  using (17) and (19), we obtain a sequence of convex approximations of problem (7) as follows

$$\min_{\boldsymbol{W}\in\mathbb{C}^{M\times Q}} \|\boldsymbol{W}\|_F^2 \tag{20a}$$

s.t. 
$$\langle \boldsymbol{G}_k(\boldsymbol{W}^{(n)}), \boldsymbol{W} \rangle + c_k(\boldsymbol{W}^{(n)}) \ge 0, \forall k \in \mathcal{K}.$$
 (20b)

where  $c_k(W^{(n)}) = f_k(W^{(n)}) - \langle G_k(W^{(n)}), W^{(n)} \rangle - 1$ .

Problem (20) is a strongly-convex quadratic program and therefore has a unique solution. Since the convergence rate of the subgradient-based method for nonsmooth optimization problem may be slow, efficient subproblem-solving algorithm is necessarily important, which will be detailed in next subsection.

## B. Dual Fast Gradient Projection Method

Since the constraints in (20) are all linear inequalities and  $W^{(n)}$  is a feasible solution to (20), the refined Slater's condition for (20) is satisfied [21]. It implies that strong duality holds, i.e., the optimal value of (20) is equal to the attained optimal value of the dual problem. Due to strong convexity of problem (20), its dual problem is Lipschitz smooth and could be solved efficiently by a fast gradient projection method.

To avoid complex notations, we consider the following general model of problem (20)

$$\min_{\boldsymbol{x} \in \mathbb{C}^{MQ}} \|\boldsymbol{x}\|_2^2 \tag{21a}$$

s.t. 
$$\operatorname{Re}(Ax) + a \ge 0.$$
 (21b)

where  $A \in \mathbb{C}^{K \times (MQ)}$  and  $a \in \mathbb{R}^{K}$ . Obviously, problem (20) could be cast into (21) by appropriate matrix concatenation.

The Lagrangian function associated with (21) is

$$L(\boldsymbol{x},\boldsymbol{z}) = \boldsymbol{x}^{\mathrm{H}}\boldsymbol{x} - \mathrm{Re}(\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}) - \boldsymbol{z}^{\mathrm{T}}\boldsymbol{a}. \tag{22}$$

Minimizing L(x, z) over x gives the optimal solution

$$\boldsymbol{x} = \frac{1}{2}\boldsymbol{A}^{\mathrm{H}}\boldsymbol{z} \tag{23}$$

and the dual objective function

$$D(z) = -\frac{1}{4}z^{\mathrm{T}}\boldsymbol{B}z - \boldsymbol{a}^{\mathrm{T}}z, \qquad (24)$$

where  $\boldsymbol{B} = \operatorname{Re}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}) = \operatorname{Re}(\boldsymbol{A})\operatorname{Re}(\boldsymbol{A})^{\mathrm{T}} + \operatorname{Im}(\boldsymbol{A})\operatorname{Im}(\boldsymbol{A})^{\mathrm{T}}$ .

Hence, the dual program of (21) has a same solution set with the following problem

$$\min_{\boldsymbol{z} \in \mathbb{R}^{K}} \frac{1}{4} \boldsymbol{z}^{\mathrm{T}} \boldsymbol{B} \boldsymbol{z} + \boldsymbol{a}^{\mathrm{T}} \boldsymbol{z}$$
(25a)

s.t. 
$$z \ge 0$$
. (25b)

Problem (25) is a continuously-differentiable convex minimization problem with a very simple constraint set. Applying Nesterov's optimal gradient scheme [22] to (25), we obtain the dual fast gradient projection (DFGP) iteration formula as follows

$$\boldsymbol{z}^{(l)} = \max\left(\tilde{\boldsymbol{z}}^{(l-1)} - \mu(\frac{1}{2}\boldsymbol{B}\tilde{\boldsymbol{z}}^{(l-1)} + \boldsymbol{a}), 0\right), \qquad (26a)$$

$$\tilde{z}^{(l)} = z^{(l)} + \frac{l-1}{l+2} \left( z^{(l)} - z^{(l-1)} \right), \tag{26b}$$

where  $\mu = \frac{2}{\lambda_1(B)}$  and  $\lambda_1(B)$  is the maximum eigenvalue of positive semidefinite matrix **B**. It is known that the algorithm converges to an  $\varepsilon$ -optimal solution to (25) within  $O(\frac{1}{\sqrt{\mu\varepsilon}})$  iterations [22].

# C. Convergence and Complexity

For clarity, the overall algorithm for the nonsmooth reformulation (7) is summarized in Algorithm 2. We first analyze the convergence of proposed algorithm. Let  $\mathcal{P}(\mathbf{W}^{(n)})$  denote the instance of problem (20) at  $\mathbf{W}^{(n)}$ . Since the cost function  $\|\mathbf{W}\|_F^2$  is independent of *n* and  $\mathbf{W}^{(n)}$  is also feasible for  $\mathcal{P}(\mathbf{W}^{(n+1)})$ , we have strict inequality  $\|\mathbf{W}^{(n+1)}\|_F^2 < \|\mathbf{W}^{(n)}\|_F^2$  unless  $\mathbf{W}^{(n+1)} = \mathbf{W}^{(n)}$ . Hence, the cost sequence  $\{\|\mathbf{W}^{(n)}\|_F^2\}$ 

output:  $W^{(n)}$ 

**Initialization** Randomly generate a feasible initial point  $W^{(0)}$ .

for 
$$n = 1, 2, ...$$
 do

**Step 1.** Calculate  $G_k(W^{(n-1)})$  according to (19) for all k.

Step 2. Solve (20) for the solution  $W^{(n)}$  using (26) and (23).

Step 3. If  $\|W^{(n)} - W^{(n-1)}\|_F \le \varepsilon$ , then STOP. end for

converges either in finite iterations or to a unique value. Noting the cost function is exactly the square of Frobenius norm of W, the variable sequence  $\{W^{(n)}\}$  converges to a unique point  $W^{(\infty)}$  as well. Since the feasible set in (7) is semi-algebraic,  $\mathcal{P}(W^{(n)})$  is an inner convex approximation of (7), and each constraint function of  $\mathcal{P}(W^{(n)})$  has a consistent directional derivative at  $W^{(n)}$  in certain direction with that of (7), we could show by using the results in [23] [24] that  $W^{(\infty)}$  is a critical point to (7) under mild constraint qualification condition.

The DFGP method in (26) is an efficient matrix-free algorithm that are based solely on matrix-vector products and comparison operations. The number of arithmetic operations per iteration for DFGP is  $O(K^2)$  or O(QMK), depending on the use of explicit or implicit matrix-vector multiplication Bz. At each step of the SCA algorithm, the number of additional arithmetic operations for computing subgradients is O(QMK).

# V. SIMULATION RESULTS

In this section, we provides numerical results to assess the performance of the proposed schemes, i.e., the SDR-G algorithm and the SCA-DFPG algorithm. We assume that for each user, the small-scale fading is frequency-flat Rayleigh, i.e., complex Gaussian distributed with zero mean and unit variance, and the shadow fading is log-normally distributed with standard deviation 0.5 dB. For simplicity, we assume that all users have a common QoS target  $\bar{\gamma}_q = 3$  dB and the noise variance of each user is  $\sigma_{k,q}^2 = 1 \forall k, q$ . The results are averaged over 500 channel realizations. The number of randomly generated candidates for each channel realization is L = 1000 and the number of iterations of DFGP method is set to 400.

#### A. Approximation Ratio Tests

We first test the proposed SDR-G procedure listed in Algorithm 1 for homogeneous channel scenario under various parameter settings. Tables 1 summarize the minimum value (Min), the maximum value (Max), the average value (Mean), and the standard deviation (Std) of empirical approximation ratios  $v_{\text{SDR-G}}^*/v_{\text{SDR-LB}}$  over 500 independent channel realizations. We can see that the maximum values of  $v_{\text{SDR-G}}^*/v_{\text{SDR-LB}}$  are lower than 3 in all test examples. Moreover, the practical results are much better than those of worst-case analysis. On the other hand, the minimum value, the maximum value and the average value of  $v_{\text{SDR-G}}^*/v_{\text{SDR-LB}}$  all increase as *K* grows

TABLE I Statistics of empirical approximation ratios  $v^*_{SDR-G}/v_{SDR-LB}$ 

Q	Μ	Κ	Min	Max	Mean	Std	θ
2	8	10	1.0003	1.8816	1.4635	0.2527	15.81
2	8	20	1.2483	2.4106	1.8943	0.1996	22.36
2	8	30	1.5017	2.8103	2.2018	0.2179	27.39
2	16	10	1.0028	1.9346	1.4697	0.2642	15.81
2	16	20	1.3508	2.6042	1.9878	0.1989	22.36
2	16	30	1.7433	2.9791	2.3608	0.2170	27.39
3	8	10	1.0003	1.9114	1.4618	0.2496	10.77
3	8	20	1.2080	2.4799	1.8881	0.1958	13.57
3	8	30	1.6496	2.7887	2.2122	0.2260	15.54
3	16	10	1.0017	1.9406	1.4758	0.2655	10.77
3	16	20	1.3305	2.6277	1.9862	0.1963	13.57
3	16	30	1.7217	2.9683	2.3685	0.2163	15.54



Fig. 1. Empirical approximation ratios for Q = 2, M = 8, K = 10.

for fixed Q and M in all test examples, which also corroborates well with the theoretic analysis.

Fig. 1 plots the empirical approximation ratio of 500 independent channel realizations for Q = 2, M = 8, K = 10. Fig. 2 shows the corresponding histogram. It can be seen that in some cases the empirical approximation ratios are very near to 1, which means the optimal solutions are obtained by the proposed algorithm for these cases.

## B. Transmitting Power Comparisons

In this part, we focus on general channel scenario and assume that there are Q = 3 orthogonal channels. We first demonstrate the convergence of the SCA-DFGP algorithm. Fig.3 plots the transmitting power consumption during each iteration for different settings in general channel scenario. The results validate the monotonicity and convergence of the SCA-DFGP algorithm. It can be seen that at the first about 10 iterations, the SCA-DFGP algorithm converges very fast and reaches the major part of the limiting value.

We will next compare the the transmitting power consumption of the proposed co-design schemes with conventional scheduling algorithms. The first benchmark is fixed scheduling (OneGroup) [3], in which all users are scheduled into a single group and receive a common message in a fixed best channel. The second benchmark is random scheduling (Equipartition), in which all users are randomly scheduled into Q groups with



Fig. 2. Histogram of the outcomes in Fig. 1.



Fig. 3. Convergence curve of the SCA-DFGP algorithm.

equal size. Moreover, the SDR lower bound (SDR-LB) is also presented.

Fig. 4 compares the average transmitting power of all the algorithms versus K for M = 32. Similarly, Fig. 5 compares the average transmitting power of all the algorithms versus M for K = 72. It can be seen from Fig. 4 and Fig. 5 that the average transmitting power consumed by the SCA-DFGP algorithm is lower than the two benchmarks. The performances of two benchmarks are very similar while the power saving of the SCA-DFGP algorithm over the two benchmarks is significantly beneficial especially when the ratio of number of users to number of antennas is large. On the other hand, the SDR-G algorithm performs poorly in large-scale antenna arrays, especially when the number of users increases, which is also confirmed by the worst-case analysis and many other studies [3] [4] [11]. Moreover, the gap between the transmitting power for the SCA-DFGP algorithm and the SDR lower bound is always less than about 3 dB in Fig. 4 and Fig. 5. Therefore, proper user scheduling is necessarily important when there are a large number of users in the system.



Fig. 4. Transmitting power versus number of users, K, for M = 32 in general channel scenario.



Fig. 5. Transmitting power versus number of antennas, M, for K = 72 in general channel scenario.

## VI. CONCLUSIONS

In this paper, the joint multicast beamforming and user scheduling problem was investigated. A mixed binary quadratically constrained quadratic program formulation and a highlystructured nonsmooth formulation were presented. Convexrelaxation-based and convex-restriction-based algorithms were proposed to solve the problem. Theoretical performance guarantee of convex-relaxation-based algorithm was proved and convergence of convex-restriction-based algorithm was established. Extensive numerical experiments were conducted to show the advantage of the proposed co-design scheme over fixed scheduling and random scheduling in terms of power consumption.

## VII. APPENDIX: PROOF OF THEOREM 1

For any  $W^* \geq 0$  and  $\boldsymbol{\xi} \sim C\mathcal{N}(\boldsymbol{0}, W^*)$ , it's easy to verify that  $|\boldsymbol{h}^{\mathrm{H}}\boldsymbol{\xi}|^2$  is an exponential random variable with mean  $\boldsymbol{h}^{\mathrm{H}}W^*\boldsymbol{h}$  and distribution function

$$\Pr\left(\left|\boldsymbol{h}^{\mathrm{H}}\boldsymbol{\xi}\right|^{2} \leq \eta \boldsymbol{h}^{\mathrm{H}}\boldsymbol{W}^{*}\boldsymbol{h}\right) = 1 - e^{-\eta} \leq \eta.$$
(27)

Let independent random variables  $\xi_q \sim C\mathcal{N}(\mathbf{0}, W_q^*)$  ( $\forall q \in Q$ ). For any  $\mu > 0$  and  $\eta > 0$ , we obtain

$$\Pr\left(\sum_{q \in Q} \left\|\boldsymbol{\xi}_{q}\right\|^{2} \leq \mu \sum_{q \in Q} \operatorname{Tr}(\boldsymbol{W}_{q}^{*}), \min_{k \in \mathcal{K}} \max_{q \in Q} \left\{\left|\boldsymbol{h}_{k,q}^{\mathrm{H}}\boldsymbol{\xi}_{q}\right|^{2}\right\} \geq \eta\right)$$
$$\geq 1 - \Pr\left(\sum_{q \in Q} \left\|\boldsymbol{\xi}_{q}\right\|^{2} \geq \mu \sum_{q \in Q} \operatorname{Tr}(\boldsymbol{W}_{q}^{*})\right)$$
$$- \sum_{k=1}^{K} \Pr\left(\max_{q \in Q} \left\{\left|\boldsymbol{h}_{k,q}^{\mathrm{H}}\boldsymbol{\xi}_{q}\right|^{2}\right\} \leq \eta\right)$$
(28a)

$$\geq 1 - \frac{1}{\mu} - \sum_{k=1}^{K} \Pr\left(\max_{q \in Q} \left\{ \left| \boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{\xi}_{q} \right|^{2} \right\} \leq \eta \right), \tag{28b}$$

where the first inequality is due to union bound of probability, and the last inequality is from the Markov's inequality.

#### A. General channel scenario

From the constraints in (9), we have

$$\max_{q \in Q} \boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{W}_{q}^{*} \boldsymbol{h}_{k,q} \geq \frac{1}{Q} \ \forall k \in \mathcal{K}.$$
(29)

Let  $q_k^* = \operatorname{argmax}_{q \in Q} \boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{W}_q^* \boldsymbol{h}_{k,q}$ . For general channel scenario, we have

$$\Pr\left(\max_{q \in Q} \left\{ \left| \boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{\xi}_{q} \right|^{2} \right\} \leq \eta \right)$$
  
$$\leq \Pr\left(\max_{q \in Q} \left\{ \left| \boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{\xi}_{q} \right|^{2} \right\} \leq Q\eta \max_{q \in Q} \boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{W}_{q}^{*} \boldsymbol{h}_{k,q} \right) \qquad (30a)$$

$$= \Pi_{q \in Q} \Pr\left(\left|\boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{\xi}_{q}\right|^{2} \le Q\eta \max_{q \in Q} \boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{W}_{q}^{*} \boldsymbol{h}_{k,q}\right)$$
(30b)

$$\leq \Pr\left(\left|\boldsymbol{h}_{k,q_{k}^{*}}^{\mathrm{H}}\boldsymbol{\xi}_{q_{k}^{*}}\right|^{2} \leq Q\eta\boldsymbol{h}_{k,q_{k}^{*}}^{\mathrm{H}}\boldsymbol{W}_{q_{k}^{*}}^{*}\boldsymbol{h}_{k,q_{k}^{*}}\right)$$
(30c)

$$=Q\eta, \tag{30d}$$

where (30b) is due to independence of random variables  $\{\xi_q\}$  and (30d) is from (27).

Thus, by setting  $\mu = \sqrt{5}$  and  $\eta = \frac{1}{\sqrt{5}KQ}$ , we have

$$\Pr\left(\sum_{q \in Q} \left\|\boldsymbol{\xi}_{q}\right\|^{2} \le \mu \sum_{q \in Q} \operatorname{Tr}(\boldsymbol{W}_{q}^{*}), \min_{k \in \mathcal{K}} \max_{q \in Q} \left\{\left|\boldsymbol{h}_{k,q}^{\mathrm{H}}\boldsymbol{\xi}_{q}\right|^{2}\right\} \ge \eta\right)$$
  
> 1 -  $\frac{1}{2}$  -  $KOn$  (31a)

$$2 = 1 - \frac{1}{\mu} - K Q \eta$$
 (31a)

$$= 1 - \frac{2}{\sqrt{5}} = 0.1056\dots$$
 (31b)

We see that with positive probability of at least 0.1, the randomly generated candidate beamformers  $\{x_q^{(l)}\}$  satisfies

$$\sum_{q \in Q} \left\| \mathbf{x}_q^{(l)} \right\|^2 \le \sqrt{5} \sum_{q \in Q} \operatorname{Tr}(\mathbf{W}_q^*)$$
(32)

and

$$\min_{k \in \mathcal{K}} \max_{q \in Q} \left\{ \left| \boldsymbol{h}_{k,q}^{\mathrm{H}} \boldsymbol{x}_{q}^{(l)} \right|^{2} \right\} \geq \frac{1}{\sqrt{5}KQ}.$$
(33)

With  $p(\{x_q\})$  defined in (12),  $\left\{\sqrt{p(\{x_q^{(l)}\})}x_q^{(l)}\right\}$  is feasible for (4), so that

$$p^{(l)} = p(\{\mathbf{x}_{q}^{(l)}\}) \sum_{q \in Q} \left\| \mathbf{x}_{q}^{(l)} \right\|^{2}$$
(34a)

$$=\frac{\sum_{q\in Q}\left\|\boldsymbol{x}_{q}^{(l)}\right\|^{2}}{\min_{k\in\mathcal{K}}\max_{q\in Q}\left\{\left|\boldsymbol{h}_{k,q}^{\mathrm{H}}\boldsymbol{x}_{q}^{(l)}\right|^{2}\right\}}$$
(34b)

$$\leq \frac{\sqrt{5}\sum_{q \in Q} \operatorname{Tr}(W_q^*)}{1/(\sqrt{5}KQ)}$$
(34c)

$$= 5KQ \cdot v_{\text{SDR-LB}}.$$
 (34d)

If one generates *L* independent realizations of  $\{\mathbf{x}_q^{(l)}\}$  from  $CN(\mathbf{0}, \mathbf{W}_q^*)$ , it is at least with probability  $1 - 0.9^L$  to obtain one candidate beamformers satisfying (34). Since  $v_{\text{SDR-G}}^* = \min_{l=1,...,L} p^{(l)}$ , it follows that

$$v_{\text{SDR-LB}} \le v^* \le v^*_{\text{SDR-G}} \le 5KQ \cdot v_{\text{SDR-LB}}.$$
 (35)

#### B. Homogeneous channel scenario

For homogeneous channel scenario, we have  $W_1^* = W_2^* = \cdots = W_Q^*$ ,  $\tilde{h}_k^{\text{H}} W_1 \tilde{h}_k \ge 1 \forall k \in \mathcal{K}$  and  $\xi_q \sim C\mathcal{N}(0, W_1^*) (\forall q \in Q)$ . Thus,

$$\Pr\left(\max_{q \in Q}\left\{\left|\tilde{\boldsymbol{h}}_{k}^{\mathrm{H}}\boldsymbol{\xi}_{q}\right|^{2}\right\} \leq \eta\right)$$
  
$$\leq \Pr\left(\max_{q \in Q}\left\{\left|\tilde{\boldsymbol{h}}_{k}^{\mathrm{H}}\boldsymbol{\xi}_{q}\right|^{2}\right\} \leq \eta\tilde{\boldsymbol{h}}_{k}^{\mathrm{H}}\boldsymbol{W}_{1}^{*}\tilde{\boldsymbol{h}}_{k}\right)$$
(36a)

$$= \Pi_{q \in Q} \Pr\left(\left|\tilde{\boldsymbol{h}}_{k}^{\mathrm{H}} \boldsymbol{\xi}_{q}\right|^{2} \le \eta \tilde{\boldsymbol{h}}_{k}^{\mathrm{H}} \boldsymbol{W}_{1}^{*} \tilde{\boldsymbol{h}}_{k}\right)$$
(36b)

$$= \Pr\left(\left|\tilde{\boldsymbol{h}}_{k}^{\mathrm{H}}\boldsymbol{\xi}_{1}\right|^{2} \leq \eta \tilde{\boldsymbol{h}}_{k}^{\mathrm{H}}\boldsymbol{W}_{1}^{*}\tilde{\boldsymbol{h}}_{k}\right)^{Q}$$
(36c)

$$\leq \eta^Q$$
. (36d)

By setting  $\mu = \sqrt{5}$  and  $\eta = \frac{1}{(\sqrt{5}K)^{(1/Q)}}$ , we have

$$\Pr\left(\sum_{q \in Q} \left\|\boldsymbol{\xi}_{q}\right\|^{2} \le \mu \operatorname{Tr}(\boldsymbol{W}_{1}^{*}), \min_{k \in \mathcal{K}} \max_{q \in Q} \left\{\left|\boldsymbol{\tilde{h}}_{k}^{\mathsf{H}} \boldsymbol{\xi}_{q}\right|^{2}\right\} \ge \eta\right)$$
$$\ge 1 - \frac{1}{\mu} - K\eta^{Q}$$
(37a)

$$= 1 - \frac{2}{\sqrt{5}} = 0.1056....$$
 (37b)

Similar to the proof for general channel scenario, we conclude that with probability of at least  $1 - 0.9^L$ , if *L* independent realizations are generated, one could obtain an approximate solution such that

$$v_{\text{SDR-LB}} \le v^* \le v_{\text{SDR-G}}^* \le 5K^{1/Q} \cdot v_{\text{SDR-LB}}.$$
 (38)

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