

LANGUAGE CLASSES ASSOCIATED WITH AUTOMATA OVER MATRIX GROUPS^{☆,☆☆,☆☆☆}

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Abstract. We investigate the language classes recognized by group automata over matrix groups. For the case of 2×2 matrices, we prove that the corresponding group automata for rational matrix groups are more powerful than the corresponding group automata for integer matrix groups. Finite automata over some special matrix groups, such as the discrete Heisenberg group and the Baumslag-Solitar group are also examined. We also introduce the notion of time complexity for group automata and demonstrate some separations among related classes. The case of linear-time bounds is examined in detail throughout our repertory of matrix group automata.

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1. INTRODUCTION

Many extensions of the classical finite automaton model have been examined. One such variant is the group automaton (finite automaton over groups), which is a nondeterministic finite automaton equipped with a register that holds an element from a group [18]. The register is initialized to the identity element of the group, and a computation is deemed successful if the register is equal to the identity element at the end of the computation after being multiplied at every step. This setup generalizes various models such as nondeterministic blind multicounter automata [8] and finite automata with multiplication [13].

The theory of group automata has been essentially developed in the case of free groups [4, 5, 14], and in the case of free Abelian groups [6, 7], where strong theorems allow to characterize the power of such models and the combinatorial properties of the languages recognized by these automata. For groups that are not of the types mentioned above, even in the case of groups of matrices of low dimension, the study of group automata

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quickly becomes nontrivial, and there are remarkable classes of linear groups for which little is known about the automaton models that they define.

In this paper, we present several new results about the classes of languages recognized by finite automata over matrix groups. We focus on matrix groups with integer and rational entries. For the case of 2×2 matrices, we prove that the corresponding group automata for rational matrix groups are more powerful than the corresponding group automata for integer matrix groups. We also explore finite automata over some special matrix groups, such as the discrete Heisenberg group and the Baumslag-Solitar group. The “zoo” of language classes associated with different groups is presented, visualizing known relationships and open problems.

We also introduce the notion of time complexity for group automata, and use this additional dimension to analyze the relationships among the language families of various automata using different groups. We develop a method for proving that automata over matrix groups where the growth rate of the group and the time are bounded can not recognize certain languages, even if one uses a very weak definition of time-bounded computation, and use this to demonstrate some new relationships between time-bounded versions of our language classes. The case of linear-time bounds is examined in detail throughout our repertory of matrix groups.

2. PRELIMINARIES

2.1. Notation and terminology

The following notation will be used throughout the paper: Q is the set of states, $q_0 \in Q$ denotes the initial state, $Q_a \subseteq Q$ denotes the set of accepting states, and Σ is the input alphabet.

By w^r , we represent the reverse of the string w . The length of w is denoted by $|w|$.

REG, CF, and RE denote the families of regular languages, context-free languages, and recursively enumerable languages, respectively.

We assume a familiarity with some basic notions from algebra and group theory (see [9, 17] for references on this topic). For a finitely generated group G and a set X of generators, the word problem language of G is the language $W(G, X)$ over $\Sigma = \{X \cup X^{-1}\}$ which consists of all words that represent the identity element of G . Most of the time, the statements about the word problem are independent of the generating set and in these cases the word problem language is denoted by $W(G)$. For a string $w = w_1 w_2 \dots w_n \in W(G)$, $w^{-1} = w_n^{-1} \dots w_1^{-1}$ where each $w_i \in \Sigma$ represents a generator.

2.2. Group automata

Group automata first appear explicitly in the paper [18] under the name of extended finite automaton. The definition is formally given as follows.

Let $K = (M, \circ, e)$ be a group under the operation denoted by \circ with the neutral element denoted by e . An *extended finite automaton* over the group $K = (M, \circ, e)$ is a 6-tuple

$$\mathcal{F} = (Q, \Sigma, K, \delta, q_0, Q_a),$$

where the transition function δ is defined as

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow \mathbb{P}(Q \times M).$$

$\delta(q, \sigma) \ni (q', m)$ means that when \mathcal{F} reads the symbol (or empty string) $\sigma \in \Sigma \cup \{\varepsilon\}$ in state q , it can move to state q' , and write $x \circ m$ in the register, where x is the old content of the register. The initial value of the register is the neutral element e of the group K . The string is accepted if, after completely reading the string, \mathcal{F} enters an accept state with the content of the register being equal to the neutral element of K .

We will prefer using the name group automaton (G -automaton) instead of extended finite automaton over group G .

Monoid automata are defined analogously where the group G is replaced by some monoid N . The class of languages recognized by G -automata will be denoted as $\mathfrak{L}(G)$.

3. MATRIX GROUPS AND ASSOCIATED LANGUAGE CLASSES

In this section, we are going to prove some new results about the classes of languages recognized by finite automata over various groups, focusing on linear groups.

3.1. Basic results

We will denote the *free group* over r generators by \mathbf{F}_r . Note that \mathbf{F}_0 is the trivial group, and \mathbf{F}_1 is isomorphic to \mathbb{Z} , the additive group of integers. The class of regular languages is characterized as the set of languages recognized by finite automata over the trivial group \mathbf{F}_0 in [5].

The relation between the classes of languages recognized by free group automata is summarized as follows.

Fact 3.1. [5] $\text{REG} = \mathfrak{L}(\mathbf{F}_0) \subsetneq \mathfrak{L}(\mathbf{F}_1) = \mathfrak{L}(\mathbb{Z}) \subsetneq \mathfrak{L}(\mathbf{F}_2)$.

A characterization of context-free languages by group automata was first stated by Dassow and Mitrana [5], and proven in [4]. Let us note that \mathbf{F}_2 contains any free group of rank $n \geq 2$ [17].

Fact 3.2. [4, 5, 14] $\mathfrak{L}(\mathbf{F}_2)$ is the family of context-free languages.

We will denote by \mathbb{Z}^k the additive group of integer vectors of dimension k . This group is isomorphic to the *free Abelian group of rank k* , and \mathbb{Z}^k -automata are equivalent to nondeterministic blind k -counter automata [11].

The following result states the hierarchy between the classes of languages recognized by \mathbb{Z}^k -automata. This result also follows from the hierarchy between the class of languages recognized by nondeterministic blind k -counter automata.

Fact 3.3. [3] $\mathfrak{L}(\mathbb{Z}^k) \subsetneq \mathfrak{L}(\mathbb{Z}^{k+1})$ for $k \geq 1$.

We denote by \mathbb{Q}^+ the multiplicative group of positive rational numbers, which is isomorphic to a free Abelian group of infinite rank. A \mathbb{Q}^+ -automaton is also equivalent to a nondeterministic finite automaton with multiplication without equality (1NFAMW) of Ibarra *et al.* [13].

The following fact characterizes the class of languages recognized for the case where the alphabet is unary.

Fact 3.4. [13] All 1NFAMW-recognizable languages over a unary alphabet are regular.

Let us mention that the class of context-free languages and the class of languages recognized by nondeterministic blind counter automata are incomparable.

Fact 3.5. CF and $\mathfrak{L}(\mathbb{Z}^k)$ are incomparable for all $k \geq 2$.

Proof. Consider the language $L = \{a^n b^n \mid n \geq 0\}$ which is a context-free language. Since context-free languages are closed under star, L^* is a context-free language whereas it cannot be recognized by any \mathbb{Z}^k -automaton for all $k \geq 1$ by [11]. On the other hand, the non-context-free language $L' = \{a^n b^n c^n \mid n \geq 0\}$ can be recognized by a \mathbb{Z}^2 -automaton. \square

3.2. Automata on groups of 2×2 and 3×3 matrices

We denote by $GL(2, \mathbb{Z})$ the general linear group of degree two over the field of integers, that is, the group of 2×2 invertible matrices with integer entries. Note that these matrices have determinant ± 1 . Restricting the matrices in $GL(2, \mathbb{Z})$ to those that have determinant 1, we obtain the special linear group of degree two over the field of integers, $SL(2, \mathbb{Z})$.

Let \mathbf{G} be the group generated by the matrices

$$M_a = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M_b = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

There exists an isomorphism φ from \mathbf{F}_2 onto \mathbf{G} by [15]. Note that M_a and M_b are integer matrices with determinant 1, which proves that \mathbf{F}_2 is a subgroup of $SL(2, \mathbb{Z})$.

Now the question is whether $\mathfrak{L}(GL(2, \mathbb{Z}))$ and $\mathfrak{L}(SL(2, \mathbb{Z}))$ correspond to larger classes of languages than the class of context-free languages. We are going to use the following fact to prove that the answer is negative.

Fact 3.6. [4] Suppose G is a finitely generated group and H is a subgroup of finite index. Then $\mathfrak{L}(G) = \mathfrak{L}(H)$.

Now we are ready to state our theorem.

Theorem 3.7. $CF = \mathfrak{L}(\mathbf{F}_2) = \mathfrak{L}(SL(2, \mathbb{Z})) = \mathfrak{L}(GL(2, \mathbb{Z}))$.

Proof. We are going to use Fact 3.6 to prove the result. Since $SL(2, \mathbb{Z})$ has index 2 in $GL(2, \mathbb{Z})$ and $GL(2, \mathbb{Z})$ is finitely generated, $\mathfrak{L}(GL(2, \mathbb{Z})) = \mathfrak{L}(SL(2, \mathbb{Z}))$. Since \mathbf{F}_2 has index 12 in $SL(2, \mathbb{Z})$ [2] and $SL(2, \mathbb{Z})$ is finitely generated, $\mathfrak{L}(SL(2, \mathbb{Z})) = \mathfrak{L}(\mathbf{F}_2)$ which is equal to the family of context-free languages by Fact 3.2. \square

Let us now investigate the group $SL(3, \mathbb{Z})$, the group of 3×3 integer matrices with determinant 1.

We start by looking at an important subgroup of $SL(3, \mathbb{Z})$, the discrete Heisenberg group. The discrete Heisenberg group \mathbf{H} is defined as $\langle a, b \mid ab = bac, ac = ca, bc = cb \rangle$, where $c = a^{-1}b^{-1}ab$ is the commutator of a and b .

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Any element $g \in \mathbf{H}$ can be written uniquely as $b^j a^i c^k$.

$$g = \begin{bmatrix} 1 & i & k \\ 0 & 1 & j \\ 0 & 0 & 1 \end{bmatrix} = b^j a^i c^k$$

It is shown in [19] that the languages $MULT = \{x^p y^q z^{pq} \mid p, q \geq 0\}$, $COMPOSITE = \{x^{pq} \mid p, q > 1\}$ and $MULTIPLE = \{x^p y^{pn} \mid p \in \mathbb{N}\}$ can be recognized by \mathbf{H} -automata, using the special multiplication property of the group.

Correcting a small error in [19], we rewrite the multiplication property of the elements of \mathbf{H} .

$$(b^x a^y c^z)(b^{x'} a^{y'} c^{z'}) = b^{x+x'} a^{y+y'} c^{z+z'+yx'}$$

We can make the following observation using the fact that $\mathfrak{L}(\mathbf{H})$ contains non-context-free languages.

Theorem 3.8. $\mathfrak{L}(SL(2, \mathbb{Z})) \subsetneq \mathfrak{L}(SL(3, \mathbb{Z}))$.

Proof. It is obvious that an $SL(2, \mathbb{Z})$ -automaton can be simulated by an $SL(3, \mathbb{Z})$ -automaton. Note that $\mathfrak{L}(SL(2, \mathbb{Z}))$ is the family of context-free languages by Theorem 3.7. Since $\mathfrak{L}(\mathbf{H}) \subseteq \mathfrak{L}(SL(3, \mathbb{Z}))$ and the non-context-free language $MULT = \{x^p y^q z^{pq} \mid p, q \geq 0\}$ can be recognized by an \mathbf{H} -automaton [19], the result follows. \square

The following result is a direct consequence of Fact 3.6.

Theorem 3.9. $\mathfrak{L}(SL(3, \mathbb{Z})) = \mathfrak{L}(GL(3, \mathbb{Z}))$.

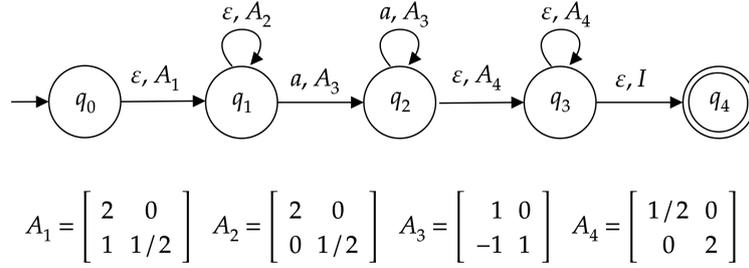


FIGURE 1. State diagram of \mathcal{G} accepting the language $L = \{a^{2^{2n+1}} \mid n \geq 0\}$.

Proof. Since $GL(3, \mathbb{Z})$ is a finitely generated group and $SL(3, \mathbb{Z})$ has finite index in $GL(3, \mathbb{Z})$, the result follows by Fact 3.6. \square

We have talked about the discrete Heisenberg group \mathbf{H} . Now let us look at a subgroup of \mathbf{H} generated by the matrices B and C , which we will call \mathbf{G}_2 .

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{G}_2 = \langle B, C \mid BC = CB \rangle$ is a free Abelian group of rank 2, and therefore it is isomorphic to \mathbb{Z}^2 .

We conclude the following about the language recognition power of \mathbb{Z}^2 and \mathbf{H} .

Theorem 3.10. $\mathfrak{L}(\mathbb{Z}^2) \subsetneq \mathfrak{L}(\mathbf{H})$.

Proof. Since \mathbb{Z}^2 is a subgroup of \mathbf{H} , $\mathfrak{L}(\mathbb{Z}^2) \subseteq \mathfrak{L}(\mathbf{H})$ follows. The inclusion is proper since \mathbf{H} can recognize the language $\text{MULT} = \{x^p y^q z^{pq} \mid p, q \geq 0\}$ [19], whereas any bounded language in $\mathfrak{L}(\mathbb{Q}^+)$ is semilinear [13]. \square

Now let us move on to the discussion about matrix groups with rational entries.

Let us denote by $GL(2, \mathbb{Q})$ the general linear group of degree two over the field of rational numbers, that is, the group of invertible matrices with rational entries. Restricting the matrices in $GL(2, \mathbb{Q})$ to those that have determinant 1, we obtain the special linear group of degree two over the field of rationals, $SL(2, \mathbb{Q})$.

We will start by proving that allowing rational entries enlarges the class of languages recognized by matrices with determinant 1.

Theorem 3.11. $\mathfrak{L}(SL(2, \mathbb{Z})) \subsetneq \mathfrak{L}(SL(2, \mathbb{Q}))$.

Proof. It is obvious that $\mathfrak{L}(SL(2, \mathbb{Z})) \subseteq \mathfrak{L}(SL(2, \mathbb{Q}))$. We will prove that the inclusion is proper.

Let us construct an $SL(2, \mathbb{Q})$ -automaton \mathcal{G} recognizing the language $L = \{a^{2^{2n+1}} \mid n \geq 0\}$. The state diagram of \mathcal{G} and the matrices are given in Figure 1. Without scanning any input symbol, \mathcal{G} first multiplies its register with the matrix A_1 . \mathcal{G} then multiplies its register with the matrix A_2 successively until nondeterministically moving to the next state. After that point, \mathcal{G} starts reading the string and multiplies its register with the matrix A_3 for each scanned a . At some point, \mathcal{G} nondeterministically stops reading the rest of the string and multiplies its register with the matrix A_4 . After successive multiplications with A_4 , \mathcal{G} nondeterministically decides moving to an accept state.

Let us trace the value of the register at different stages of the computation. Before reading the first input symbol, the register has the value

$$\begin{bmatrix} 2^{x+1} & 0 \\ 2^x & \frac{1}{2^{x+1}} \end{bmatrix}$$

as a result of the multiplications with the matrix A_1 and x times the matrix A_2 . Multiplication with each A_3 leaves 2^{x+1} and $\frac{1}{2^{x+1}}$ unchanged while subtracting $\frac{1}{2^{x+1}}$ from 2^x for each scanned a . As a result of y multiplications with A_3 , the register will have the value

$$\begin{bmatrix} 2^{x+1} & 0 \\ 2^x - \frac{y}{2^{x+1}} & \frac{1}{2^{x+1}} \end{bmatrix}.$$

For the rest of the computation, \mathcal{G} will multiply its register with A_4 until nondeterministically moving to the final state. As a result of z multiplications with A_4 , the register will have the value

$$\begin{bmatrix} \frac{2^{x+1}}{2^z} & 0 \\ (2^x - \frac{y}{2^{x+1}}) \frac{1}{2^z} & \frac{2^z}{2^{x+1}} \end{bmatrix}.$$

The final value of the register is equal to the identity matrix when $y = 2^{2x+1}$ and $z = x + 1$, which is possible only when the length of the input string is 2^{2x+1} for some $x \geq 0$. In the successful branch, the register will be equal to the identity matrix and \mathcal{G} will end up in the final state having successfully read the input string. For input strings which are not members of L , either the computation will end before reading the whole input string, or the final state will be reached with the register value not equaling the identity matrix.

Since the matrices used during the computation are 2 by 2 rational matrices with determinant 1, $L \in \mathfrak{L}(SL(2, \mathbb{Q}))$. $\mathfrak{L}(SL(2, \mathbb{Q}))$ contains a unary nonregular language, which is not true for $\mathfrak{L}(SL(2, \mathbb{Z}))$ by Theorem 3.7 and we conclude the result. \square

Let us note that the set of languages recognized by \mathbb{Q}^+ -automata is a proper subset of the set of languages recognized by $SL(2, \mathbb{Q})$ -automata.

Theorem 3.12. $\mathfrak{L}(\mathbb{Q}^+) \subsetneq \mathfrak{L}(SL(2, \mathbb{Q}))$.

Proof. Let $L \in \mathfrak{L}(\mathbb{Q}^+)$ and let \mathcal{G} be a \mathbb{Q}^+ -automaton recognizing L . We will construct an $SL(2, \mathbb{Q})$ -automaton \mathcal{G}' recognizing L . Let $S = \{s_1, \dots, s_n\}$ be the set of elements multiplied with the register during the computation of \mathcal{G} . We define the mapping φ as follows.

$$\varphi : s_i \mapsto \begin{bmatrix} s_i & 0 \\ 0 & \frac{1}{s_i} \end{bmatrix}$$

The elements $\varphi(s_i)$ are 2×2 rational matrices with determinant 1. Let δ and δ' be the transition functions of \mathcal{G} and \mathcal{G}' respectively. We let $(q', s_i) \in \delta(q, \sigma) \iff (q', \varphi(s_i)) \in \delta'(q, \sigma)$ for every $q, q' \in Q$, $\sigma \in \Sigma$ and $s_i \in S$. The resulting \mathcal{G}' recognizes L .

The inclusion is proper since $L = \{a^{2^{2n+1}} \mid n \geq 0\} \in \mathfrak{L}(SL(2, \mathbb{Q}))$ by Theorem 3.11, and $\mathfrak{L}(\mathbb{Q}^+)$ does not contain any unary nonregular languages by Fact 3.4, noting that \mathbb{Q}^+ -automata are equivalent to 1NFAMW's. \square

We will now look at a special subgroup of $GL(2, \mathbb{Q})$.

For two integers m and n , the *Baumslag-Solitar group* $BS(m, n)$ is defined as $BS(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle$. We are going to focus on $BS(1, 2) = \langle a, b \mid bab^{-1} = a^2 \rangle$.

Consider the matrix group G_{BS} generated by the matrices

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}.$$

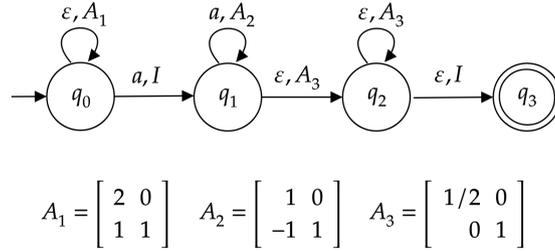


FIGURE 2. State diagram of \mathcal{G} recognizing $\text{UPOW} = \{a^{2^n} \mid n \geq 0\}$.

Consider the isomorphism $a \mapsto A, b \mapsto B$. The matrices A and B satisfy the property $BAB^{-1} = A^2$,

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix},$$

and we conclude that G_{BS} is isomorphic to $BS(1, 2)$.

We will prove that there exists a $BS(1, 2)$ -automaton which recognizes a non-context-free language.

Theorem 3.13. $\mathcal{L}(BS(1, 2)) \not\subseteq \text{CF}$.

Proof. Let us construct a $BS(1, 2)$ -automaton \mathcal{G} recognizing the language $\text{UPOW} = \{a^{2^n} \mid n \geq 0\}$. The state diagram of \mathcal{G} and the matrices are given in Figure 2. Without scanning any input symbol, \mathcal{G} multiplies its register with the matrix A_1 successively. \mathcal{G} nondeterministically moves to the next state reading the first input symbol without modifying the register. After that point, \mathcal{G} starts reading the string and multiplies its register with the matrix A_2 for each scanned a . At some point, \mathcal{G} nondeterministically stops reading the rest of the string and multiplies its register with the element A_3 . After successive multiplications with A_3 , \mathcal{G} nondeterministically decides to move to an accept state.

As a result of i multiplications with A_1 , the register has the value

$$\begin{bmatrix} 2^i & 0 \\ 2^i - 1 & 1 \end{bmatrix}$$

before reading the first input symbol. Multiplication with each A_2 leaves 2^i unchanged while subtracting 1 from $2^i - 1$ for each scanned a . The register will have the value

$$\begin{bmatrix} 2^i & 0 \\ 2^i - 1 - j & 1 \end{bmatrix}$$

as a result of j multiplications with the matrix A_2 .

For the rest of the computation, \mathcal{G} will multiply its register with A_3 resulting in the register value

$$\begin{bmatrix} \frac{2^i}{2^k} & 0 \\ 2^i - 1 - j & 1 \end{bmatrix}$$

since each multiplication with A_3 divides 2^i by 2.

The register contains the identity matrix at the end of the computation if $i = k$ and $j = 2^i - 1$ which is possible if the input string is of the form $a^{1+2^i-1} = a^{2^i}$. In the successful branch, the register will be equal to the identity matrix and \mathcal{G} will end up in the final state having successfully read the input string.

For input strings which are not members of UPOW , either the computation will end before reading the whole input string or the final state will be reached with the register value being different from the identity matrix.

Note that $A_1 = B^{-1}A^{-1}$, $A_2 = A$ and $A_3 = B$, where A and B are the generators of the group G_{BS} and recall that G_{BS} is isomorphic to $BS(1, 2)$. Since UPOW is a unary nonregular language, it is not context-free and we conclude the result. \square

Note that $\mathcal{L}(\mathbb{Z}) \subsetneq \mathcal{L}(BS(1, 2))$ since the subgroup generated by a in $BS(1, 2)$ is isomorphic to \mathbb{Z} and $\mathcal{L}(BS(1, 2))$ contains a unary nonregular language.

3.3. Automata on matrices of higher dimensions

In [18], it is proven that $\mathbf{F}_2 \times \mathbf{F}_2$ -automata are as powerful as Turing machines.

Fact 3.14. [18] $\mathcal{L}(\mathbf{F}_2 \times \mathbf{F}_2)$ is the family of recursively enumerable languages.

We make the following observation.

Theorem 3.15. $RE = \mathcal{L}(\mathbf{F}_2 \times \mathbf{F}_2) = \mathcal{L}(SL(4, \mathbb{Z}))$.

Proof. The first equality is Fact 3.14. Recall from Section 3.2 that φ is an isomorphism from \mathbf{F}_2 onto \mathbf{G} , the matrix group generated by the matrices M_a and M_b . Let \mathbf{G}' be the following group of matrices

$$\left\{ \begin{bmatrix} M_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_2 \end{bmatrix}, M_1, M_2 \in \mathbf{G} \right\}.$$

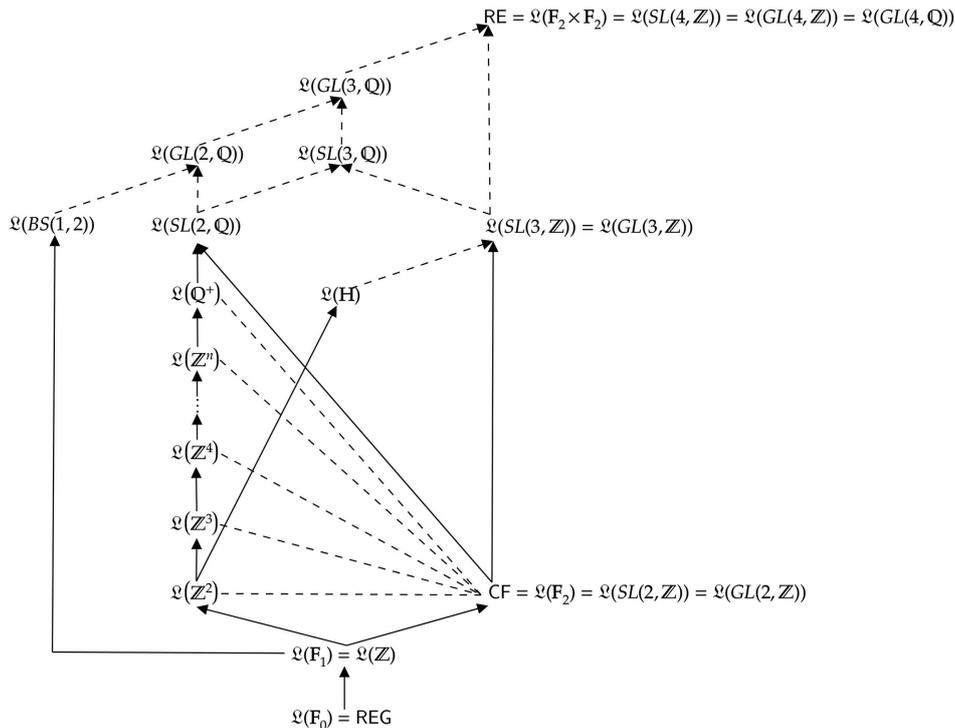


FIGURE 3. Language classes associated with groups.

We will define the mapping $\psi : \mathbf{F}_2 \times \mathbf{F}_2 \rightarrow \mathbf{G}'$ as $\psi(g_1, g_2) = (\varphi(g_1), \varphi(g_2))$ for all $(g_1, g_2) \in \mathbf{F}_2 \times \mathbf{F}_2$ which is an isomorphism from $\mathbf{F}_2 \times \mathbf{F}_2$ onto \mathbf{G}' .

This proves that $\mathbf{F}_2 \times \mathbf{F}_2$ is isomorphic to a subgroup of $SL(4, \mathbb{Z})$. The fact that $\mathfrak{L}(\mathbf{F}_2 \times \mathbf{F}_2)$ is the set of recursively enumerable languages helps us to conclude that $\mathfrak{L}(SL(4, \mathbb{Z}))$. \square

Let us also state that the classes of languages recognized by automata over supergroups of $SL(4, \mathbb{Z})$ such as $GL(4, \mathbb{Z})$ or $SL(4, \mathbb{Q})$ are also identical to the class of recursively enumerable languages.

Theorem 3.16. $\mathfrak{L}(G) = \text{RE}$, where G is any matrix group whose matrix entries are computable numbers and $SL(4, \mathbb{Z})$ is a subgroup of G .

Proof. Note that any finite automaton over a matrix group can be simulated by a nondeterministic Turing machine which keeps track of the register simply by multiplying the matrices and checking whether the identity matrix is reached at the end of the computation, provided that the matrix entries are computable numbers. Since $\text{RE} = \mathfrak{L}(SL(4, \mathbb{Z}))$ and G contains $SL(4, \mathbb{Z})$ as a subgroup, $\mathfrak{L}(G)$ is the set of recursively enumerable languages. \square

We summarize the results in Figure 3. Solid arrows represent proper inclusion, dashed arrows represent inclusion and dashed lines represent incomparability.

4. TIME COMPLEXITY

In the previous section, we compared various automaton models solely on the basis of the groups they employed as a computational resource. The theory of computational complexity deals with various different types of such resources, the allowed runtime of the machines being the most prominent among them. Some of the automata we saw in Section 3 (e.g. Fig. 2) have arbitrarily long computations, and it is a legitimate question to ask whether our results, for instance, the relationships in Figure 3, would still hold if one imposed common time bounds on the automata. We study such questions in this section.

4.1. Definitions

A group automaton \mathcal{G} recognizing language L is said to be *strongly $t(n)$ time-bounded* if for any input string x with $|x| = n$, every computation of \mathcal{G} on x takes at most $t(n)$ steps. We will denote the set of languages recognized by strongly $t(n)$ -time bounded G -automata by $\mathfrak{L}(G)_{t(n)}^s$.

Although the strong mode of recognition defined above is standard in studies of time complexity, we will be able to prove the impossibility results of the next subsection even when the machines are subjected to the following, looser requirement:

A group automaton \mathcal{G} recognizing language L is said to be *weakly $t(n)$ time-bounded* if for each accepted input string $x \in L$ with $|x| = n$, \mathcal{G} has a successful computation which takes at most $t(n)$ steps. So any input string is allowed to cause longer computations, as long as none of those are accepting for inputs which are not members of L . We will denote the set of languages recognized by weakly $t(n)$ -time bounded G -automata by $\mathfrak{L}(G)_{t(n)}^w$.

A machine is *real-time* if every transition consumes an input symbol.

Note that the statement $\mathfrak{L}(G)_{t(n)}^s \subseteq \mathfrak{L}(G)_{t(n)}^w$ is true by definition.

Let X be a generator set for the group G . The *length* of $g \in G$, denoted $|g|_X$, is the length of the shortest representative for g in $(X \cup X^{-1})^*$. Let $B_G^X(n) = \{g \in G, |g|_X \leq n\}$ be the set of all elements in G which can be represented by a word of length at most n . The *growth function of a group G* with respect to a generating set X , denoted $g_G^X(n)$, is the cardinality of the set $B_G^X(n)$, that is $g_G^X(n) = |B_G^X(n)|$. The growth function is asymptotically independent of the generating set, and we will denote the growth function of a group G by $g_G(n)$.

For a positive integer n , two strings $w, w' \in \Sigma^*$ are *n -dissimilar for L* if $|w| \leq n$, $|w'| \leq n$, and there exists a string $v \in \Sigma^*$ with $|wv| \leq n$, $|w'v| \leq n$ such that $wv \in L$ iff $w'v \notin L$. Let $A_L(n)$ be the maximum k such that there exist k distinct strings that are pairwise n -dissimilar.

A finite set of strings S is said to be a set of *uniformly n -dissimilar strings* for L if for each string $w \in S$, there exists a string v such that $|wv| \leq n$ and $wv \in L$ and for any string $w' \in S$ such that $w \neq w'$, $|w'v| \leq n$ and $w'v \notin L$. Let $U_L(n)$ be the maximum k such that there exist k distinct strings that are uniformly n -dissimilar.

Note that the following is always true by definition, since the strings in a uniformly n -dissimilar set are pairwise n -dissimilar.

Lemma 4.1. $U_L(n) \leq A_L(n)$ for all $n \geq 0$.

4.2. Limitations of machines on slow groups running in short time

Theorem 4.2. Let G be a group with growth function $g_G(n)$. $L \notin \mathfrak{L}(G)_{t(n)}^w$ if $g_G(t(n)) \in o(U_L(n))$.

Proof. Suppose for a contradiction that there exists a weakly $t(n)$ time-bounded G -automaton \mathcal{G} recognizing L in time $t(n)$. For a sufficiently large n , let S be the set of uniformly n -dissimilar strings such that $|S| = U_L(n)$. For every string $w_i \in S$, there exists a string v_i such that $w_i v_i \in L$ and $w_j v_i \notin L$ for all $w_j \in S$ with $i \neq j$.

Let S_{acc} be the set of accepted extended strings of the form $w_i v_i \in L$ with $|w_i v_i| \leq n$ where $w_i \in S$ and $w_j v_i \notin L$ for all $w_j \in S$ with $i \neq j$ and $|w_j v_i| \leq n$. Let C be the set of $t(n)$ time bounded accepting computation paths for the strings in S_{acc} . The computation $c_{w_i v_i} \in C$ on the string $w_i v_i$ can be written as

$$c_{w_i v_i} = c_{w_i v_i}^{w_i} c_{w_i v_i}^{v_i}$$

where $c_{w_i v_i}^{w_i}$ represents the computation up to the end of the prefix w_i and $c_{w_i v_i}^{v_i}$ represents the rest of the computation on the string v_i .

A configuration of a group automaton is a pair consisting of a state and a group element. Let us count the number of configurations that can be reached at the end of the computation $c_{w_i v_i}^{w_i}$. Since the number of states is constant, the number of configurations that can be reached is dependent on the number of different group elements that can appear in the register. After reading a prefix w_i with $|w_i| = m \leq n$, the product of the labels on the edges can be given by $l = g_{i_1} g_{i_2} \dots g_{i_k}$ for some $k \leq t(m)$, since the computation in consideration is time bounded. l can be expressed as a product of κ generators, where κ is at most $C \cdot k$ for some constant C , since each group element labeling a transition in \mathcal{G} is composed of at most some constant number of generators, which is independent of the length of the string. The number of elements in G which can be represented as a product of at most κ generators is given by $g_G(\kappa)$ by the definition of the growth function of G . Hence, the number of different values that can appear in the register after reading a string of length exactly m is less than or equal to $g_G(\kappa)$. Since $\kappa \leq C \cdot k$ and $k \leq t(m)$ and $g_G(t(n)) \in o(U_L(n))$, we can conclude that

$$g_G(\kappa) \leq g_G(C \cdot t(m)) \in o(U_L(n)).$$

Now it is easy to see that the number of different configurations that can be reached at the end of a computation $c_{w_i v_i}^{w_i}$ is $o(U_L(n))$. Note that the cardinality of the set S , and thus that of S_{acc} , is equal to $U_L(n)$. Due to the pigeonhole principle, the same configuration must be reached at the end of two computations $c_{w_i v_i}^{w_i}$ and $c_{w_j v_j}^{w_j}$ for some $i \neq j$. This will result in the acceptance of the strings $w_i v_j$ and $w_j v_i$, which are not members of L . We arrive at a contradiction and conclude that L cannot be recognized by any weakly $t(n)$ time-bounded G -automaton. \square

In the next lemma, we set a lower bound on maximum cardinality of the set of uniformly n -dissimilar strings in the word problem language of some group G .

Lemma 4.3. Let G be a finitely generated group with growth function $g_G(n)$. Then $U_{W(G)}(n) \geq g_G(\lfloor \frac{n}{2} \rfloor)$.

Proof. Let X be the generator set of G . The number of distinct elements g in G which can be represented by a word of length less than or equal to $\lfloor \frac{n}{2} \rfloor$ is $g_G(\lfloor \frac{n}{2} \rfloor)$, which is the cardinality of the set $B_G^X(\lfloor \frac{n}{2} \rfloor) = \{g \in G, |g|_X \leq \lfloor \frac{n}{2} \rfloor\}$. Let S be the set containing the string representations of the elements in $B_G^X(\lfloor \frac{n}{2} \rfloor)$. Every $w_i \in S$

can be extended with w_i^{-1} so that the extended string represents the identity element of G and has length less than or equal to n . Since the strings in $W(G)$ are those which belong to $(X \cup X^{-1})^*$ and represent the identity element of G , the extended string $w_i w_i^{-1} \in W(G)$. For every string $w_j \in S$ such that $i \neq j$, $w_j w_i^{-1} \notin W(G)$ since it is not possible for $w_j w_i^{-1}$ to represent the identity element of G . We conclude that the set S is uniformly n -dissimilar. Since $|S| = |B_G^X(\lfloor \frac{n}{2} \rfloor)| = g_G(\lfloor \frac{n}{2} \rfloor)$, it follows that $U_{W(G)}(n) \geq g_G(\lfloor \frac{n}{2} \rfloor)$. \square

The following theorem is about the language recognition power of finite automata over polynomial-growth groups which are weakly polynomial time-bounded.

Theorem 4.4. *Let G and H be groups with polynomial and exponential growth functions $g_G(n)$ and $g_H(n)$, respectively. For any polynomial $t(n)$, $\mathfrak{L}(H) \not\subseteq \mathfrak{L}(G)_{t(n)}^w$.*

Proof. Since $U_{W(H)}(n) \geq g_H(\lfloor \frac{n}{2} \rfloor)$ by Lemma 4.3, and $g_H(n)$ is an exponential function, $U_{W(H)}(n)$ is also at least exponential. $g_G(t(n))$ is a polynomial function, since both $g_G(n)$ and $t(n)$ are polynomial. Hence, $W(H) \notin \mathfrak{L}(G)_{t(n)}^w$ by Theorem 4.2, and the result follows since $W(H)$ is trivially in $\mathfrak{L}(H)$. \square

Theorem 4.5. *Let G be a group with a polynomial growth function. For any polynomial $t(n)$, $\text{CF} \not\subseteq \mathfrak{L}(G)_{t(n)}^w$.*

Proof. It is known that the word problem of the free group of rank 2, $W(\mathbf{F}_2)$, has an exponential growth function [12]. Assuming that G is a group with polynomial growth function, $W(\mathbf{F}_2)$ cannot be recognized by any weakly $t(n)$ time-bounded G -automaton by Theorem 4.2. Since $W(\mathbf{F}_2)$ is a context-free language, the proof is complete. \square

4.3. Group automata under linear time bounds

In this section, we focus on linear-time computation.

Let X be a generator set. For each symbol $x \in X$, the functions P_x and Q_x are defined as follows.

$$\begin{aligned} P_x : X^* &\rightarrow X^* & w &\mapsto wx \\ Q_x : X^*x &\rightarrow X^* & wx &\mapsto w \end{aligned}$$

Let K_X be the submonoid of all partial functions on X^* generated by P_x and Q_x for all $x \in X$. K_X is called the *polycyclic monoid* on X . A K_X -automaton is equivalent to a pushdown automaton, where P_x and Q_x can be interpreted as pushing and popping symbols on the stack. The equivalence between the two models is due to the nature of the functions P_x and Q_x , and is described in detail in [14]. The resemblance between the free group and K_X is used to prove that $\mathfrak{L}(\mathbf{F}_2) = \text{CF}$ in [4, 14].

Our aim is to show that \mathbf{F}_2 -automata working in linear time can recognize all context-free languages. It is stated in [21] that K_X -automata which consume at least one input symbol at each step are as powerful as K_X -automata without any time bound. However, it is not straightforward to see whether the same is true for \mathbf{F}_2 -automata.

Theorem 4.6. $\mathfrak{L}(\mathbf{F}_2)_{O(n)}^w = \text{CF}$.

Proof. We are going to use the construction of Kambites [14] to prove that any context-free language can be recognized by a weakly linear-time bounded \mathbf{F}_2 -automaton.

Let L be a context-free language and let $\mathcal{M} = \{Q, \Sigma, K_X, \delta, q_0, Q_a\}$ be a polycyclic monoid automaton recognizing L . K_X is the polycyclic monoid on X where the cardinality of the set X is n for some $n \geq 2$. Let e be the identity element of K_x . The construction of Kambites provides an \mathbf{F}_{n+1} -automaton $\mathcal{G} = \{Q', \Sigma, \mathbf{F}_{n+1}, \delta', q'_0, Q'_a\}$ recognizing the language L . The generator set for \mathbf{F}_{n+1} is X' , where $X' = X \cup \#$.

Let us analyze the construction in more detail.

- $Q' = Q_- \cup Q_+$ where $Q_- = \{q_- | q \in Q\}$ and $Q_+ = \{q_+ | q \in Q\}$
- $q'_0 = q_+$ where $q = q_0$.

- $Q'_a = \{q_- | q \in Q_a\}$.
- $\delta'(p_+, \sigma) = (q_+, x\#)$ if $\delta(p, \sigma) = (q, x\#)$ where x is a positive generator for all $\sigma \in \Sigma$.
- $\delta'(p_-, \sigma) = (q_+, x'\#)$ if $\delta(p, \sigma) = (q, x'\#)$ where x' is a negative generator for all $\sigma \in \Sigma$.
- $\delta'(p_+, \sigma) = (q_+, e)$ if $\delta(p, \sigma) = (q, e)$ for all $\sigma \in \Sigma$.
- $\delta'(q_+, \epsilon) = (q_-, e)$ for each $q \in Q$.
- $\delta'(q_-, \epsilon) = (q_-, \#^{-1})$ for each $q \in Q$.

We will prove that \mathcal{G} actually runs in linear time. There are two transitions where the automaton is allowed to move without consuming any input symbols.

For each state $q \in Q$, there are two states q_+ and q_- in \mathcal{G} which are connected with an edge labeled (ϵ, e) . These transitions do not change the register value, and cannot contribute more than half of the runtime of the machine, since at least one input symbol has to be consumed between any two executions of such transitions.

ϵ -loops exist in the machine \mathcal{G} for each state q_- where the loop is labeled by $(\epsilon, \#^{-1})$. Although this looks worrisome at first for the purpose of bounding the runtime, the number of times these loops are traversed is actually bounded, as the following argument suggests. Suppose that the register is multiplied with l_1, l_2, \dots, l_m while reading some input string w of length n , resulting in the register value $l = l_1 l_2 \dots l_m (\#^{-1})^k$, where $k \in \mathbb{N}$, at the end of the computation. If w is accepted by the machine, l should satisfy the following, as well as being equal to the identity element:

$$l_i = \begin{cases} (\#^{-1})^p x_i \# \text{ for some } p \in \mathbb{N}, & \text{if } x_i \text{ is a negative generator} \\ x_i \#, & \text{if } x_i \text{ is a positive generator} \end{cases}$$

This is called a *permissible padding* in [14]. By looking at the transition function of \mathcal{G} , one can see that the register is multiplied by a $\#$ only when an input symbol is consumed. Hence, the number of $\#$'s that occur in l is less than or equal to the length of the string. The register is multiplied with $\#^{-1}$ without consuming any input symbol. In order for the $\#$'s and $\#^{-1}$'s to cancel each other, they should be equal in number. Therefore, it can be concluded that the ϵ -loops are traversed at most n times.

We can conclude that any context-free language can be recognized by a weakly linear-time bounded free group automaton. Since \mathbf{F}_2 contains every free group of countable rank, the proof is complete. □

We state the following theorem, which is the linear-time equivalent of Fact 3.6 [4].

Theorem 4.7. *Suppose G is a finitely generated group and H is a subgroup of finite index. Then $\mathfrak{L}(G)_{\mathcal{O}(n)}^w = \mathfrak{L}(H)_{\mathcal{O}(n)}^w$.*

Proof. We know that the statement is true in general when there is no time bound by [4]. The proof in [4] still works when all automata in the constructions are required to work in linear time. □

Now we can show that Theorem 3.7 also holds for linear-time bounded group automaton.

Theorem 4.8. $\text{CF} = \mathfrak{L}(\mathbf{F}_2)_{\mathcal{O}(n)}^w = \mathfrak{L}(SL(2, \mathbb{Z}))_{\mathcal{O}(n)}^w = \mathfrak{L}(GL(2, \mathbb{Z}))_{\mathcal{O}(n)}^w$.

Proof. The proof is identical with the proof of Theorem 3.7 by using Theorem 4.7. □

By using the results proven in Section 4.2, we can demonstrate the language recognition power of weakly linear-time bounded \mathbf{H} -automata.

Theorem 4.9. $\mathfrak{L}(\mathbf{H})_{\mathcal{O}(n)}^w \subsetneq \mathfrak{L}(SL(3, \mathbb{Z}))_{\mathcal{O}(n)}^w$.

Proof. $\mathfrak{L}(\mathbf{H})_{\mathcal{O}(n)}^w \subseteq \mathfrak{L}(SL(3, \mathbb{Z}))_{\mathcal{O}(n)}^w$ since \mathbf{H} is a subgroup of $SL(3, \mathbb{Z})$. Since the Heisenberg group has polynomial growth function [16], there exists a context-free language which can not be recognized by any \mathbf{H} -automaton in polynomial time by Theorem 4.5. Since $\text{CF} = \mathfrak{L}(SL(2, \mathbb{Z}))_{\mathcal{O}(n)}^w$ by Theorem 4.8, the result follows. □

Theorem 4.10.

- i. For $k \geq 5$, $\mathfrak{L}(\mathbf{H})_{O(n)}^w$ and $\mathfrak{L}(\mathbb{Z}^k)_{O(n)}^w$ are incomparable.
- ii. $\mathfrak{L}(\mathbf{H})_{O(n)}^w$ and CF are incomparable.

Proof.

- i. In [19], a weakly linear-time bounded \mathbf{H} -automaton which recognizes the language $\text{MULT} = \{x^p y^q z^{pq} \mid p, q \geq 0\}$ is constructed. The language MULT can not be recognized by any \mathbb{Z}^k -automaton, since any bounded language in $\mathfrak{L}(\mathbb{Q}^+)$ is semilinear by [13].

In [10], it is implicitly proven there exists a uniformly n -dissimilar set of size $\Theta(n^k)$ for the language $L_k = \{0^{a_1} 10^{a_2} 1 \dots 0^{a_k} 10^{a_1} 10^{a_2} 1 \dots 0^{a_k} 1\}$ for all integers k . For $k = 5$, there exists a uniformly n -dissimilar set of size $\Theta(n^5)$ for the language L_5 and $U_{L_5}(n) \geq n^5$. Since $g_{\mathbf{H}}(n)$ is a polynomial of order 4 [16] and $t(n) = O(n)$, $g_{\mathbf{H}}(t(n)) \in o(U_{L_5}(n))$. By Theorem 4.2, we conclude the result.

- ii. The language $\text{MULT} = \{x^p y^q z^{pq} \mid p, q \geq 0\}$ is not a context-free language. Since \mathbf{H} has a polynomial growth function [16], there exists a context-free language which can not be recognized by any \mathbf{H} -automaton in polynomial-time by Theorem 4.5. \square

Let us note that L_5 can be recognized by a \mathbb{Z}^5 -automaton in real time. The existence of the languages L_k can be used to prove the linear-time nondeterministic counter hierarchy, with the help of Theorem 4.2.

Theorem 4.11. $\mathfrak{L}(\mathbb{Z}^k)_{O(n)}^w \subsetneq \mathfrak{L}(\mathbb{Z}^{k+1})_{O(n)}^w$ for $k \geq 1$.

Proof. The language $L_{k+1} = \{0^{a_1} 10^{a_2} 1 \dots 0^{a_{k+1}} 10^{a_1} 10^{a_2} 1 \dots 0^{a_{k+1}} 1\}$ can be recognized by a \mathbb{Z}^{k+1} -automaton in real time. While scanning the first $k+1$ segments of 0's, the i th counter is increased for each scanned 0 as 0^{a_i} is read. In the remainder of the computation, the i th counter is decreased for each scanned 0 when 0^{a_i} is read.

There exists a uniformly n -dissimilar set of size $\Theta(n^{k+1})$ for the language L_{k+1} , so $U_{L_{k+1}}(n) \geq n^{k+1}$. Since $t(n) = O(n)$ and $g_{\mathbb{Z}^k}(n) = n^k$ [12], $g_{\mathbb{Z}^k}(t(n)) \in o(U_{L_5}(n))$. We conclude by Theorem 4.2. \square

A celebrated result of the field of computational complexity, the nondeterministic time hierarchy theorem, will enable us to demonstrate that the computational power $\mathbf{F}_2 \times \mathbf{F}_2$ -automata is dependent on the time allotted for their execution.

Fact 4.12. [20] If $g(n)$ is a time-constructible function, and $f(n+1) = o(g(n))$, then there exists a language which cannot be recognized by any nondeterministic Turing machine in time $f(n)$, but can be recognized by a nondeterministic Turing machine in time $g(n)$.

Assume that any recursively enumerable language can be recognized by some linear-time $\mathbf{F}_2 \times \mathbf{F}_2$ -automaton. One can easily build a nondeterministic Turing machine that simulates such a $\mathbf{F}_2 \times \mathbf{F}_2$ -automaton with only a polynomial slowdown. But this would mean that any recursively enumerable language can be recognized by some nondeterministic TM in polynomial time, contradicting Fact 4.12, which implies that there exist languages which can only be recognized by nondeterministic Turing machines which run in at least exponential time. We have proven the following theorem.

Theorem 4.13. $\mathfrak{L}(\mathbf{F}_2 \times \mathbf{F}_2)_{O(n)}^w \subsetneq \text{RE}$.

Using the ability of Turing machines to simulate any finite automaton over a computable matrix group, the statement of the above theorem can be extended as follows.

Theorem 4.14. $\mathfrak{L}(G)_{O(n)}^w \subsetneq \text{RE}$ for any matrix group G whose matrix entries are computable numbers.

Proof. In Theorem 3.16, we have mentioned that Turing machines can simulate any finite automaton over a computable matrix group. By the nondeterministic time hierarchy theorem, it can be shown that there exist some languages which can not be recognized by any finite automata over matrix groups in linear time. \square

Theorem 4.15. $\mathfrak{L}(\mathbf{F}_2)_{O(n)}^w \subsetneq \mathfrak{L}(\mathbf{F}_2 \times \mathbf{F}_2)_{O(n)}^w$.

Proof. It is obvious that an \mathbf{F}_2 -automaton can be simulated by an $\mathbf{F}_2 \times \mathbf{F}_2$ -automaton. $\mathfrak{L}(\mathbf{F}_2)_{O(n)}^w = \text{CF}$ by Theorem 4.8. The inclusion is proper since the non-context-free language $L = \{a^n b^n c^n \mid n \geq 0\}$ can be recognized by an $\mathbf{F}_2 \times \mathbf{F}_2$ -automaton in real time by using the two registers as two counters. \square

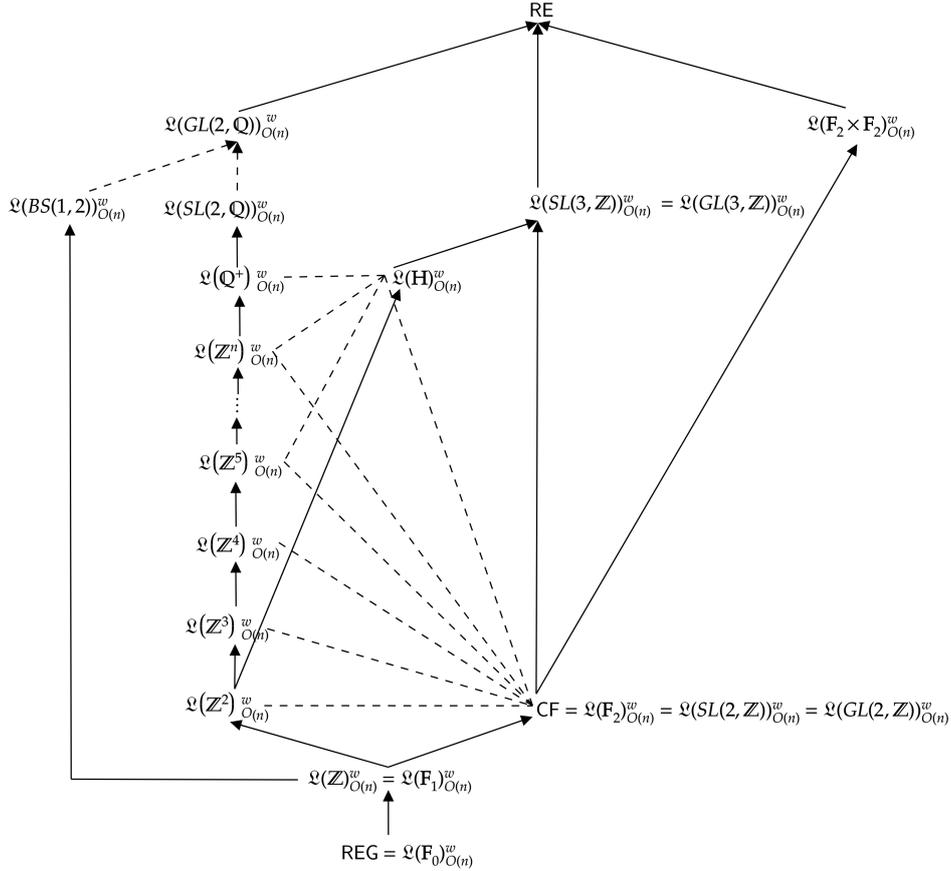


FIGURE 4. Language classes recognized by weakly linear-time bounded group automata.

In the rest of the section, the linear-time counterparts of the relationships in Figure 3 will be stated.

Theorem 4.16.

- i. $\mathfrak{L}(\mathbf{Q}^+)_{O(n)}^w \subsetneq \mathfrak{L}(\mathbf{SL}(2, \mathbf{Q}))_{O(n)}^w$.
- ii. $\mathfrak{L}(\mathbf{Z})_{O(n)}^w \subsetneq \mathfrak{L}(\mathbf{BS}(1, 2))_{O(n)}^w \not\subseteq \text{CF}$.
- iii. $\mathfrak{L}(\mathbf{SL}(2, \mathbf{Z}))_{O(n)}^w \subsetneq \mathfrak{L}(\mathbf{SL}(3, \mathbf{Z}))_{O(n)}^w$.
- iv. $\mathfrak{L}(\mathbf{Z}^2)_{O(n)}^w \subsetneq \mathfrak{L}(\mathbf{H})_{O(n)}^w$.
- v. CF and $\mathfrak{L}(\mathbf{Z}^k)_{O(n)}^w$ are incomparable for all $k \geq 2$.
- vi. $\mathfrak{L}(\mathbf{SL}(3, \mathbf{Z}))_{O(n)}^w = \mathfrak{L}(\mathbf{GL}(3, \mathbf{Z}))_{O(n)}^w$.
- vii. $\text{REG} = \mathfrak{L}(\mathbf{F}_0)_{O(n)}^w \subsetneq \mathfrak{L}(\mathbf{F}_1)_{O(n)}^w = \mathfrak{L}(\mathbf{Z})_{O(n)}^w \subsetneq \mathfrak{L}(\mathbf{F}_2)_{O(n)}^w$.

Proof.

- i., ii., iii., iv. Analogous results where no time bound was imposed on the machines were proven in Theorems 3.12, 3.13, 3.8, and 3.10, respectively. The group automata recognizing the witness languages

$L = \{a^{2^{2n+1}} | n \geq 0\}$, $UPOW = \{a^{2^n} | n \geq 0\}$ and $MULT = \{x^p y^q z^{pq} | p, q \geq 0\}$ operate in weakly linear time in all cases.

- v. The equivalent result for the general case is given in Fact 3.5. The non-context-free language $L' = \{a^n b^n c^n | n \geq 0\}$ can be recognized by a \mathbb{Z}^2 -automaton in real time.
- vi. The equivalent result for the general case is given in Theorem 3.9. The result follows by Theorem 4.7.
- vii. The equivalent result for the general case is given in Fact 3.1. \mathbf{F}_0 is the trivial group, and any regular language can be recognized by a deterministic finite automaton, which can be seen as finite automaton over \mathbf{F}_0 , in real time. Since \mathbf{F}_1 is isomorphic to \mathbb{Z} , the equality is obvious. Since the nonregular language $L = \{a^n b^n | n \geq 0\}$ can be recognized by a \mathbb{Z} -automaton in real time, the proper inclusion follows. Lastly, since $\mathfrak{L}(\mathbf{F}_2)_{O(n)}^w$ is equivalent to CF by Theorem 4.8, the last proper inclusion is still valid. \square

The results are summarized in Figure 4.

5. OPEN QUESTIONS

Does there exist an $SL(3, \mathbb{Z})$ -automaton recognizing $W(\mathbb{Z}^3)$?¹

Can we prove a stronger version of Theorem 4.5, which is independent of the time component? For instance, for the case of \mathbf{F}_2 , is it true that $W(\mathbf{F}_2) \notin \mathfrak{L}(\mathbf{H})$ in general?

Can we describe the necessary properties of a group G so that $\mathfrak{L}(G)$ contains $W(\mathbf{F}_2)$?

Little is known about $BS(1, 2)$ -automata. Does $\mathfrak{L}(BS(1, 2))$ contain every context-free language?

Which, if any, of the subset relationships in Figure 3 are proper inclusions?

Can we add other classes above RE in Figure 3 by examining groups on matrices with uncomputable entries?

Theorem 4.2 uses the definition of uniform n -dissimilarity requiring that $g_G(t(n)) \in o(U_L(n))$. Would the theorem be still true if we replace $U_L(n)$ by $A_L(n)$? The gap between $U_L(n)$ and $A_L(n)$ might be large as mentioned in [10]. Consider the language $L = \{a^i b^j | i \neq j\}$. It is stated in [10] that a set of uniformly n -dissimilar strings for L can not contain more than two strings. However, $A_L(n) \notin O(1)$, since L is not a regular language.

Can real-time \mathbf{F}_2 -automata recognize every context-free language?

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¹Corollary 2 of [3] states that the word problem of a finitely generated Abelian group H is recognized by a G -automaton if and only if H has a finite index subgroup isomorphic to a subgroup of G . That corollary could be used to give an affirmative answer to this open question. Unfortunately, the corollary is wrong: Let H be an Abelian group and let $G = \mathbf{F}_2 \times \mathbf{F}_2$. $\mathfrak{L}(\mathbf{F}_2 \times \mathbf{F}_2)$ contains the word problem of any finitely generated Abelian group. Since $\mathbf{F}_2 \times \mathbf{F}_2$ is finitely generated, any finite index subgroup of $\mathbf{F}_2 \times \mathbf{F}_2$ is also finitely generated. Any finite index subgroup of $\mathbf{F}_2 \times \mathbf{F}_2$ is either free or has a subgroup of finite index that is a direct product of free groups [1]. Any subgroup of an Abelian group is again Abelian. Hence, it is not possible that G has a finite index subgroup isomorphic to a subgroup of H .

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