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ON CHRISTOFFEL CLASSES

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Abstract. We characterize conjugation classes of Christoffel words (equivalently of standard words) by the number of factors. We give several geometric proofs of classical results on these words and sturmian words.

Mathematics Subject Classification. 68R15.

1. Introduction

Sturmian sequences have a long history, through the work of Bernoulli in the 18th century, of Smith, Christoffel and Markoff in the 19th century, Morse and Hedlund in the 20th century and the explosion of researches at the end of it. See the books by Allouche and Shallit [1] and Berstel and Séébold [3]. They are related to continued fractions, discrete geometry, symbolic dynamics, formal languages and combinatorics on words.

Christoffel words are a finitary version of Sturmian sequences, related to continued fractions of rational numbers. They are a variant of the so called standard words [3], which appear in Christoffel's article [6].

The present article, besides some new results, rests on two principles: first, most of the theory of Sturmian sequences may be done on its finitary counterpart, the theory of Christoffel words and their conjugates; secondly, most of the proofs use only elementary arguments of planar geometry, in the spirit of [4]. We shall illustrate here the second principle (for the first, it should be done elsewhere). This is done for some already known results in the Appendix. We also give some new results: in particular, the conjugation classes of Christoffel words (equivalently

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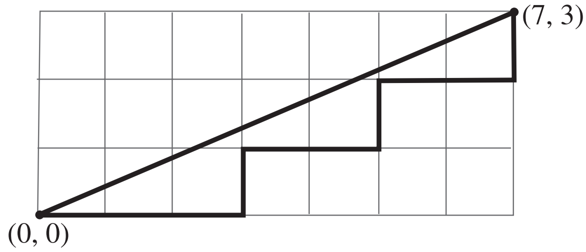

FIGURE 1. The lower Christoffel word *aaabaabaab* of slope $\frac{3}{7}$·

of standard words) are characterized by the number of factors, see Theorem 4.1, which is a finitary version of a well-known result of Morse-Hedlund on Sturmian sequences.

Furthermore, Theorem 5.1 gives the exact position of the $k + 1$ factors of length k. This result has as consequence that in a Sturmian sequence, the $k+1$ factors of length k appear in some window of length $2k$ (it could not be shorter).

This article also sheds some light onto the circular structure of Christoffel words, which justifies the title. For related work on conjugacy and Sturmian sequences, see the interesting word of Chuan [7–10].

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2. Christoffel words

A word w on a two-letter alphabet is called a lower Christoffel word if it is obtained by discretizing a segment in the plane, as in Figure 1.

Formally, the definition goes as follows: each word w on an ordered two-letter alphabet $\{a, b\}$ defines naturally a continuous (even piecewise linear) path in the plane, from the origin to some point $(p, q) \in \mathbb{N}^2$; letter a corresponds to a segment $[(i, j), (i+1, j)]$, letter b to a segment $[(i, j), (i, j+1)]$; thus p (resp. q) is the number of $a's$ (resp. $b's$) in w.

Now let $(p, q) \in \mathbb{N}^2$ with $gcd(p, q) = 1$. Consider a word w whose a–degree is p and whose b–degree is q. We say that w is a *lower Christoffel word* if the path of w is under the segment $[(0, 0), (p, q)]$, and if both delimit a polygon with no integral interior point. We say that w is the lower Christoffel word of *slope* $\frac{q}{p}$. Observe that w is of length $p + q$.

One defines similarly *upper Christoffel words*. A *Christoffel word* is by definition a lower or an upper Christoffel word. Note that a single letter is by definition also a Christoffel word, but we disregard in the sequel this trivial case.

Note that if w, w' are the lower and upper Christoffel words associated to (p, q), then $w' = \tilde{w}$ (the reversal of w); moreover, $w = amb, w' = bma$, where m is the word that encodes the sequence of vertical and horizontal intersections of the

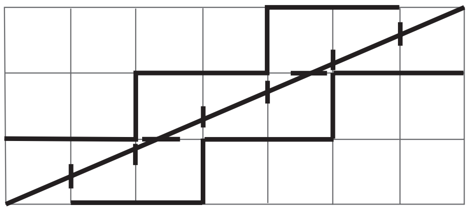

FIGURE 2. $m = aabaabaa$.

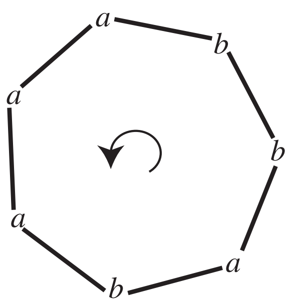

FIGURE 3. The circular word $(aaababb)$.

segment with the axes of the integer lattice; in particular, by symmetry, m is a *palindrome*. See Figure 2.

Let us call *cutting word* a word m that is obtained in this way. In other words, m is a cutting word (on the alphabet $\{a, b\}$) if and only if amb (or equivalently bma) is a Christoffel word. These words have been studied extensively; they are called *central* in [3], and the notation PER is used by A. de Luca for the set they form. See [2] for other discretization procedures for segments.

3. Pirillo's theorem

Recall that two words u, v are *conjugate* if for some words f, g, one has $u = fg, v = gf$. Conjugation is an equivalence relation. An equivalence class is called *a conjugation class*, or a *circular word*. The conjugation class of w is denoted (w). See Figure 3.

Theorem 3.1. *A word m on the two-letter alphabet $\{a, b\}$ is a cutting word if and only if amb and bma are conjugate.*

Remark. Pirillo's statement in [15, 16] is the following: a word m is a palindrome prefix of some standard Sturmian sequence if and only if mab and mba are conjugate.

This statement is equivalent to the theorem; indeed, it follows from the general theory of Sturmian words (see [3]) that m is a palindrome prefix of a standard Sturmian sequence if and only if amb and bma are Christoffel words. Moreover, it is easy to see that: mab and mba conjugate $\Leftrightarrow$ amb and bma conjugate.

The fact that the lower and upper Christoffel words of the same slope are conjugate (which is the direct part of Pirillo's theorem) was already known by H. Cohn [11] (Lem. 6.1); see also [12] proof of Proposition 10.

4. Characterization of Christoffel classes

A word v is a *factor* of a word w if for some words p, q, one has $w = pvq$. A word v is called *factor* of a circular word (w) if v is a factor of some conjugate of w; note that v may be factor of (w) without being factor of w, *e.g.* aa is factor of (aba), but not of aba.

Theorem 4.1. *Let w be a word of length $n \geq 2$. The following statements are equivalent.*

- *(i) w is conjugate to a Christoffel word.*
- *(ii) For $k = 0, \ldots, n-1$, (w) has $k+1$ factors of length k.*
- *(iii) (w) has $n-1$ factors of length $n-2$ and w is primitive.*

We call *Christoffel class* the conjugation class of a Christoffel word. Recall that a word is *primitive* if it is not a power of some other word; equivalently, the associate circular word is not fixed by any nontrivial rotation.

We shall use the following lemma, which is a finitary version of a well-known result for (infinite) sequences.

Lemma 4.1. *Let w be a word of length n. The following statements are equivalent:*

- *(i)* w is primitive;
- *(ii)* for $k = 0, \ldots, n-1, (w)$ has at least $k+1$ factors of length k.

We prove this lemma, since we could not find a reference for it, although the technique is classical.

Proof. If w is not primitive, then $w = u^p, p \geq 2$. Then $k = |u| < n$, and (w) has $\leq k$ factors of length k.

For the converse, denote by a_k the number of factors of length k of (w). Then $1 = a_0 \leq a_1 \leq \ldots \leq a_n$ since each factor of length $i-1$ has a right extension into a factor of length i, if $i \leq n$. Suppose that $a_k \leq k$ for some $k \in \{0, \ldots, n-1\}$. Then for some $l \leq k$, one has $a_{l-1} = a_l \leq l$. Consider the *factor graph* of order $l-1$ of (w), whose vertices are the factors of length $l-1$ of (w), with an edge $u \xrightarrow{a} v$, if ua is a factor of length l of (w), a a letter, and if $ua = bv$ for some letter b. By hypothesis, each factor of length $l-1$ has a unique right extension into a factor of length l. Hence each vertex has exactly one outgoing edge. Hence, each strongly connected component of the graph is a simple closed path. Since the sequence $w^\infty = ww \ldots w \ldots$ is the sequence of the labels of the edges of some infinite path

$w.w = a\,a\,b\,a\,a\,b\,a\,a\,b\,a\,b.\,a\,a\,b\,a\,a\,b\,a\,a\,b\,a\,b$

FIGURE 4. The 4 factors of length 3.

in the graph, we see that w^∞ has a period not greater than the number of vertices, that is, hence w^∞ has a period $\leq k$. Hence w is not primitive. □

Proof of the theorem.

(i) ⇒ (ii) This will be proved independently in the next section.
(ii) ⇒ (iii) Is clear, by the lemma.
(iii) ⇒ (i) By the lemma, (w) has at least n factors of length $n-1$; but it cannot have more, so that (w) has exactly n factors of length $n-1$.

Now, the circular word (w) has n occurrences of factors of length $n-2$; moreover, there are $n-1$ distinct such factors. We deduce that (w) has exactly one factor of length $n-2$ that appears twice (call it m), and the others appear only once.

Since (w) has n factors of length $n-1$, each factor of length $n-2$ has a unique right (resp. left) extension into a factor of length $n-1$ except one, which has two extensions and which we denote by r (resp. l). Necessarily, r (resp. l) must appear twice. Hence $l = m = r$.

Since m appears exactly twice and since length of w = length of $m + 2$, we have $(w) = (amb) = (cmd)$; by the property of double extension, we deduce that $b \neq d,\ a \neq c$; by counting letters, we see that $\{a, b\} = \{c, d\}$, hence $a = d,\ b = c$ and $a \neq b$. Thus $(amb) = (bma)$, amb and bma are conjugate and we conclude using Pirillo's theorem. □

5. Factors of a Christoffel class

We want to prove the following result.

Theorem 5.1. *Let w be a Christoffel word of length n and $k \in \{0, 1 \ldots, n-1\}$. Let p (resp. s) the prefix (resp. suffix) of length k of w. Then (w) has $k+1$ distinct factors of length k and they coincide with the $k+1$ factors of sp, which are all distinct.*

Note that sp is of length $2k$ and that a word of length $2k$ has $k+1$ factors of length k, when they are counted with multiplicities; here, they are all distinct.

As an example, take $w = aabaabaabab$ and $k = 3$. See Figure 4.

Proof of the theorem.

1. Let C denote the set of conjugates of w. Fix k as in the Theorem. Note that the set of factors of length k of (w), or equivalently of ww, is equal to the set of prefixes of length k of the conjugates of w.

   Define C_k to be the set of conjugates of w whose first letter is a letter in s (s is defined in the statement), together with w itself; in other words, the

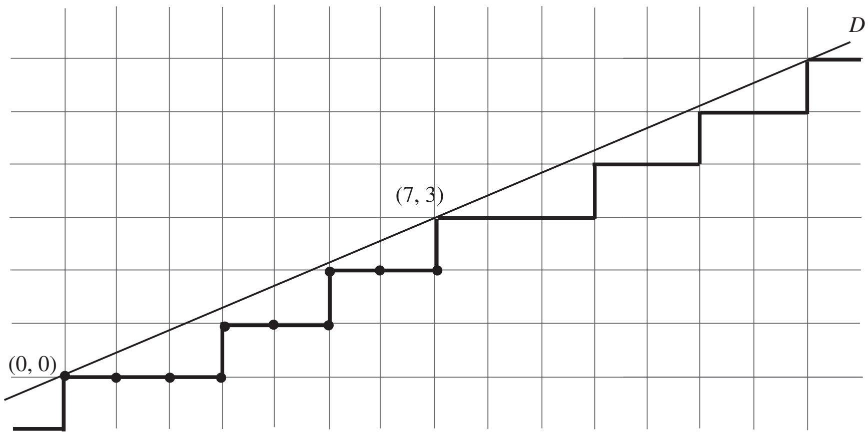

FIGURE 5. The 10 conjugates of $x^3yx^2yx^2y$.

elements of C_k are the prefixes of length n of s_2w, for some factorization $s = s_1s_2$. Clearly, $|C_k| = k + 1$, since w is primitive.

We shall show that the set of prefixes of length k of the words in C is equal to the set of prefixes of length k of the words in C_k, and that the latter are distinct. This will prove the theorem.

2. To this end, we define a mapping $\varphi : C\backslash w \to C\backslash w'$ (w' is the greatest conjugate in the lexicographic order of w), such that: if $w_2 = \varphi(w_1)$ and p_1, p_2 are the prefixes of length k of w_1, w_2, then $(*)$ either $p_1 = p_2$ or $p_1 > p_2$ in lexicographical order; moreover, $(**)$ the latter case occurs if and only if $w_1 \in C_k$.

   This will prove 1.
3. It will turn out from the definition of φ that: $w_2 = \varphi(w_1) \Rightarrow w_1 = myxn, w_2 = mxyn$; this will prove $(*)$. Moreover, $|m| < k \Leftrightarrow w_1 \in C_k$; this will prove $(**)$. Note that we use here the alphabet $\{x, y\}$, referring to the coordinates in the x, y-plane, see below.
4. We use the geometric realization of words in order to define φ.

   We identify a primitive word w with the bi-infinite sequence (we shall say "sequence") $\cdots w_. w \cdots w \cdots \in \{x, y\}^{\mathbb{Z}}$, where the dot indicates the position of the zero. The latter sequence will be identified with the corresponding bi-infinite path in the x, y-plane, together with some integer point on this path, which serves to identify the origin of the path and distinguish between conjugates of w. All the conjugates of w give the same bi-infinite path, but with different origins. See Figure 5, where the origins are the fat points.
5. Now, consider all the points $(np, nq) \in \mathbb{Z}$ (here $(p, q) = (7, 3)$).

   They are all on line D and the path goes vertically towards these points, and leaves them horizontally (see Fig. 5); this corresponds to the factor

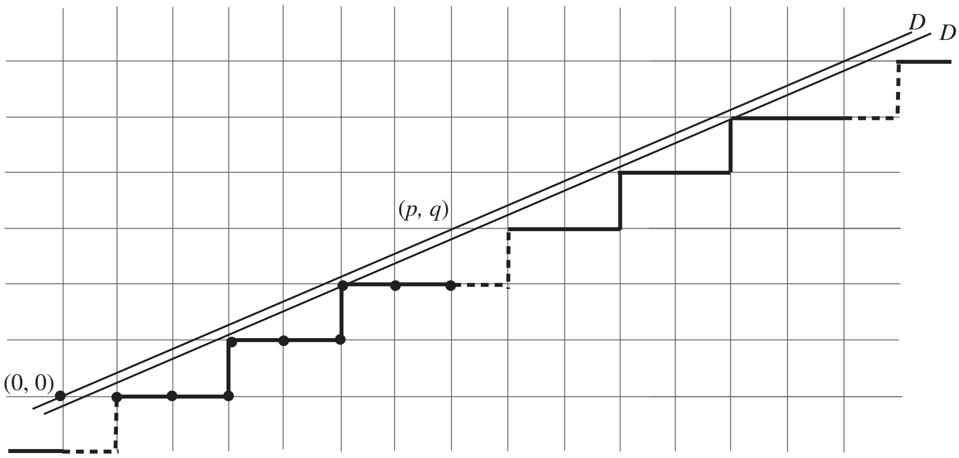

FIGURE 6. Geometric definition of the mapping φ.

yx in $\dots www \dots$ We replace these yx by xy and change correspondingly the path. See Figure 6.

The new path is identical to the previous one, after a translation, which amounts to replace line D by line D', which passes through the points of the ancient path closest to D, but not on D.

Changing the path, but keeping the same fat points, we obtain the mapping $\varphi : C\backslash w \to C\backslash w'$, since we identify conjugates with fat points. It is readily verified that φ satisfies 3. □

The proof shows also the following result.

Corollary 5.1. *Let w be a lower Christoffel word and $w = w_1 < w_2 < \ldots < w_n$ be its conjugates ordered lexicographically. Then $w_1 = xmy, w_n = ymx$ and for each $i = 1, \ldots, n-1$, one has for some words $u, v, w_i = uxyv, w_{i+1} = uyxv$.*

We may illustrate the corollary by writing a matrix with $w = w_1$ in the first row, w_2 in the second etc. Then each line differs from the next only by one factor xy which is replaced by yx. See Figure 7, where dots indicate the replacement.

Note that this matrix, for general words w, appears in the so called Burrows-Wheeler transformation. It allows to S. Mantaci, A. Restivo, M. Sciortino to prove that a word w is in a Christoffel class if and only if the last column of this matrix is formed by $y's$, followed by $x's$, see [14].

On this matrix, the mapping φ of the proof of Theroem 5.1 appears as scanning through the rows: φ of a conjugate of w is the conjugate in the row above it. Note that in the example of Figure 7, if we take $k = 4$, the set C_4 of the proof of Theorem 5.1 is

$$C_4 = \{yx^2yx^3yx^2, x^2yx^3yx^2y, xyx^3yx^2yx, yx^3yx^2yx^2, x^3yx^2yx^2y\};$$

$$
\begin{array}{cccccccccc}
x & x & x & y & x & x & y & x & x & y \\
x & x & y & x & x & x & y & x & x & y \\
x & x & y & x & x & y & x & x & x & y \\
x & x & y & x & x & y & x & x & y & x \\
x & y & x & x & x & y & x & x & y & x \\
x & y & x & x & y & x & x & x & y & x \\
x & y & x & x & y & x & x & y & x & x \\
y & x & x & x & y & x & x & y & x & x \\
y & x & x & y & x & x & x & y & x & x \\
y & x & x & y & x & x & y & x & x & x
\end{array}
$$

FIGURE 7. Matrix of conjugates.

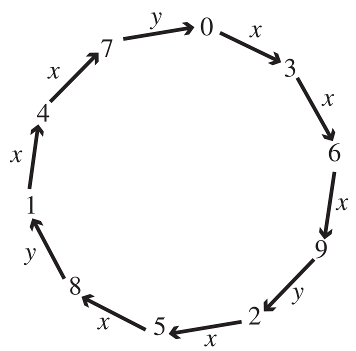

FIGURE 8. Cayley graph of a $\mathbb{Z}/10\mathbb{Z}$ with generator 3.

the factors of length 4 of ww are the prefixes of length 4 of the words in C_4, that is $yx^2y, x^2yx, xyx^2, yx^3, x^3y$.

A closer look at this matrix shows that it has a cyclic structure, and also many symmetries. This may be deduced from the following construction of Christoffel words, which appears already in Christoffel's article [6] (see also the equivalent formulation by finite interval rotations [14] p. 244). We give the construction on an example, for the Christoffel word $w = x^3yx^2yx^2y$.

The graph of Figure 8 is the Cayley graph of $\mathbb{Z}/10\mathbb{Z}$: its vertices are $0, 1, \ldots, 9$, and the edges correspond to the generator 3 of $\mathbb{Z}/10\mathbb{Z}$, which is the number of $y's$ in the Christoffel word w. Each vertex of the graph corresponds to a conjugate (equivalently, a nontrivial suffix) of w, and the numbering of the vertex corresponds to the distance to the line D of the corresponding integer point in Figure 9.

The word w is recovered by putting x on an edge $i \to j$ if $i < j$, and y otherwise, and reading the edges, beginning form the vertex 0.

The following consequence of the theorem was indicated to us by Valérie Berth. Let α be irrational > 0 and consider the line $y = \alpha x$. We obtain a bi-infinite

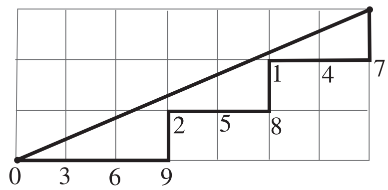

FIGURE 9. Distances to the segment.

sequence s by discretizing from below this line. It is a consequence of the general theory of Sturmian sequences that s has $k+1$ factors of length k for any $k \in \mathbb{N}$. If we write $s = \dots a_{-2}a_{-1}a_1a_2a_3\dots$, where the origin is between a_{-1} and a_1, then: *these* $k+1$ *factors are exactly the* $k+1$ *factors* of $a_{-k}\dots a_{-2}a_{-1}a_1a_2\dots a_k$, which are distinct.

Indeed, let t be the cutting sequence corresponding to the line $y = \alpha x$. Then, denoting by $\tilde{t}$ the reversal of t, we have $s = \tilde{t}yxt$. Now, for each palindrome word m which is a prefix of $t, w = x\,m\,y$ is a lower Christoffel word (see [5] Th. 4.1, [3] Cor. 2.2.29); and there are arbitrary long such words, so we may assume that $|w| > k$. Then m is a suffix of $\tilde{t}$ and $m\,y\,x\,m$ is a factor of $s = \tilde{t}\,y\,x\,t$, and the $y\,x$ factors match. Since $w\,w = x\,m\,y\,x\,m\,y$ and $|w| > k$, the theorem implies the above assertion.

We also obtain the following corollary, since Sturmian sequences of the same slope have the same factors.

Corollary 5.2. *For each Sturmian sequence and each nonnegative integer* k*, some factor of length* $2k$ *of the sequence contains the* $k+1$ *factors of length* k *of the sequence.*

Remark. Another proof of Corollary 5.2 using the Rauzy graph (see [3]) is easily obtained.

6. Appendix: Some geometrical proofs of known results

a) We first prove the direct part of Pirillo's theorem: *if* w, w' *are the lower and upper Christoffel word of the same slope, then they are conjugate.*

It is easy to verify the following fact: if l, l' are two parallel lines, as in the leftmost part of Figure 10, then there exists at most one discrete path, with steps as the ones in Section 2, between them: indeed, then the three other configurations cannot occur (each square in Fig. 10 is a unit square and the fat points are integer points).

Now consider the Christoffel word w of slope $\frac{q}{p}$; we construct the segment $(0,0), (2p, 2q)$, see Figure 11. Let A, B be the points on the path that are the furthest from this segment. Then AB is parallel to the segment. By the previous fact, the path from A to B is necessarily the one encoded by the upper Christoffel word w'. Hence w' is a factor of ww, which implies that w, w' are conjugate.

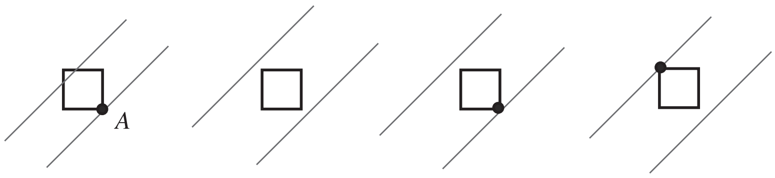

FIGURE 10. The first configuration forbids the others.

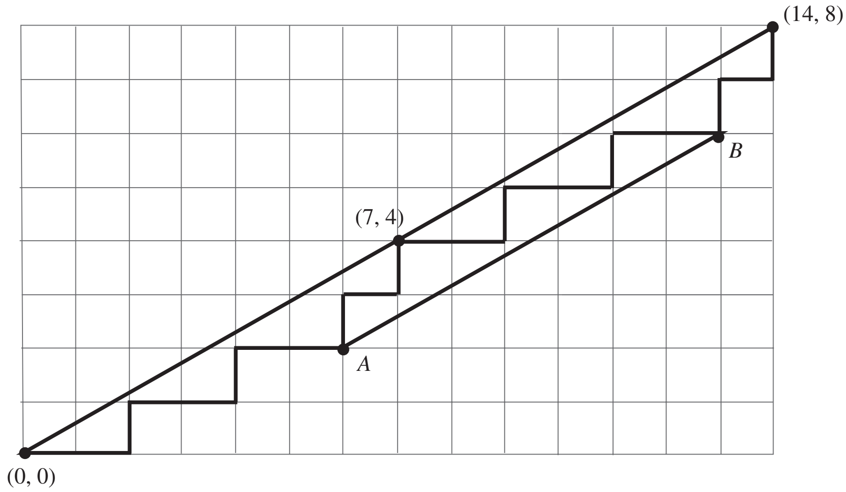

FIGURE 11. Conjugation of the upper and lower Christoffel words.

b) With the same geometric argument, we obtain a little bit more. Recall from [4] the *standard factorization* of a lower Christoffel word w: it is the factorization $w = uv$ corresponding to cutting the path from $(0,0)$ to (p,q) at the closest integer point A' to the segment $[(0,0),(p,q)]$; see Figure 12. The two words u, v are necessarily lower Christoffel words.

Likewise, the upper Christoffel word w' has a standard factorization which corresponds to the closest point A'' in his path; by symmetry, since $w' = \tilde{w}$ (the *reversal* of w), its standard factorization is $w' = \tilde{v}\tilde{u}$. We have $\tilde{v} = ynx, \tilde{u} = ymx$ where m, n are palindromes, since $\tilde{v}, \tilde{u}$ are upper Christoffel words.

We have also a factorization $w = fg$ corresponding to the furthest point A. Now this point is necessarily the southeast corner of the unit square whose northwest corner is A''. Thus we see that f, g and $\tilde{v}, \tilde{u}$ are almost equal: $f = xnx,\ g = ymy$. Thus $v = x\tilde{n}y = xny,\ u = xmy$. Hence we obtain

Proposition 6.1. The lower and upper Christoffel words w, w' are conjugate by palindromes. More precisely $w = xmx\,yny,\ w' = yny\,xmx$, where the standard factorization of w is $w = xmy \cdot xny$ and m, n are palindromes.

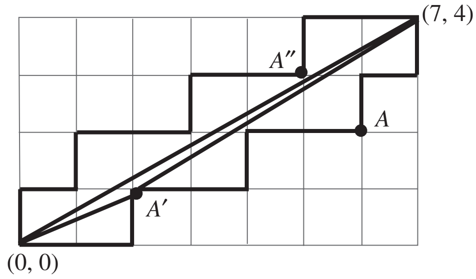

FIGURE 12. $x^2y \cdot x^2yx^2yxy$.

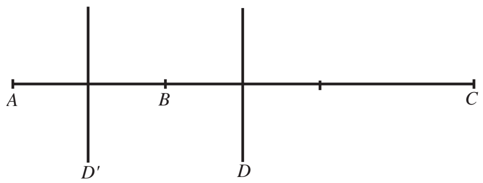

FIGURE 13. Two symmetries imply a translation.

Remark. These results could be obtained as a consequence of the de Luca-Mignosi characterization of *standard words* (a word mxy is standard $\Leftrightarrow$ xmy is a Christoffel word), see [13]. They show indeed that a word w is standard if and only if, for some palindromes m, n, r one has $w = mxy = nr$.

c) We prove now geometrically the well-known fact that *if $w = xuy$ is a Christoffel word of slope q/p, with p, q relatively prime, then u has the two periods s, t with $s + t = p + q$ and sp, $tq \equiv 1 \,\mathrm{mod.}(p + q)$* (see [3, 13] Prop. 2.2.12). Note that s, t are necessarily relatively prime.

We use Figure 12 and the notations of Part *b* above. We have $w = xuy = xmy\,xny$, hence $u = myx\,n$, which shows that u has the palindrome prefix m and the palindrome suffix n. Now, u is palindrome, and if a palindrome of length k has a prefix (or suffix) of length l, then it has the period $k - l$; this is because the product of two axial symmetries is a translation, see Figure 13: D is the bisector of segment AC, and D' that of AB. The product of the symmetry by D' followed by that of D maps B onto C, and A onto $A + \overrightarrow{BC}$; hence it is the translation with respect vector $\overrightarrow{BC}$.

Thus u has the periods which are the sums t and s of the coordinates of the points A' and A'' in Figure 12. Let $(x', y'), (x'', y'')$ be these coordinates. Then the parallelogram constructed on these points has no interior integer points. Hence its area is one, that is, $\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix} = 1$. Moreover, $p = x' + x'', q = y' + y''$. Thus we conclude in view of the following lemma.

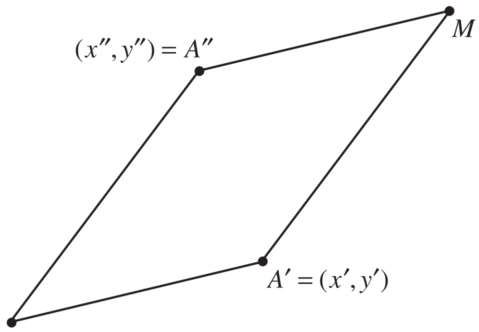

FIGURE 14. Parallelogram.

Lemma 6.1. *Let $(x', y'), (x'', y'') \in \mathbb{N}^2$ be as in Figure 14 and suppose the parallelogram $0A'MA''$ does not contain any integer interior point. Then $(x'+y')(y'+y'')$ and $(x''+y'')(x'+x'')$ are both congruent to 1 mod.$(x'+y'+x''+y'')$.*

Proof. The area of a parallelogram with integer vertices, and no integer interior point, is 1. Thus, we have $1 = \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix} = x'y'' - x''y'$. Thus $x'y'' = 1 + x''y'$ and $(x'+y')(y'+y'') = x'y' + x'y'' + y'^2 + y'y'' = x'y' + 1 + x''y' + y'^2 + y'y'' = 1 + (x'+x''+y'+y'')y'$, which proves the first congruence. The second is obtained similarly. □

d) The proof in c). shows also the following fact: *if the Christoffel word w is of slope $\frac{q}{p}, gcd(p,q) = 1$, and if $w = uv$ is its standard factorization, then $|u| = t, |v| = s$ with $sp, tq \equiv 1$ mod.$(p+q)$.*

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