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HAMILTONICITY IN PARTLY CLAW-FREE GRAPHS

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Abstract. Matthews and Sumner have proved in [10] that if G is a 2-connected claw-free graph of order n such that $\delta(G) \geq (n-2)/3$, then G is Hamiltonian. We say that a graph is almost claw-free if for every vertex v of G, $\langle N(v) \rangle$ is 2-dominated and the set A of centers of claws of G is an independent set. Broersma *et al.* [5] have proved that if G is a 2-connected almost claw-free graph of order n such that $\delta(G) \geq (n-2)/3$, then G is Hamiltonian. We generalize these results by considering the graphs satisfying the following property: for every vertex $v \in A$, there exist exactly two vertices x and y of $V \backslash A$ such that $N(v) \subseteq N[x] \cup N[y]$. We extend some other known results on claw-free graphs to this new class of graphs.

Keywords. Graph theory, claw-free graphs, almost claw-free graphs, Hamiltonicity, matching.

Mathematics Subject Classification. 05C45.

1. Introduction

In this paper, we will consider only finite undirected graphs without loops and multiple edges. We use the terminology and notations in [3]. In addition we'll consider only finite simple graphs $G = (V, E)$. If $S \subset V$, then $\langle S \rangle$ denotes the subgraph of G induced by S, and $G - S$ stands for $\langle V \backslash S \rangle$. If H is an induced subgraph of G, $V(H)$ and $E(H)$ are respectively the set of vertices and the set of edges of the graph H. The cardinality of a maximum independent set of G will be denoted $\alpha(G)$. $N(v)$ is the set of the neighbors of a vertex v, and $N[v] = N(v) \cup \{v\}$. The cardinality of $N(v)$ is the degree $d(v)$ of the vertex v, and $\delta(G)$ denotes the minimum degree of G. We denote by $\sigma_k(G)$ the minimum value of the degree-sum

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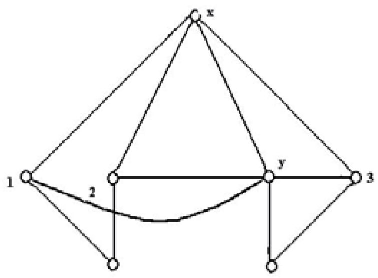

Figure 1.

of any k pairwise non adjacent vertices. The connectivity of G is $k(G)$. If H is a subgraph of G and S is a subset of V or a subgraph of G, then $N_H(S)$ denotes the set of all vertices of H having a neighbor in S. A dominating set of G is a subset S of V such that every vertex of G belongs to S or is adjacent to a vertex in S. The graph G is k-dominated if G has a dominating set of cardinality k. If G has Hamiltonian cycle (a cycle containing once every vertex of G), then G is called Hamiltonian and if G has an even number of vertices then G is called an even graph.

A graph G is H-free if it contains no induced subgraph isomorphic to H.

The r-edge graph $K_{1,r}$ is called the star, and the unique vertex of degree r is called the center of the star. When r is equal to 3, then $K_{1,3}$ is the claw.

Excluding this configuration, we obtain the well known class of claw-free graphs.

The class of claw-free graphs has been the topic of study of several authors. Indeed, matching properties of claw-free graphs were observed, interesting results on Hamiltonian properties were proved, and a lot of NP-complete problems were solved in polynomial time. For more details on this class of graphs see [7].

It is interesting to investigate classes of graphs containing claw-free graphs, and to generalize results on claw-free graphs to these superclasses. These last years, there have been a lot of results in this way and authors were interested in classes of graphs that do not contain "too" many claws. The work we propose deals with the same subject.

Our main goal is to extend some results obtained for claw-free graphs to a new larger class that admits some induced claws. This class will be called the class of partly claw-free graphs.

Definition 1.1. Let $G = (V, E)$ be a graph and let A be the set of centers of claws of G. The graph G is partly claw-free if it satisfies the following property: for every vertex $v \in A$, there exist exactly two vertices x and y of $V \backslash A$ such that $N(v) \subseteq N[x] \cup N[y]$, we say that $N(v)$ is 2-dominated in $V \backslash A$.

Example 1.2. The graph of Figure 1 is partly claw-free, but is not claw-free, since it admits 2 claws $\langle\{x; 1, 2, 3\}\rangle$ and $\langle\{y; 1, 2, 3\}\rangle$.

For this new class of graphs, we can do the following remarks:

(R1) A partly claw-free graph is $K_{1,5}$-free.

Proof. Suppose (R1) false. Let $H = \langle\{v; a, b, c, d, e\}\rangle$ be an induced subgraph of G, isomorphic to $K_{1,5}$ and suppose that v is the unique vertex of H of degree 5. So, every 2-dominating set of $N_H(v)$ has a vertex which is adjacent to at least three vertices among $\{a, b, c, d, e\}$. Then every 2-dominating set of $N_H(v)$ contains a center of a claw, that contradicts the definition of a partly claw-free. □

(R2) A graph G is locally claw-free if for every vertex v of G, $\langle N(v)\rangle$ is claw-free. A partly claw-free graph is not necessarily locally claw-free. (see Fig. 1)

(R3) A graph G is almost claw-free if for every vertex v of G, $\langle N(v)\rangle$ is 2-dominated and the set A of centers of claws of G is an independent set. The class of partly claw-free graphs contains the class of almost claw-free graphs introduced by Ryjacek [13], and is different of the class of graphs whose centers claws are independent [9].

2. Toughness

Definition 2.1. The graph G is t-tough ($t \geq 0$) if $|S| \geq t.w(G - S)$ for every subset S of V with $w(G-S) > 1$, where $w(G-S)$ denotes the number of connected components of $G - S$. The toughness of G, denoted τ(G), is the maximum value of t for which G is t-tough. ($\tau(K_n) = \infty$ for all $n \geq 1$).

Let G be a noncomplete graph and let $k(G)$ be the connectivity of G, then $\tau(G) \leq k(G)/2$ [6]. If G is claw-free, then the equality holds, as was shown by Matthews and Sumner [10]:

Theorem 2.2. *If G is a noncomplete claw-free graph, then $\tau(G) = k(G)/2$.*

In the same paper, they have given their well known conjecture that every 4-connected claw-free graph is Hamiltonian. Another conjecture by Thomassen [17] states that all 4-connected line graphs are Hamiltonian. As all line graphs are claw-free graphs, the second conjecture appears much weaker than the first, but Ryjacek [14] proved that the two conjectures are actually equivalent.

In this section, we prove the following result which generalizes Theorem 2.2 in case $k(G) \leq 2$.

Theorem 2.3. *If G is a noncomplete partly claw-free graph, then $\tau(G) \geq \min\{1, k(G)/2\}$.*

Proof. Only minor changes are needed to adapt the proof given for almost claw-free graphs in [5].

In any noncomplete graph G, $\tau(G) \leq k(G)/2$. If G is not connected, then $\tau(G) = k(G)/2 = 0$. Suppose $G \neq K_n$ is a connected partly claw-free graph and S is a cutset of G such that $\tau(G) = |S|/w(G-S) < \min\{1, k(G)/2\}$. Let $H_1 ....H_p$ be

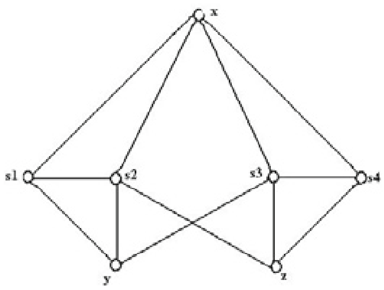

FIGURE 2.

the components of $G-S$. There exist at least $k(G)$ disjoint paths from $u \in V(H_i)$ to $v \in V(H_j)$ for any $i, j \in \{l, ..., p\}$ with $i \neq j$. Each of these paths contains a vertex of S. Hence for each $i \in \{l, ..., p\}$ there are at least $k(G)$ edges joining vertices of H_i to distinct vertices of S. Thus there are at least $p.k(G)$ edges from $G-S$ to S, counting at most one from any component of $G-S$ to a particular vertex of S. Suppose every vertex $v \in$S has neighbors in at most two components of $G-S$. Then there are at most 2 $|S|$ edges from $G-S$ to S counting at most one from any component of $G-S$ to a particular vertex of S. Then $p.k(G) \leq 2|S|$ or, $k(G)/2 \leq |S|/p = \tau(G)$, and that leads to a contradiction.

Hence, S contains a center v of a claw with neighbors in at least three components of $G-S$.

Since G is partly claw-free, there exist exactly two vertices x and y of $V \backslash A$ such that $N(v) \subseteq N[x] \cup N[y]$. This implies that there exists a vertex y of $V \backslash A$, and, moreover, that v has neighbors in at least three components of $G-S$, and y is adjacent to vertices in precisely two of these components. Thus $y \in S$. But then $T = S - \{y\}$ is a cutset of G with $w(G-T) = w(G-S) - 1$, so that $\tau(G) \leq |T|/w(G-T) = (|S|-1)/(w(G-S)-1) < |S|/w(G-S) = \tau(G)$, a contradiction. Hence $\tau(G) \geq \min\{1, k(G)/2)\}$. □

For partly claw-free graphs with connectivity exceeding two, a similar result to that of Theorem 2.2 cannot be obtained. For instance, the graph of Figure 2, depicted in [5], is a 3-connected partly claw-free. The set of centers of claws is $A = \{s_2, s_3\}$. But for $S = \{s_1, s_2, s_3, s_4\}$, $|S| = 4$, $w(G-S) = 3$ and hence:

$$\tau(G) \leq \frac{|S|}{w(G-S)} = \frac{3}{4} < \frac{3}{2} = \frac{k(G)}{2}\cdot$$

3. Perfect matching

The following result appears in [15]:

Theorem 3.1. *If G is an even connected claw-free graph, then G has a perfect matching.*

Further, the same author, extend Theorem 3.1, showing the following:

Theorem 3.2. *If G is an even connected graph that does not have a perfect matching, then there is a set $S \subseteq V$, such that $w_0(G-S) > |S|$, where $\omega_0(G-S)$ is the number of odd components of $G-S$, and every vertex of S is adjacent to vertices in at least three odd components of $G-S$.*

Theorem 3.1 has been extended to the class of almost claw-free graphs:

Theorem 3.3 [13]**.** *If G is an even connected almost claw-free graph, then G has a perfect matching.*

The aim of this section is to generalize Theorem 3.1 to the class of partly claw-free graphs. We prove:

Theorem 3.4. *Every even connected partly claw-free graph has a perfect matching.*

Proof. To prove Theorem 3.4, consider an even connected partly claw-free graph G without any perfect matching.

Let $S \subset V$, and suppose that S has the properties given in Theorem 3.2, and let $x \in S$. The vertex x is adjacent to at least three vertices v_1, v_2 and v_3 belonging to three odd components of $G-S$, hence $\langle\{x; v_1, v_2, v_3\}\rangle$ is an induced claw. Since G is partly claw-free, then two among the three vertices v_1, v_2 and v_3 are dominated by a same vertex y in S, and then by Theorem 3.2, y is a center of a claw, which contradicts the definition of a partly claw-free graph, and then G has a perfect matching. $\square$

4. Hamiltonian cycles

Matthews and Sumner have proved in [11]:

Theorem 4.1. *If G is a 2-connected claw-free graph with $\delta(G) \geq \frac{(n-2)}{3}$, then G is Hamiltonian.*

The following generalization of Theorem 4.1 was independently obtained by Broersma [4] and Zhang [18]:

Theorem 4.2. *If G is a 2-connected claw-free graph with $\sigma_3(G) \geq n-2$, then G is Hamiltonian.*

More generally, Zhang [18] proved:

Theorem 4.3. *if G is a k-connected claw-free graph with $\sigma_{k+1}(G) \geq n-k$ $(k \geq 2)$ then G is Hamiltonian.*

Theorem 4.1 was extended to classes of graphs containing a restricted number of claws by Flandrin and Li [8], to almost claw-free graphs by Broersma *et al.* [5], and for graphs whose centers claws are independant by Li *et al.* [9].

An analogous theorem to Theorem 4.2 has been obtained for almost claw-free graphs by Broersma *et al.* [5].

We have proved the following two results. The first generalizes Theorem 4.1, and the second is analogous to Theorem 4.2. These results are independent of the aforementioned result of Flandrin and Li and generalize the results of Broersma *et al.*

Theorem 4.4. *If G is a 2-connected partly claw-free graph with $\delta(G) \geq \frac{(n-2)}{3}$, then G is Hamiltonian.*

Theorem 4.5. *If G is a 2-connected partly claw-free graph with $\sigma_3(G) \geq n$, then G is Hamiltonian.*

5. Proofs of Theorems 4.4 and 4.5

We first introduce some additional notations and prove two auxiliary results.

Let C be a cycle of G with a given orientation. By $\underline{C}$, we denote the same cycle with the reversed orientation. If $u, v \in V(C)$ then uCv; denotes the consecutive vertices on C from u to v in the direction specified. The same vertices in the reverse order, are given by $u\underline{C}v$. We will consider uCv and $u\underline{C}v$ both as paths and as vertex sets. We use u^+ to denote the successor of a vertex u on C and u^- to denote its predecessor.

Lemma 5.1. *Let C be a longest cycle with a given orientation in a partly claw-free graph G. Let $y \in V\backslash V(C)$ and let x be a neighbor of y on C such that $x \in A$ and $x^-x^+ \notin E(G)$. Then there exists a vertex $d \in N(x^-) \cap N(x^+) \cap (V\backslash A)$, and if $d \in V(C)$ then: either $d^+ = x^-$ or $d^- = x^+$, or there is a path Q_1 between d^- and d^+ and a path Q_2 between $x^-(x^+)$ and x such that $V(Q_1) \cap V(Q_2) = \emptyset$ and $V(Q_1) \cup V(Q_2) = \{x^-, x, x^+, d^-, d, d^+\}$.*

Proof. Suppose first that y and x^- have a common neighbor $v \in V\backslash A$. It is clear that the choice of C implies that $v \in V(C)$, and yv^-, $yv^+ \notin E(G)$. Since v is not a center of a claw, then $v^-v^+ \in E(G)$, and we can extend C by replacing v^-vv^+ by v^-v^+, and x^-x by x^-vyx, and get a cycle C' longer than C, which contradicts the fact that C is the longest cycle of G. Hence y and x^- have no common neighbor $v \in V\backslash A$. By symmetry, y and x^+ have no common neighbor in $V\backslash A$. Since $N(x)$ is 2-dominated in $V\backslash A$, there is a vertex $d \in V\backslash A$ dominating both x^+ and x^-.

If $d \in V(C)$ then If $d^+ = x^-$ or $d^- = x^+$, we are done. Suppose now that $d^+ \neq x^-$ and $d^- \neq x^+$ and consider the subgraph of G induced by $\{d^-, d, d^+, x^+\}$, since d is not a center of a claw, at least one of the edges d^-d^+, d^-x^+ and d^+x^+ belongs to G. If $d^-d^+ \in E(G)$, then put $Q_1 = d^-d^+$ and $Q_2 = x^-dx^+x$. If $d^-x^+ \in E(G)$, then put $Q_1 = d^-x^+dd^+$ and $Q_2 = x^-x$. If $d^+x^+ \in E(G)$, then put $Q_1 = d^-dx^+d^+$ and $Q_2 = x^-x$. The similar statement for x^+ follows by symmetry. □

In the sequel, let G be a non Hamiltonian 2-connected partly claw-free graph, let C be a longest cycle in G, and let H be a component of $G - V(C)$. As in [5], we shall use the following notations. We denote by $x_1, ..., x_k$ the vertices of $N_C(H)$ occurring on C in the order of their indices, and let $S_i = x_i^+Cx_{i+1}^-$ and $s_i = |S_i|$.

Clearly, $k \geq 2$. Let l_i denote the length of a longest path between x_i and x_{i+1} with all internal vertices in $H (i = 1, ..., k$; indices mod $k)$.

Lemma 5.2. $\sum_1^k s_i \geq \sum_1^k l_i + k$

Proof. Let i∈{1,2,...,k} and let $L_i = y_1 y_2 y_3 ... y_t$ be a path of length l_i between x_i and x_{i+1} and with all internal vertices in H.

We distinguish several cases.

Case 1. $x_{i+1}^- x_{i+1}^+ \in E$

Case 1.A. $x_i^- x_i^+ \in E$ then $s_i \geq 1 + l_i$, since otherwise the cycle $C' = x_i^- x_i^+ x_i L_i x_{i+1} x_{i+1}^- x_{i+1}^+ C x_i^-$ will be longer than C.

Case 1.B. $x_i^- x_i^+ \notin E$ then by Lemma 5.1, there is a vertex $d \in N(x_i^-) \cap N(x_i^+) \cap (V \backslash A)$. Suppose first that $d \notin V(C)$. If $d \in L_i, \exists j, 1 \leq j \leq t, d = y_j$, then the cycle $C\prime = x_i^+ C x_i y_1 .... y_j x_i^+$ will be longer than C. So $d \notin L_i$ and then $s_i \geq 1 + l_i$, since replacing $x_i^- x_i S_i x_{i+1} x_{i+1}^+$ by $x_i^- d x_i^+ x_i L_i x_{i+1} x_{i+1}^- x_{i+1}^+$ in C, we get a cycle C' longer than C. Hence $d \in V(C)$. If $d^- = x_i^+$, then $s_i \geq 2 + l_i$ since otherwise replacing $x_i^- x_i S_i x_{i+1} x_{i+1}^+$ by $x_i^- d x_i^+ x_i L_i x_{i+1} x_{i+1}^- x_{i+1}^+$ in C, we obtain a cycle C' longer than C. If $d^+ = x_i^+$, then $s_i \geq l_i$. If $d = x_{i+1}$, then the cycle $x_{i+1} x_i^+ C x_{i+1}^- x_{i+1}^+ C x_i L_i x_{i+1}$ is longer than C. If $d \in S_i$, then $s_i \geq 2 + l_i$ since otherwise replacing in C, $d^- d d^+$ and $x_i^- x_i S_i x_{i+1} x_{i+1}^-$ by Q_1, Q_2, L_i and $x_{i+1}^- x_{i+1}^+$ we obtain a cycle C' longer than C.

Case 2. $x_{i+1}^- x_{i+1}^+ \notin E$.

By Lemma 5.1, there is a vertex $d_2 \in N(x_{i+1}^-) \cap N(x_{i+1}^+) \cap (V \backslash A)$

Case 2.A. $x_i^- x_i^+ \in E$. This case is symmetric to the case 1.B.

Case 2.B. $x_i^- x_i^+ \notin E$. By Lemma 5.1, there is a vertex $d_1 \in N(x_i^-) \cap N(x_i^+) \cap (V \backslash A)$. Using similar arguments as the case 1.B, we obtain that $d_1, d_2 \in V(C)$.

Clearly $d_1, d_2 \notin \{x_i^-, x_i^+, x_i, x_{i+1}, x_{i+1}^-, x_{i+1}^+\}$. Suppose $d_1 = d_2$ and assume without loss of generality $d_1 \in x_{i+1}^+ C x_i^-$. Consider the subgraph $\langle\{d_1, x_i^-, x_i^+, d_1^-\}\rangle$, since d_1 is not a center of a claw, then, at least one of $d_1^- x_i^-$ or $d_1^- x_i^+$ is an edge of G, and, then the cycles $d_1 C x_i^- d_1^- \underline{C} x_{i+1} L_i x_i C x_{i+1}^- d_1$ or $d_1 C x_i L_i x_{i+1}^- \underline{C} x_i^+ d_1^- \underline{C} x_{i+1}^+ d_1$ respectively contradict the choice of C. Hence $d_1 \neq d_2$. Suppose $d_1 d_2 \in E(C)$ and assume without loss of generality $d_1 \in x_{i+1}^+ C x_i^-$. If $d_2 = d_1^-$, then the cycle $x_i \underline{C} d_1 x_i^+ C x_{i+1}^- d_2 \underline{C} x_{i+1} L_i x_i$ is longer than C. If $d_2 = d_1^+$, then the cycle $x_i \underline{C} d_2 x_{i+1}^- \underline{C} x_i^+ d_1 \underline{C} x_{i+1} L_i x_i$ is longer than C. Hence $d_1 d_2 \notin E(C)$. Using similar arguments as above, we obtain the following lower bounds on s_i in the nine possible cases:

(i) $d_1^- = x_i^+$

(i.1) $d_2^+ = x_{i+1}^-$ then $s_i \geq 3 + l_i$

(i.2) $d_2^- = x_{i+1}^+$ then $s_i \geq 1 + l_i$
(i.3) $d_2^+ \neq x_{i+1}^-$, $d_2^- \neq x_{i+1}^+$ then $s_i \geq 3 + l_i$ if $d_2 \in S_i$
$s_i \geq 3 + l_i$ if $d_2 \notin S_i$;

(ii) $d_1^+ = x_i^-$
(ii.1) $d_2^+ = x_{i+1}^-$ symmetric to the case (i.2)
(ii.2) $d_2^- = x_{i+1}^+$ then $s_i \geq -1 + l_i$
(ii.3) $d_2^+ \neq x_{i+1}^-$, $d_2^- \neq x_{i+1}^+$ then $s_i \geq 1 + l_i$ if $d_2 \in S_i$
$s_i \geq l_i$ if $d_2 \notin S_i$;

(iii) $d_1^- x_i^+$ and $d_1^+ \neq x_i^-$
(iii.1) $d_2^+ = x_{i+1}^-$ symmetric to the case (i.3)
(iii.2) $d_2^- = x_{i+1}^+$ symmetric to the case (ii.3)
(iii.3) $d_2^+ \neq x_{i+1}^-$, $d_2^- \neq x_{i+1}^+$ then $s_i \geq 1 + l_i$.

We obtain the result by summing up all s_i $(i = 1, .., k)$. □

5.1. Proof of Theorem 4.4

Using Lemmas 5.1 and Lemma 5.2, only minor changes are needed to adapt the proof given in [5] for almost claw-free graphs.

Assume $\delta(G) \geq \frac{n-2}{3}$. Using Lemma 5.2, we obtain:
$n \geq \sum_1^k s_i + k + 1 \geq \sum_1^k l_i + 2k + 1 \geq 4k + 1 \geq 9$.
Suppose $V(H) = \{v\}$, then:

$$\frac{n-2}{3} \leq \delta(G) \leq d(v) \leq k \leq \frac{n-1}{4},$$

and that leads to a contradiction.

Hence no component of $G - V(C)$ is an isolated vertex. We may assume $|V(H)| \geq 2$. Among the pairs $v_1, v_2 \in V(H)$ for which

$$|Nc(v_1)| + |Nc(v_2)| \text{is as large as possible} \tag{5.1}$$

choose a pair $\{u, v\}$ such that

$$|N_C(u) \cup N_C(v)| \text{is as large as possible.} \tag{5.2}$$

If $|N_C(u) \cup N_C(v)| \leq 1$, then (1) and (2) imply $|N_C(H)| \leq 1$, a contradiction. Hence $|N_C(u) \cup N_C(v)| \geq 2$. Moreover, by the 2-connectedness of G, we may assume u and v are chosen in such a way that $uy_1, vy_2 \in E(G)$ for two distinct vertices $y_1, y_2 \in V(C)$. Let $p = |N_C(u)|, q = |N_C(v)|$ and $r = |N_C(u) \cap N_C(v)|$. Assume, without loss of generality that $p \geq q$, and let $1(u, v)$ denote the length of a longest path between u and v in H. Denote $N_C(u) \cup N_C(v)$ by $\{x_1, ..., x_t\}$, where the vertices occur on C in the order of their indices. Then, using Lemma 5.2 for this subset $\{x_1, ..., x_t\}$ of $N_C(H)$, we obtain:

$$\begin{aligned} n &\geq |V(H)| + |V(C)| \\ &\geq |V(H)| + \sum_1^t s_i + t \\ &\geq |V(H)| + \sum_1^t l_i + 2t \geq |V(H)| + 4t + \max\{2, r\}.l(u, v). \qquad (5.3) \end{aligned}$$

We distinguish two cases.

Case 1. $p + q \leq \delta(G) - 1$.
By the choice of u and v, $d_H(v_1)+d_H(v_2) \geq 2\delta(G)-(p+q) \geq \delta(G)+1 \geq (n+1)/3$ for all $v_1, v_2 \in V(H)$. By Theorem 2.3, G is 1-tough. Using the result of Bauer and Schmeichel [2], $|V(C)| \geq 2\delta(G) + 2$, hence $|V(H)| \leq n - (2\delta(G) + 2) \leq (n - 2)/3$. Thus $d_H(v_1) + d_H(v_2) \geq |V(H)| + 1$ for all $v_1, v_2 \in V(H)$, implying, by using the result of Ore [12], that H is Hamiltonian-connected. In particular, $l(u, v) = |V(H)| - 1$. Using (3), we have:

$$n \geq |V(H)| + 4t + 2l(u, v) \geq 3|V(H)| + 4t - 2.$$

Clearly,

$$\delta(G) + 1 - q \leq |V(H)|. \qquad (5.4)$$

Hence

$$\begin{aligned} n &\geq 3\delta(G) + 4t - 3q + 1 = 3\delta(G) + t + 3(t - q) + 1 \\ &\geq 3\delta(G) + 3 \geq n + 1 \end{aligned}$$

which leads to a contradiction.

Case 2. $p + q \geq \delta(G)$.
Using (3) and (4), we have:

$$\begin{aligned} n &\geq |V(H)| + 4t + \max\{2, r\}.l(u, v) \\ &\geq \delta(G) + 1 - q + 4(p + q - r) + \max\{2, r\}.l(u, v) \\ &= \delta(G) + 1 + 2(p + q) + (p + q - r) + p - 3r + \max\{2, r\}.l(u, v) \\ &\geq 3\delta(G) + 3 + p - 3r + \max\{2, r\}.l(u, v) \\ &\geq n + 1 - 2r + \max\{2, r\}.l(u, v). \end{aligned}$$

This clearly yields a contradiction in case $l(u, v) \geq \min\{2, r\}$. For the remaining cases assume $l(u, v) = 1$ and $r \geq l(u, v) + 1$.
Then $N_H(u) \cap N_H(v) = \emptyset$ and hence $|V(H)| \geq 2\delta(G) - (p + q)$.

By (3),

$$\begin{aligned} n &\geq |V(H)| + 4t + r \\ &\geq 2\delta(G) - (p+q) + 4(p+q-r) + r \\ &= 2\delta(G) + (p+q) + (p+q-r) + (p+q-2r) \\ &\geq 3\delta(G) + 2 \geq n. \end{aligned}$$

This implies $p = q = r = 2, \delta(G) = 4, n = 14$ and $|V(H)| = 4$. Now, u and v have neighbors w_1 and w_2 in H, respectively, such that $w_1w_2, uw_1, vw_2 \notin E(G)$ (since $l(u,v) = 1$). Furthermore, $d_H(w_1) + d_H(w_2) = 2$ since $|V(H)| = 4$, while on the other hand the choice of u and v, implies $d_H(w_1) + d_H(w_2) \geq 2\delta(G) - (p+q) = 8 - 4 = 4$, it is a contradiction. □

5.2. Proof of Theorem 4.5

Assume $\sigma_3(G) \geq n$. By Theorem 2.3, G is 1-tough. We use Lemma b [1].

Lemma b. *Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n$. Then every longest cycle of G is a dominating cycle (a cycle such that every edge of G has at least one end point in this cycle). Moreover, if G is non Hamiltonian, then G contains a longest cycle C such that* $\max\{d(v)/v \in V(G) - V(C)\} \geq \sigma_3(G)/3$.

Let C be a dominating cycle such that there is a vertex $v \in V(G) - V(C)$ with $d(v) \geq \sigma_3(G)/3 \geq n/3$. By Lemma 5.2 (with $d(v) = k$),

$$n \geq \sum_1^k s_i + k + 1 \geq \sum_1^k l_i + 2k + 1 \geq \frac{4}{3}n + 1$$

a contradiction. □

6. Concluding remarks

In this article we have defined a new class of graphs generalizing the class of claw-free graphs. For this larger class of graphs, two new results on Hamiltonicity have been given. The first one generalizes a result proved for claw-free graphs and the second is an analogue of one given for the claw-free graphs.

The graph of Figure 3 depicted in [11] shows that Theorem 4.4 is the best possible, but we do not know whether Theorem 4.5 is.

As every almost claw-free graph is partly claw-free graph, the results proved for almost claw-free in [5] will be corollaries of Theorems 4.4 and 4.5.

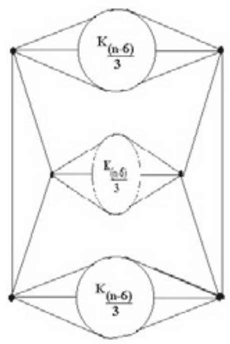

FIGURE 3.

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