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THE AVERAGE LOWER REINFORCEMENT NUMBER OF A GRAPH

Tufan Turaci[1] and Ersin Aslan[2]

Abstract. Let $G = (V(G), E(G))$ be a simple undirected graph. The reinforcement number of a graph is a vulnerability parameter of a graph. We have investigated a refinement that involves the average lower reinforcement number of this parameter. The *lower reinforcement number*, denoted by $r_{e^*}(G)$, is the minimum cardinality of *reinforcement set* in G that contains the edge e^* of the complement graph $\overline{G}$. The *average lower reinforcement number* of G is defined by $r_{av}(G) = \frac{1}{|E(\overline{G})|} \sum_{e^* \in E(\overline{G})} r_{e^*}(G)$.

In this paper, we define the average lower reinforcement number of a graph and we present the exact values for some well−known graph families.

Mathematics Subject Classification. 05C40, 05C69, 68M10, 68R10.

1. Introduction

Graph theory has seen an explosive growth due to interaction with areas like computer science, operation research, *etc.* Especially, it has become one of the most powerful mathematical tools in the analysis and study of the architecture of a network. It is known that communication systems are often exposed to failures and attacks [26, 32]. The problem of quantifying the vulnerability of graphs has received much attention recently, especially in the field of computer, communication and spy networks. In a network, the vulnerability parameters measure the resistance of the network to disruption of operation after the failure of certain stations or links [27]. A network is described as an undirected and unweighted graph in which vertices represent the processing and edges represent the communication channel between them [26, 27].

In the literature, various measures have been defined to measure the robustness of network and a variety of graph theoretic parameters have been used to derive formulas to calculate network vulnerability. The best known measure of reliability of a graph is its connectivity. The vertex (edge) connectivity is defined to be the minimum number of vertices (edges) whose deletion results in a disconnected or trivial graph [15]. Then toughness [12], integrity [7], domination number [17], bondage number [4,5], reinforcement number [23] *etc.* have been proposed for measuring the vulnerability of networks. Recently, some average vulnerability parameters such as average lower independence number [3, 10, 18], average lower domination number [2, 6, 18, 30], average connectivity number [9, 19], average lower connectivity number [1] and average lower bondage number [31] have been defined.

Keywords and phrases. Graph vulnerability, connectivity, network design and communication, domination number, reinforcement number, average lower reinforcement number.

[1] Department of Mathematics, Faculty of Science, Karabük University, 78050, Karabük, Turkey. tufanturaci@karabuk.edu.tr

[2] Turgutlu Vocational Training School, Celal Bayar University, 45400 Manisa, Turkey. ersin.aslan@cbu.edu.tr

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Let $G=(V(G),E(G))$ be a simple undirected graph. We begin by recalling some standard definitions that we need throughout this paper. For any vertex $v \in V(G)$, *the open neighborhood of* v is $N_G(v)=\{u \in V(G) | \ uv \in E(G)\}$ and *closed neighborhood of* v is $N_G[v]=N_G(v) \cup \{v\}$. The *degree of vertex* v in G denoted by $d_G(v)$, that is the size of its open neighborhood [17]. The *maximum degree of* G is $\max\{d_G(v) | \ v \in V(G)\}$ and denoted by $\Delta(G)$. The *complement* $\overline{G}$ of a graph G has $V(G)$as its vertex sets, but two vertices are adjacent in $\overline{G}$ if only if they are not adjacent in G. The largest integer not greater than x is denoted by $\lfloor x \rfloor$. A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G)-S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all *dominating sets* of G is called the *domination number* of G and it is denoted by $\gamma(G)$ [17].

The study of domination in graphs is an important research area, perhaps also the fastest-growing area within graph theory. The reason for the steady and rapid growth of this area may be the diversity of its applications to both theoretical and real world problems [17]. When investigating the domination number of a given graph G, one may want to learn the answer of the following question: how many edges need to be added to decrease the domination number of the original graph? One of the vulnerability parameters known as reinforcement number in a graph G answers this question. The *reinforcement number* $r(G)$ was introduced by Kok and Mynhardt [23] and is defined as follows:

$$r(G)=\min\{|R| \ : \ R \subseteq E(\overline{G}), \ \gamma(G)>\gamma(G+R)\}.$$

If $\gamma(G)=1$, then $r(G)=0$ is defined. Furthermore, a set $R \subseteq E(\overline{G})$ is a *reinforcement set* if $\gamma(G)>\gamma(G+R)$. The reinforcement problem applies to a variety of settings modeled by graphs where dominators have costs but where edges can be added to the graph (incurring less cost), eliminating the need for some of the dominators. For example, in a network it might be very expensive to set up a new mirror of a database, but relatively cheap to add a link [8]. The reinforcement number has received much research attention (see, for example, [8, 22]), and its many variations have also been well described and studied in graph theory, including total reinforcement [20, 29], independence reinforcement [33], fractional reinforcement [11, 13], weak reinforcement [14], strong reinforcement [16], roman reinforcement [28], signed reinforcement [24], k-reinforcement [8] and so on. Furthermore, the complexities of reinforcement numbers have been studied in [8, 21, 25].

In 2004, Henning introduced the concept of average domination and average independence [18]. Finding largest dominating sets and independent sets in graphs is the problem which is closely in relation with the concept of average domination and average independence. Also, the average lower domination and average lower independence number are the theoretical vulnerability parameters for a network that is represented by a graph [6, 18]. The average lower domination number of a graph G, denoted by $\gamma_{av}(G)$, is defined as:

$$\gamma_{av}(G)=\frac{1}{|V(G)|}\sum\nolimits_{v \in V(G)} \gamma_v(G),$$

where the lower domination number, denoted by $\gamma_v(G)$, is the minimum cardinality of a dominating set of the graph G that contains the vertex v [18].

The average parameters have been found to be more useful in some circumstance than the corresponding measures based on worst-case situation [19]. Thus incorporating the concept of the reinforcement number and the idea of the average lower domination number, we will introduce a new graph parameter called the average lower reinforcement number, denoted by $r_{av}(G)$. This paper is organized as follows: In Section 2, we define the average lower reinforcement number and determine upper bounds, lower bounds and exact solutions of the average lower reinforcement number for any graph. In Section 3, we compute the average lower reinforcement number of well−known families of graphs. In Section 4, we discuss the use of the average number in order to distinguish between two graphs. Finally, in Section 5, we present our conclusions.

2. The average lower reinforcement number

In this section, we introduce a new graph theoretical parameter, the average lower reinforcement number For an edge e^* of a graph $\overline{G}$, the *lower reinforcement number*, denoted by $r_{e^*}(G)$, is the minimum cardinality

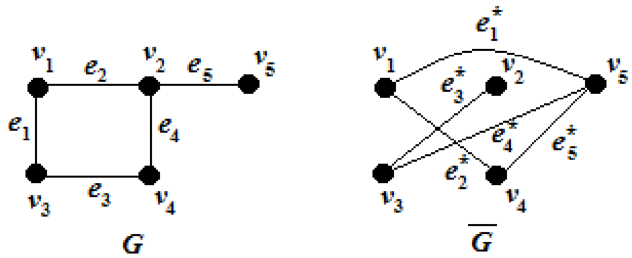

FIGURE 1. The graphs G and its complement graph $\overline{G}$.

TABLE 1. The reinforcement sets and lower reinforcement number of every edge $e_i^* \in E(\overline{G})$.

Edges	The reinforcement sets with minimum cardinality	$r_{e_i^*}(G)$
e_1^*	$\{e_1^*, e_3^*\}, \{e_1^*, e_2^*\}$	2
e_2^*	$\{e_2^*, e_3^*\}, \{e_2^*, e_1^*\}, \{e_2^*, e_5^*\}$	2
e_3^*	$\{e_3^*\}$	1
e_4^*	$\{e_4^*, e_3^*\}$	2
e_5^*	$\{e_5^*, e_3^*\}, \{e_5^*, e_2^*\}$	2

of *reinforcement set* in G that contains the edge e^*. Clearly, $r(G) = \min\{r_{e^*}(G) \mid e^* \in E(\overline{G})\}$. Furthermore, *the average lower reinforcement number* is defined as:

$$r_{av}(G) = \frac{1}{\left|E(\overline{G})\right|} \sum\nolimits_{e^* \in E(\overline{G})} r_{e^*}(G).$$

If $\gamma(G) = 1$ then, $r_{av}(G) = 0$ is defined.

Corollary 2.1. *If K_n is a complete graph of order n, then $r_{av}(K_n) = 0$.*

Corollary 2.2. *If $K_{1,n}$ is a star graph of order $n+1$, then $r_{av}(K_{1,n}) = 0$.*

Definition 2.3 ([17])**.** The wheel graph $W_{1,n}$ with n spokes is a graph that contains an n-cycle and one additional central vertex v_c that is adjacent to all vertices of the cycle. Wheel graph $W_{1,n}$ has $(n+1)$-vertices and $2n$-edges.

Corollary 2.4. *If $W_{1,n}$ is a wheel graph of order $n+1$, then $r_{av}(W_{1,n}) = 0$.*

Example 2.5. Let the graph G be 4-cycle with one additional vertex and edge. The graph G and its complement graph $\overline{G}$ as shown in Figure 1.

The reinforcement sets and lower reinforcement number of every edge $e_i^* \in E(\overline{G})$ is presented in Table 1.

Since we have $r_{e_1^*}(G) = 2$, $r_{e_2^*}(G) = 2$, $r_{e_3^*}(G) = 1$, $r_{e_4^*}(G) = 2$ and $r_{e_5^*}(G) = 2$, $r_{av}(G) = (2+2+1+2+2)/5 = 1.8$.

Theorem 2.6. *If G is a connected graph of order n, then*

$$r(G) \leqslant r_{av}(G) \leqslant \frac{(r(G))^2 + \left(\binom{n}{2} - |E(G)| - r(G)\right)(r(G)+1)}{\binom{n}{2} - |E(G)|}.$$

Proof. Let R be the set including the minimum reinforcement sets. We have two cases depending on the cardinality of R.

Case 1. Let $|R| = 1$. Clearly the minimum reinforcement set is unique and it is denoted by R^*. Let $e_1^*, e_2^*, \ldots, e_{|R^*|}^*$ be elements of R^*. Then, we get $r_{e_i^*}(G) = r(G)$ for every $e_i^* \in R^*$, where $i \in \{1, 2, \ldots, |R^*|\}$. It is not difficult to see that $r_{e_i}(G) = r(G) + 1$ for every $e_i \in E(K_n) \backslash (E(G) \cup R^*)$. Thus, we have

$$\begin{aligned} r_{av}(G) &= \frac{1}{\left|E\left(\overline{G}\right)\right|}\left(\sum_{e_i^* \in R^*} r_{e_i^*}(G) + \sum_{e_i \in E(K_n)\backslash(E(G)\cup R^*)} r_{e_i}(G)\right) \\ &= \frac{1}{\binom{n}{2} - |E(G)|}\left(r(G)\,|R^*| + \left(\binom{n}{2} - |E(G)| - |R^*|\right)(r(G)+1)\right). \end{aligned}$$

Clearly, $|R^*| = r(G)$. So, we have $r_{av}(G) = \frac{(r(G))^2 + \left(\binom{n}{2} - |E(G)| - r(G)\right)(r(G)+1)}{\binom{n}{2} - |E(G)|}$ is also an upper bound.

Case 2. Let $|R| > 1$. If the union of reinforcement set is equal to $E(\overline{G})$, then the lower reinforcement number is $r(G)$ for every edge of $E(\overline{G})$. Thus we get $r(G) = r_{av}(G)$. Clearly, if the union of reinforcement set is not equal to $E(\overline{G})$, then we get $r(G) < r_{av}(G)$. So, we have $r(G) \leqslant r_{av}(G)$, that is, $r(G)$ is a lower bound for the average lower reinforcement number.

By Cases 1 and 2, we get $r(G) \leqslant r_{av}(G) \leqslant \frac{(r(G))^2 + \left(\binom{n}{2} - |E(G)| - r(G)\right)(r(G)+1)}{\binom{n}{2} - |E(G)|}$.

The proof is completed. □

Theorem 2.7. *If G is a connected graph of order n with the domination number $\gamma(G) = 2$ and a vertex of G with maximum degree is unique, then*

$$r_{av}(G) = \frac{(n - \Delta(G) - 1)^2 + \left(\binom{n}{2} + \Delta(G) + 1 - n - |E(G)|\right)(n - \Delta(G))}{\binom{n}{2} - |E(G)|}.$$

Proof. Let G be the connected graph with the domination number $\gamma(G) = 2$, and let v be the vertex with maximum degree. Since the vertex v is unique, the $r(G)$-reinforcement set is unique. Then this set only includes edges which are incident to the vertex v. The number of these edges is $n - \Delta(G) - 1$, and let E^* be the set that includes these edges. So, these edges are labeled by $e_1^*, e_2^*, \ldots, e_{n-\Delta(G)-1}^*$. Clearly, $r_{e_i^*}(G) = n - \Delta(G) - 1$ is obtained for every edge e_i^* of E^*. Furthermore, we get $r_{e_i}(G) = n - \Delta(G)$ for every edge $e_i \in E(G) \backslash E^*$, where $i \in \{1, 2, \ldots, (\binom{n}{2}) + \Delta(G) + 1 - n - |E(G)|)\}$.

Thus, we have

$$\begin{aligned} r_{av}(G) &= \frac{1}{\left|E\left(\overline{G}\right)\right|}\left(\sum_{e_i^* \in E^*} r_{e_i^*}(G) + \sum_{e_i \in E(G)\backslash E^*} r_{e_i}(G)\right) \\ &= \frac{1}{\binom{n}{2} - |E(G)|}\left((n - \Delta(G) - 1)^2 + \left(\binom{n}{2} + \Delta(G) + 1 - n - |E(G)|\right)(n - \Delta(G))\right). \end{aligned}$$

As a result, the proof is completed. □

Definition 2.8 ([17]). Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$be graphs. Let G be a join graph $G_1 + G_2$. The vertices and edges of join graph G are $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv| \ u \in V(G_1), v \in V(G_2)\}$, respectively.

Theorem 2.9 ([23]). *Let G_1 be a connected graph of order n_1 and size m_1, and let G_2 be a connected graph of order n_2 and size m_2. Let the domination numbers $\gamma(G_1) \geqslant 2$ and $\gamma(G_2) \geqslant 2$. Then, $r(G_1 + G_2) = \min\{n_1 - \Delta(G_1) - 1, n_2 - \Delta(G_2) - 1\}$.*

Theorem 2.10. *Let G_1 be a connected graph of order n_1 and size m_1, and let G_2 be a connected graph of order n_2 and size m_2. If the domination numbers $\gamma(G_1) \geqslant 2$ and $\gamma(G_2) \geqslant 2$, then*

(a) *If $p_1 > p_2$, then $r_{av}(G_1 + G_2) \geqslant \frac{p_2\left(n_2^2 - n_2 - 2m_2\right) + (p_2+1)\left(n_1^2 - n_1 - 2m_1\right)}{n_1^2 + n_2^2 - (n_1+n_2) - 2(m_1+m_2)}$;*

(b) *If $p_2 > p_1$, then $r_{av}(G_1 + G_2) \geqslant \frac{p_1\left(n_1^2 - n_1 - 2m_1\right) + (p_1+1)\left(n_2^2 - n_2 - 2m_2\right)}{n_1^2 + n_2^2 - (n_1+n_2) - 2(m_1+m_2)}$;*

(c) *If $p_1 = p_2$, then $r_{av}(G_1 + G_2) \geqslant p_1$,*

where $p_1 = n_1 - \Delta(G_1) - 1$ and $p_2 = n_2 - \Delta(G_2) - 1$.

Proof. Let $G = G_1 + G_2$. Let $e_1^*, e_2^*, \ldots, e_X^*$ be edges of $\overline{G_1}$, where $X = \left(n_1^2 - n_1 - 2m_1\right)/2$, and let $e_1^{**}, e_2^{**}, \ldots, e_Y^{**}$ be edges of $\overline{G_2}$, where $Y = \left(n_2^2 - n_2 - 2m_2\right)/2$. Furthermore, we have $r(G) = \min\{p_1, p_2\}$ by Theorem 2.3. We analyze three cases depending on p_1 and p_2.

Case 1. Suppose that $p_1 > p_2$. We know $r(G) = p_2$ by Theorem 2.9. If the degree of every vertex of G_2 is $\Delta(G_2)$, that is the graph G_2 is regular, then obviously the cardinality of every reinforcement set is p_2. Furthermore, every edge of $\overline{G_2}$ belongs to any reinforcement set. So, we have $r_{e_i^{**}}(G) = p_2$ for every $e_i^{**} \in E\left(\overline{G_2}\right)$. If the graph G_2 is not regular, some edges may not belong to any reinforcement set. Clearly, we have $r_{e_i^{**}}(G) = p_2 + 1$ for these edges. As a result, we get $r_{e_i^{**}}(G) = p_2$ or $r_{e_i^{**}}(G) = p_2 + 1$ for every $e_i^{**} \in E\left(\overline{G_2}\right)$, depending on e_i^{**} is or not in a reinforcement set. It is not difficult to see that we get $r_{e_i^*}(G) = p_2 + 1$ for every $e_i^* \in E\left(\overline{G_1}\right)$ since e_i^* is not in any reinforcement set. Thus,

$$\begin{aligned}
r_{av}(G) &= \frac{1}{\left|E\left(\overline{G}\right)\right|}\left(\sum_{e_i^* \in E\left(\overline{G_1}\right)} r_{e_i^*}(G) + \sum_{e_i^{**} \in E\left(\overline{G_2}\right)} r_{e_i^{**}}(G)\right) \\
&\geqslant \frac{1}{\left|E\left(\overline{G_1}\right)\right| + \left|E\left(\overline{G_2}\right)\right|}\left((p_2+1)\left|E\left(\overline{G_1}\right)\right| + p_2\left|E\left(\overline{G_2}\right)\right|\right) \\
&\geqslant \frac{(p_2+1)\left(\frac{n_1^2 - n_1 - 2m_1}{2}\right) + p_2\left(\frac{n_2^2 - n_2 - 2m_2}{2}\right)}{\frac{n_1^2 + n_2^2 - (n_1+n_2) - 2(m_1+m_2)}{2}} \\
&\geqslant \frac{p_2\left(n_2^2 - n_2 - 2m_2\right) + (p_2+1)\left(n_1^2 - n_1 - 2m_1\right)}{n_1^2 + n_2^2 - (n_1+n_2) - 2(m_1+m_2)}.
\end{aligned}$$

As a result, we get $r_{av}(G_1 + G_2) \geqslant \frac{p_2\left(n_2^2 - n_2 - 2m_2\right) + (p_2+1)\left(n_1^2 - n_1 - 2m_1\right)}{n_1^2 + n_2^2 - (n_1+n_2) - 2(m_1+m_2)}$.

Case 2. Suppose that $p_2 > p_1$. The proof of Case 2 is similar to that of Case 1 and is omitted.

Case 3. Suppose that $p_1 = p_2$. Clearly, $r(G) = p_1 = p_2$. If G_1 and G_2 are regular, then we have $r_{e_i^*}(G) = p_1 = p_2$ for every $e_i^* \in E\left(\overline{G}\right)$. As a result, $r_{av}(G_1+G_2) = p_1$, which is also a lower bound. Then, we get $r_{av}(G_1+G_2) \geqslant p_1$.

By Cases 1, 2 and 3 the proof is completed. □

3. The average lower reinforcement number of some well−known graph families

In this section, we calculate the average lower reinforcement number of well−known graphs such as the path graph P_n, cycle graph C_n and complete bipartite graph $K_{a,b}$. Now, we recall a basic result for the reinforcement numbers of path and cycle graph.

Theorem 3.1 ([23])**.** *Let $n \geqslant 4$ be an integer and write n as $n = 3k+i$, where k is an integer and $i \in \{1,2,3\}$. Then $r(P_n) = r(C_n) = i$.*

Theorem 3.2. *Let P_n be a path graph of order n, where $n \geqslant 4$. Then,*

$$r_{av}(P_n) = \begin{cases} \dfrac{28n^2-90n+90}{9n^2-27n+18}, & \text{if} \quad \text{n} \equiv 0 \ (\text{mod}3); \\ \dfrac{16n^2-56n+40}{9n^2-27n+18}, & \text{if} \quad \text{n} \equiv 1 \ (\text{mod}3); \\ \dfrac{22n^2-76n+64}{9n^2-27n+18}, & \text{if} \quad \text{n} \equiv 2 \ (\text{mod}3). \end{cases}$$

Proof. Let k, m and s be integer, and also let $\{v_1, v_2, \ldots, v_{n-1}, v_n\}$ be the vertices of P_n. Some notations are used in order to make the proof of theorem understandable. Let $v_k$and $v_m$be any two vertices of P_n. If these two vertices are not adjacent in the graphP_n, then the edge between these two vertices is denoted by $e^*_{(v_k)(v_m)}$ in the graph $\overline{P_n}$. Observe that $\left|E\left(\overline{P_n}\right)\right| = \frac{n^2-3n+2}{2}$. Let $E\left(\overline{P_n}\right) = R^* \cup E^*$, where the set R^* is the set of edges that are the elements of reinforcement sets. If $e^*_{(v_k)(v_m)} \in R^*$, then $r_{e^*_{(v_k)(v_m)}}(P_n) = r(P_n)$. If $e^*_{(v_k)(v_m)} \notin R^*$, then we have $r_{e^*_{(v_k)(v_m)}}(P_n) = r(P_n) + 1$. In order to calculate the average lower reinforcement number of P_n, we analyze three cases depending on n.

Case 1. $n \equiv 0\,(\text{mod } 3)$.
We have $r(P_n) = 3$ by Theorem 3.1. The edges do not belong to any reinforcement set as follows:

$$E^* = \left\{ e^{**}_{(v_{3k})(v_{3m+1})} \middle| \ 1 \leqslant k \leqslant \frac{n-6}{3} \ \text{and} \ k+1 \leqslant m \leqslant \frac{n-3}{3} \right\} \cup \left\{ e^{**}_{(v_1)(v_n)} \right\}.$$

Clearly $|E^*| = \frac{n^2-9n+36}{18}$. Since $\left|E(\overline{P_n})\right| = |E^*| + |R^*|$ it is follows that $|R^*| = \frac{4n^2-9n-9}{9}$. Let $e^* \in R^*$ and $e^{**} \in E^*$. So, we have $r_{e^*}(P_n) = 3$ and $r_{e^{**}}(P_n) = 4$ for every $e^* \in R^*$ and $e^{**} \in E^*$, respectively.

Thus, we get

$$\begin{aligned} r_{av}(P_n) &= \frac{1}{|R^* \cup E^*|} \left(\sum_{e^* \in R^*} r_{e^*}(P_n) + \sum_{e^{**} \in E^*} r_{e^{**}}(P_n) \right) \\ &= \frac{1}{\left(\frac{n^2-3n+2}{2}\right)} \left(3\,|R^*| + 4\,|E^*|\right) \\ &= \frac{28n^2-90n+90}{9n^2-27n+18}. \end{aligned}$$

Case 2. $n \equiv 1 \,(\text{mod } 3)$.
We have $r(P_n) = 1$ by Theorem 3.1. The edges belong to any reinforcement set as follows: $R^* = \left\{ e^*_{(v_k)(v_{k+3m+2})} \middle| \; (1 \leqslant k \leqslant n-2 \textit{ and } k \neq 3s) \textit{ and } 0 \leqslant m \leqslant \left\lfloor \frac{n-1-k}{3} \right\rfloor \right\}$.

Clearly $|R^*| = \frac{n^2+n-2}{9}$. Since $\left|E(\overline{P_n})\right| = |E^*| + |R^*|$ it is follows that $|E^*| = \frac{7n^2-29n+22}{18}$. Let $e^* \in R^*$ and $e^{**} \in E^*$. Thus we have $r_{e^*}(P_n) = 1$ and $r_{e^{**}}(P_n) = 2$ for every $e^* \in R^*$and $e^{**} \in E^*$, respectively. Then the rest of proof is similar to Case 1. As a result, we get $r_{av}(P_n) = \frac{16n^2-56n+40}{9n^2-27n+18}$.

Case 3. $n \equiv 2 \,(\text{mod } 3)$.
We have $r(P_n) = 2$ by Theorem 3.1. The edges do not belong to any reinforcement set as follows: $E^* = E_1 \cup E_2$, where $E_1 = \{ e^{**}_{(v_k)(v_{k+3m+4})} | \; 1 \leqslant k \leqslant n-4 \textit{ and } 0 \leqslant m \leqslant \left\lfloor \frac{n-4-k}{3} \right\rfloor \}$ and $E_2 = \{ e^{**}_{(v_{3k})(v_{3m})} | \; 1 \leqslant k \leqslant \frac{n-5}{3} \textit{ and } k+1 \leqslant m \leqslant \frac{n-2}{3} \}$. Clearly $|E_1| = \frac{n^2-5n+6}{6}$, and $|E_2| = \frac{n^2-7n+10}{18}$. Since $\left|E(\overline{P_n})\right| = |E^*| + |R^*|$ it is follows that $|R^*| = \frac{5n^2-5n-10}{18}$. Let $e^* \in R^*$ and $e^{**} \in E^*$. Thus we have $r_{e^*}(P_n) = 2$ and $r_{e^{**}}(P_n) = 3$ for every $e^* \in R^*$and $e^{**} \in E^*$, respectively. Then the rest of proof is similar to Case 1. As a result, we get $r_{av}(P_n) = \frac{22n^2-76n+64}{9n^2-27n+18}$.

By Cases 1, 2 and 3 the proof is completed. □

Theorem 3.3. *Let C_n be a cycle graph of order n, where $n \geqslant 4$. Then,*

$$r_{av}(C_n) = \begin{cases} 3, & \text{if} \quad n \equiv 0 \text{ (mod3)}; \\ \dfrac{5n-17}{3n-9}, & \text{if} \quad n \equiv 1 \text{ (mod3)}; \\ \dfrac{7n-23}{3n-9}, & \text{if} \quad n \equiv 2 \text{ (mod3)}. \end{cases}$$

Proof. Let k, m and s be integer, and also let $\{v_1, v_2, \ldots, v_{n-1}, v_n\}$ be the vertices of C_n. Some notations are used in order to make the proof of theorem understandable. Let $v_k$and $v_m$be any two vertices of C_n. If these two vertices are not adjacent in the graph C_n, then the edge between these two vertices is denoted by $e^*_{(v_k)(v_m)}$ in the graph $\overline{C_n}$. Observe that $\left|E\left(\overline{C_n}\right)\right| = \frac{n^2-3n}{2}$. Let $E\left(\overline{C_n}\right) = R^* \cup E^*$ where the set R^* is the set of edges that are the elements of reinforcement sets. If $e^*_{(v_k)(v_m)} \in R^*$, then $r_{e^*_{(v_k)(v_m)}}(C_n) = r(C_n)$. If $e^*_{(v_k)(v_m)} \notin R^*$, then we have $r_{e^*_{(v_k)(v_m)}}(C_n) = r(C_n) + 1$. In order to calculate the average lower reinforcement number of C_n, we analyze three cases depending on n. □

Case 1. $n \equiv 0 \,(\text{mod } 3)$.
We have $r(C_n) = 3$ by Theorem 3.1. It is clear that we get $r_{e^*}(C_n) = 3$ for every $e^* \in E(\overline{C_n})$. As a result, we have $r_{av}(C_n) = 3$.

Case 2. $n \equiv 1 \,(\text{mod } 3)$. We have $r(C_n) = 1$ by Theorem 3.1. The edges belong to any reinforcement set as follows: $R^* = \{ e^*_{(v_k)(v_{k+3m+2})} | \; 1 \leqslant k \leqslant n-2 \; \wedge \; 0 \leqslant m \leqslant \left\lfloor \frac{n-2-k}{3} \right\rfloor \}$. Clearly $|R^*| = \frac{n^2-n}{6}$. Since $\left|E(\overline{C_n})\right| = |E^*| + |R^*|$ it is follows that $|E^*| = \frac{n^2-4n}{3}$. Let $e^* \in R^*$ and $e^{**} \in E^*$. Then we have $r_{e^*}(C_n) = 1$ and $r_{e^{**}}(C_n) = 2$ for every $e^* \in R^*$ and $e^{**} \in E^*$, respectively.

Thus, we get

$$\begin{aligned} r_{av}(C_n) &= \frac{1}{|R^* \cup E^*|} \left(\sum_{e^* \in R^*} r_{e^*}(C_n) + \sum_{e^{**} \in E^*} r_{e^{**}}(C_n) \right) \\ &= \frac{1}{\left(\frac{n^2-3n}{2}\right)} \left(|R^*| + 2\,|E^*|\right) \\ &= \frac{5n-17}{3n-9}. \end{aligned}$$

Case 3. $n \equiv 2 \pmod 3$.
We have $r(C_n) = 2$ by Theorem 3.1. The edges do not belong to any reinforcement set as follows: $E^* = E_1 \cup E_2$, where $E_1 = \{e^{**}_{(v_k)(v_{k+3m+4})} | \ 2 \leqslant k \leqslant n-4 \ \mathit{and} \ 0 \leqslant m \leqslant \lfloor \frac{n-4-k}{3} \rfloor\}$ and $E_2 = \{e^{**}_{(v_k)(v_{k+3m+4})} | \ k = 1 \ \mathit{and} \ 0 \leqslant m \leqslant \lfloor \frac{n-6}{3} \rfloor\}$. Clearly $|E^*| = \frac{n^2-5n}{3}$. Since $|E(\overline{C_n})| = |E^*| + |R^*|$ it is follows that $|R^*| = \frac{n^2-2n}{3}$. Let $e^* \in R^*$ and $e^{**} \in E^*$. We have $r_{e^*}(C_n) = 2$ and $r_{e^*}(C_n) = 3$ for every $e^* \in R^*$ and $e^{**} \in E^*$, respectively. Then the rest of proof is similar to Case 2. As a result, we get $r_{av}(C_n) = \frac{7n-23}{3n-9}$.

By Cases 1, 2 and 3 the proof is completed.

Theorem 3.4. *If $K_{a,b}$ is a complete bipartite graph of order $a+b$, where $a \geqslant 2$ and $b \geqslant 2$, then*

$$r_{av}(K_{a,b}) = \begin{cases} a-1, & \text{if } a = b; \\ \dfrac{a(a-1)^2 + ab(b-1)}{a^2+b^2-a-b}, & \text{if } a < b. \end{cases}$$

Proof. We have $|E(\overline{K_{a,b}})| = |E(K_{a+b})| - ab$. So, $|E(\overline{K_{a,b}})| = \frac{a^2+b^2-a-b}{2}$. If $a \geqslant 2$ *and* $b \geqslant 2$, then we have $\gamma(K_{a,b}) = 2$ [17]. We study two cases depending on a and b. □

Case 1. Suppose that $a = b$. Let v_i be a vertex of $K_{a,b}$, let R be the set including the minimum reinforcement sets, and let R_{v_i} be a minimum reinforcement set that contains the edges which are incident the vertex v_i of the complement graph $\overline{K_{a,b}}$. Due to $\gamma(K_{a,b}) = 2$, we have $r(K_{a,b}) = n - \Delta(G) - 1$ by Theorem 2.7. So, we get $r(K_{a,b}) = a - 1$. Clearly the union of the sets R_{v_i} is equal to $E(\overline{K_{a,b}})$, and $|R_{v_i}| = a-1$ for every $v_i \in \overline{K_{a,b}}$. So, we get the lower reinforcement number is $a-1$ for every edge of $E(\overline{K_{a,b}})$. Therefore $r_{av}(K_{a,b}) = a - 1$.

Case 2. Suppose that $a < b$. Let $V(K_{a,b}) = S_1 \cup S_2$, where the set S_1 includes vertices with degree b and the set S_2 includes vertices with degree a. Let $v_1, v_2, \ldots, v_{|S_1|}$ be vertices of S_1. Clearly, we have $r(K_{a,b}) = a-1$ by Case 1 of Theorem 3.4, and there are $(a^2-a)/2$ edges that can be added between each pair of vertices of S_1. Let E_1 includes these edges, and let e_i be an edge of E_1. Then we get $r_{e_i}(K_{a,b}) = a-1$ for every edge of E_1. Similarly, let $v_1^*, v_2^*, \ldots, v_{|S_2|}^*$ be vertices of S_2. Clearly, there are $(b^2-b)/2$ edges that can be added between each pair of vertices of S_2. Let E_2 includes these edges, and let e_i^* be an edge of E_2. Clearly, we get $r_{e_i^*}(K_{a,b}) = r(K_{a,b}) + 1 = a$ for every edge of E_2.

Thus we get

$$\begin{aligned} r_{av}(K_{a,b}) &= \frac{1}{|E_1 \cup E_2|}\left(\sum_{e_i \in E_1} r_{e_i}(K_{a,b}) + \sum_{e_i^* \in E_2} r_{e_i^*}(K_{a,b})\right) \\ &= \frac{2}{a^2+b^2-a-b}\left(\left(\frac{a^2-a}{2}\right)(a-1) + \left(\frac{b^2-b}{2}\right)a\right) \\ &= \frac{a(a-1)^2 + ab(b-1)}{a^2+b^2-a-b}. \end{aligned}$$

By Cases 1 and 2 the proof is completed.

4. The vulnerability and the average lower reinforcement number

In this section, the notation of vulnerability of a graph is considered under the reinforcement set. Given two graphs, one can ask the following question: is the average lower reinforcement number a suitable parameter, regarding vulnerability? In other words, does the average lower reinforcement number distinguish between them? Let G_1 and G_2 be the graphs presented in Figure 2. It can be easily seen that the connectivity, domination number, average lower domination number and reinforcement number of these graphs are equal such as

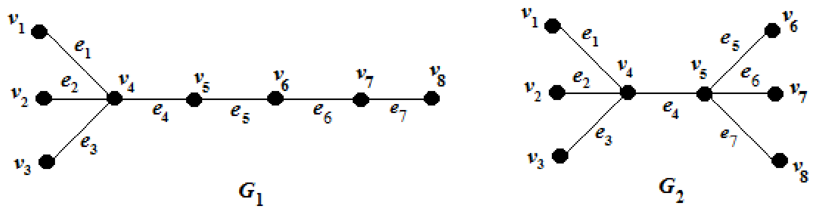

FIGURE 2. The graphs G_1 and G_2 with 8-vertices and 7-edges.

$k(G_1) = k(G_2) = 1$, $\gamma(G_1) = \gamma(G_2) = 2$, $\gamma_{av}(G_1) = \gamma_{av}(G_2) = 2.75$ and $r(G_1) = r(G_2) = 3$. Furthermore, the vertices and edges number of these graphs are equal such as $|V(G_1)| = |V(G_2)| = 8$ and $|E(G_1)| = |E(G_2)| = 7$.

The average lower reinforcement numbers of these two graphs G_1 and G_2 are $r_{av}(G_1) = \frac{81}{21}$, and $r_{av}(G_2) = \frac{78}{21}$. The results could be checked by readers. Thus, the average lower reinforcement number may be used for distinguish between these two graphs G_1 and G_2.

Another example, as one can see, when $k \geqslant 1$, $r(P_{3k+1}) = 1$, $r(P_{3k+2}) = 2$ and $r(P_{3k+3}) = 3$. But, $r_{av}(P_n) = \frac{16n^2-56n+40}{9n^2-27n+18}$ $(if\ n \equiv 1(\text{mod}3))$, $r_{av}(P_n) = \frac{22n^2-76n+64}{9n^2-27n+18}$ $(if\ n \equiv 2(\text{mod}3))$, and $r_{av}(P_n) = \frac{28n^2-90n+90}{9n^2-27n+18}$ $(if\ n \equiv 0(\text{mod}3))$. It is easy to see that the reinforcement number of P_n is always constant value. On the other hand, the average lower reinforcement number of P_n is not constant, that is always variable value. Similarly, this situation holds for the cycle C_n for $n \equiv 1(\text{mod}3)$ and $n \equiv 2(\text{mod}3)$.

These examples means that the average lower reinforcement number can be more efficient compared with the other vulnerability parameters.

5. Conclusion

In this paper, we have presented a new graph theoretical parameter, called the average lower reinforcement number. The present parameter has been constructed by summing the lower reinforcement number of every edge of a complement graph $\overline{G}$ divided by the number of edges of $\overline{G}$. Additionally, the stability of popular interconnection networks has been studied and their average lower reinforcement numbers have been computed. Then upper bounds, lower bounds and exact formulas of the average lower reinforcement number have been obtained for any given graph G.

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