RAIRO-Oper. Res. 49 (2015) 215–242
DOI: 10.1051/ro/2014031

RAIRO Operations Research
www.rairo-ro.org

LEAD TIME AND ORDERING COST REDUCTIONS IN BUDGET AND STORAGE SPACE RESTRICTED PROBABILISTIC INVENTORY MODELS WITH IMPERFECT ITEMS

Aref Gholami-Qadikolaei[1], Abolfazl Mirzazadeh[1] and Reza Tavakkoli-Moghaddam[2]

Abstract. Regarding today's business environment restrictions, one of significant concern of inventory manager is to determine optimal policies of inventory/production systems under some restrictions such as budget and storage space. Therefore here, a budget constraint on total inventory investment and a maximum permissible storage space constraint are added simultaneously to a stochastic continuous review mixed backorder and lost sales inventory system. This study also assumes that the received lot may contain some defective units with a beta-binomial random variable. Two lead time demand (LTD) distribution approach are proposed in this paper, one with normally distributed demand and another with distribution free demand. For each approach, a Lagrange multiplier method is applied in order to solve the discussed constrained inventory models and a solution procedure is developed to find optimal values. This study, also, shows that the respective budget and storage space constrained inventory models to be minimized are jointly convex in the decision variables. Numerical examples are also presented to illustrate the models.

Keywords. Stochastic inventory system, lead time, inventory constraints, minimax distribution free procedure, imperfect items, ordering cost reduction, Lagrange multiplier method.

Mathematics Subject Classification. 90B05, 90C15, 90C30, 90C47.

Received December 5, 2012. Accepted April 30, 2014.

[1] Department of Industrial Engineering, Kharazmi University, Mofatteh Ave, Tehran 1571914911, Iran. a.mirzazadeh@aut.ac.ir

[2] Department of Industrial Engineering, College of Engineering, University of Tehran, Tehran, Iran

Article published by EDP Sciences

1. Introduction

Today, lead time plays an important role in the logistics management. In the inventory management literature, lead time is treated as predetermined constant or stochastic parameters. If it is assumed that lead time can be decomposed into several components, such as setup time, process time, and queue time, it can be assumed that each component may be reduced at a crashing cost. One of first papers dealing with a variable lead time in an inventory model is due to Lia and Shyu [16]. The authors present a stochastic model in which the order quantity is predetermined and lead time is a unique decision variable. Ben-Daya and Rauf [6] extend this model by considering both lead time and order quantity as decision variables. Moon and Gallego [19] assume unfavorable lead time demand distribution and solve both continuous review and periodic review models with a mixture of backorders and lost sales using minimax distribution free approach. Ouyang *et al.* [21] generalize Ben-Daya and Rauf [6] model in which the lead time demand is considered a normal distribution or distribution free. Later, Hariga and Ben-Daya [15] extend Ouyang *et al.* [21] study and present both continuous review and periodic review inventory models in partial and perfect lead time demand distribution information environment in which reorder point treated as a variable. Also Wu *et al.* [28] study on negative exponential crashing cost with mixture of distribution in their study. Chung *et al.* [10] investigate a periodic review inventory model with controllable lead time and setup cost. Chandra and Grabis [7] develop a model with lead time-dependent procurement costs and assume that shortening lead time results in increasing procurement costs. The relationship between lead time and procurement costs is established with the help of a linear and nonlinear procurement cost function. Lin *et al.* [18] proposes an integrated vendor-buyer inventory model wherein a lot received for buyer main contain some defective items due to vendor's imperfect manufacturing system. Glock [12] investigates on integrated inventory system with stochastic demand and variable, lot size-dependent lead time and assumes that lead time consists of production, setup and transportation time. Lin [17] proposes a stochastic inventory model including defective items wherein the demands of different customer are not identical in the lead time.

However, in some practical situations, ordering/setup cost can also be controlled and reduced through various efforts such as worker training, procedural changes and specialized acquisition for the classical inventory/production model. Moreover, it has been observed in many manufacturing settings including job shops, batch shops and flow shops, whose ordering (or setup) cost can be reduced by investing capital. Porteus [22] is pioneer in presenting the concept of investing in reducing setup cost on the classical EOQ model without backorders which contains a set up as a function of capital expenditure. Billington [3] proposes a no-backorder inventory model including a setup cost as a function of capital expenditure. Kim *et al.* [24] present several classes of setup reduction functions and describe a general solution procedure on the EPQ model. Paknejad *et al.* [23] present a quality-adjusted lot-sizing model with stochastic demand and constant lead time and study the benefits of lower setup cost in the model. Sarker and Coats [26] propound an inventory model with setup cost reduction under probabilistic lead time constraint and finite number of investment possibilities to reduce setup cost. Wu and Lin [27] develop a model with lead time and setup cost reduction for partial backorder system and consider amount received is uncertain. Ben-Daya and Hariga [1] study a model wherein both lead time and crashing cost maybe reduced at a crashing cost.

In the inventory management literature, many research works have been devoted to the study of restrictions such as storage space or inventory investment constraints on the inventory system performances. This literature can be divided into two categories, the first deal with considering a budget constraint on the inventory investment and the second with storage space constraint:

Dealing with the first line of research, in practical environment, it is very likely that the firm has a fixed amount of capital to invest in goods carried as inventory. Hence, if it is assumed that inventory investment budget is restricted then the critical issue is to make sure that optimal decision variables are satisfied the constraint. Brown and Gerson [4] proposed some models for stochastic inventory system with the limitation on total inventory investment. Schrady and Choe [31] proposed a model with the total time weighted shortages with limitations on inventory investment. Bera *et al.* [5] presents a multi-product continuous review inventory model under a budget constraint wherein the system purchasing cost is paid when an order arrives. Our model formulation differs from the one developed in the Bera *et al.* [5], since, the system purchasing cost is paid at the time an order is placed.

In the second category, as a consequence of high cost of land acquisition in the most societies, most of inventory systems have limited storage space to stock goods. Hariga [14] presents a stochastic space constrained continuous review inventory system for a single item and random demand wherein the order quantity and reorder point are decision variables. Xu and Leung [25] propose an analytical model in a two-party vendor managed system where the retailer restricts the maximum space allocated to the vendor. Moon *et al.* [20] proposed three extended models with variable capacity. First, they presented an EOQ model with random yields. Second, they developed a multi-item EOQ model with storage space and solved model with Lagrange multiplier method. Third, they applied a distribution free approach to the (Q,r) with variable capacity.

One of the assumptions of the inventory management literature is that the quality of the product in a lot is perfect. In practice, however, a received lot may contain some defective items. If there is a possibility that a lot contains defective items, the firm may issue a larger order than was originally planned so as to guarantee satisfaction of customer demand. Salameh and jabber [30] develop an EOQ model where imperfect quality items are salvaged at a discount price. Incoming raw material containing items of imperfect quality that occurs as random fraction with a known probability distribution, Undergo a screening in which defective items in the lot removed by the end of screening period and sold at a discounted price.

Chang [8] reformulates the EOQ model of Salameh and Jaber [30] to capture the uncertainty in the defective rate using fuzzy set theory. Ben-Daya and Noman [2] develop continuous review inventory inspection models in which Inspection polices include no inspection, sampling inspection and 100% inspection. They assume when a lot is received, the buyer uses some type of inspection policy. The fraction of nonconforming is assumed to be a random variable following a beta distribution. Taheri-Tolgari *et al.* [32] present a discounted cash-flow technique for an inventory model for imperfect item with considering inspection errors. They assume some produced items might not be perfect and first stage inspector of product quality control might make some inspection errors during separation of defective and perfect items.

However, in the all of previously mentioned models, the problem of determining optimal continuous review policies for budget and storage space constrained stochastic mixed backorder and lost sales inventory system with lead time and ordering cost reduction has not been explored. This paper considers the amount received is random variables due to rejection during inspection. Imperfect units are assumed to be beta-binomial random variable. The imperfect items found out during inspection are all discarded Purchasing cost is paid at the time of order placing. For this reason, maximum inventory investment will occur at the time an order is placed. Considering this assumption, a budget constraint is established in this paper. Storage space constraint is random since the inventory level when an order arrives is a random variable. Hence, a chance-constrained programming technique is utilized to make it crisp. The piece-wise linear crashing cost function is considered in the existing models. This kind of crashing cost function is widely used in project management in which the duration of some activities can be reduced by assigning more resources to the activities. This study, first assumes the lead time demand follows a normal distribution and then relaxes the assumption about the form of the distribution function of the lead time demand by only assuming the known mean and variance of it. Lagrange multiplier technique is applied in order to solve the constrained problems. This study, also, shows that the discussed budget and storage space constrained inventory models to be minimized are jointly convex in the decision variables. A solution procedure is proposed to find optimal order quantity, lead time, ordering cost and reorder point. In addition, numerical examples are presented to illustrate the models

This paper is organized as follows. Notations and assumptions are given in Section 2. In Section 3, the constrained inventory model that the lead time demand has perfect information is formulated and the models of partial information for the lead time demand are examined in Section 4. A numerical example and sensitivity analysis is provided to illustrate solution procedure in Section 5, and Section 6 contains some concluding remarks and future research.

2. Notation and assumption

Following notations have been used in this paper:

Q Order quantity.

r Reorder point.

k	Safety factor.
β	The fraction of demand which is backordered during stockout period, $0 \leqslant \beta \leqslant 1$.
D	Average demand per year.
π	Stockout cost per unit short.
π_0	Marginal profit per unit.
p	Defective rate in an order lot, $p\epsilon[0, 1)$ and is a random variable.
$g(p)$	Probability density function (p.d.f.) of p.
C_p	Purchasing cost per unit.
h	Holding cost per year per unit.
f	Space used per unit.
A	Fixed ordering cost per order.
A_0	Original ordering cost.
L	Length of lead time.
$C(L)$	Total lead time crashing cost per order.
δ	Inspection cost per unit.
B	Maximum inventory investment.
F	Maximum available space.
γ	Smallest acceptable probability that total used space is within maximum available space.
φ	Smallest acceptable probability that total investment is within maximum available investment.
Y	Number of defective item in a lot, as a random variable.
X	Demand during lead time.
X^+	Maximum value of x and 0.
$E(\bullet)$	Mathematical expectation.

The mathematical models developed in this paper are based on the following assumptions:

1. Shortages are allowed and partially backlogged.
2. Planning horizon is infinite.
3. Demand rate D is a random variable with mean μ_D and standard deviation σ_D
4. Inventory is continuously reviewed. Replenishments are made whenever the inventory level falls to the reorder point r and replenishment rate is infinite.
5. The cost equations are approximations because inventory levels and demands are treated as continuous instead of discrete quantities.
6. Shortages cost do not depend on time.
7. The time that the system is out of stock during a cycle is small in comparison to the cycle length.
8. There are no orders outstanding at the time the reorder point is reached.
9. The purchasing cost is paid at the time of order placing.
10. The reorder level is larger than the mean of the lead time demand.
11. Inspection is error free.
12. Inspection time is negligible.
13. The reorder point r is the expected demand during lead time plus safety stock (SS) and $SS = k\times$ (standard deviation of lead time) *i.e.* $r = E(x) + k\sigma_x$ where k is safety factor satisfying $P(x > r) = P(z > k) = \alpha$, z represents the standard normal random variable and α represents the allowable stockout probability during lead time.

14. The lead time L consists of n mutually independent components. The ith component has a minimum duration a_i, the normal duration b_i and a crashing cost c_i per unit time. Further, for convenience, we rearrange c_i such that $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_n$
15. If we let $L_0 = \sum_{j=1}^{n} b_j$ and L_i be the length of lead time with components $1, 2, \ldots, i$ crashed to their minimum duration, then L_i can be expressed as $L_i = \sum_{j=1}^{n} b_j - \sum_{j=1}^{i} (b_j - a_j)$, $i = 1, 2, \ldots, n$; and the lead time crashing cost $C(L)$ per cycle for a given $L \epsilon [L_i, L_{i-1}]$ is given by $C(L) = c_i (L_{i-1} - L) + \sum_{j=1}^{i-1} c_j (b_j - a_j)$
16. The components of lead time are crashed one at a time starting with component 1 (because, it has a minimum unit crashing cost) and then component 2 and so on.
17. Consider lead time is constant and Mean and variance of demand during lead time x is (Tersine [33]):

$$E(x) = \mu_D L \text{ and } \operatorname{Var}(x) = \sigma_x^2 = L\sigma_D^2 \tag{2.1}$$

3. Model formulation for normal demand approach:

In this paper, an arrival order may contain some defective items and we consider defective rate in an order lot is a random variable (Y), which can be expressed as follows:

$$P(y|p) = \binom{Q}{y} p^y (1-p)^{Q-y}, \ y = 0, 1, 2, \ldots Q. \tag{3.1}$$

In this case:

$$E(y|p) = Qp. \tag{3.2}$$

And

$$\operatorname{Var}(y|p) = Qp(1-p). \tag{3.3}$$

Hence, non-conditioning on p, the expected value of Y is:

$$E(y) = E(E(y|p)) = QE(p). \tag{3.4}$$

And variance of Y is calculated as follow:

$$\operatorname{Var}(y) = E(\operatorname{Var}(y|p)) + \operatorname{Var}(E(y|p)) \tag{3.5}$$

$$E(\operatorname{Var}(y|p)) = E(Q(p(1-p))) = QE(p(1-p)) \tag{3.6}$$

$$\operatorname{Var}(E(y|p)) = E\left(Q^2 p^2\right) - (E(p))^2 = Q^2 E\left(p^2\right) - Q^2 E^2(p) = Q^2 \operatorname{Var}(p) \tag{3.7}$$

$$\rightarrow \operatorname{Var}(y) = QE(p(1-p)) + Q^2 \operatorname{Var}(p) \tag{3.8}$$

and

$$\rightarrow E\left(y^2\right) = Q^2 E\left(p^2\right) + QE\left(p\left(1-p\right)\right). \tag{3.9}$$

When inventory of an item reaches to reorder level management places an order of amount Q which Y items are defectives. The amount of shortages per cycle is a random variable since X (demand during lead time) is a random variable and can be expressed as follows:

$$(x-r)^+ = \mathrm{Max}\,(x-r,0) = \begin{cases} x-r, & x \geqslant r \\ 0, & x \leqslant r \end{cases} \tag{3.10}$$

So, the number of backorders per cycle is $\beta(x-r)^+$ and the number of lost sales is $(1-\beta)\,(x-r)^+$. The net inventory level just before order arrives is $r-x+(1-\beta)\,(x-r)^+$ and the maximum inventory level at the beginning of the cycle (considering y defective items) is $Q-y+r-x+(1-\beta)\,(x-r)^+$. So the total inventory per cycle is:

$$\frac{Q-y}{D}\left(\frac{Q-y}{2} + r - x + (1-\beta)\,(x-r)^+\right). \tag{3.11}$$

Therefore, the inventory cost per cycle given that there are y defective items in an arriving order of size Q is expressed as follows:

$$\begin{aligned}&\left[A + c_i\left(L_{i-1}-L\right) + \sum\nolimits_{j=1}^{i-1} c_j\left(b_j - a_j\right)\right] + h\frac{Q-y}{D}\left[\frac{Q-y}{2} + r - x + (1-\beta)\,(x-r)^+\right] \\ &\qquad + \left[\left(\pi + \pi_\circ\left(1-\beta\right)\right)(x-r)^+\right] + \delta Q, \\ &\qquad\qquad L \in \left[L_i, L_{i-1}\right].\end{aligned} \tag{3.12}$$

The expected inventory cost per cycle is as follows:

$$\begin{aligned}&\left[A + c_i\left(L_{i-1}-L\right) + \sum\nolimits_{j=1}^{i-1} c_j\left(b_j - a_j\right)\right] \\ &+ h\frac{E(Q-y)^2}{2D} + h\frac{E\left(Q-y\right)}{D}\left[r - E(x) + (1-\beta)\,E\left(x-r\right)^+\right] \\ &+ \left[\left(\pi + \pi_\circ\left(1-\beta\right)\right)E\left(x-r\right)^+\right] + \delta Q, \quad L \in \left[L_i, L_{i-1}\right].\end{aligned} \tag{3.13}$$

The expected length of the cycle time is as follows:

$$E\left(T\right) = \frac{E\left(Q-y\right)}{D}\cdot \tag{3.14}$$

Hence, using the result of a basic theorem from renewal reward processes [29] the expected annual cost can be computed as the expected cost per cycle divided by expected cycle time:

$$
\begin{aligned}
EAC\left(Q, r, L\right) = & \frac{D}{E\left(Q-y\right)}\left[A + c_i\left(L_{i-1}-L\right)+\sum\nolimits_{j=1}^{i-1} c_j\left(b_j-a_j\right)\right] \\
& + h\frac{E\left(Q-y\right)^2}{2E\left(Q-y\right)} + h\left[r - E(x) + \left(1-\beta\right)E\left(x-r\right)^{+}\right] \\
& + \frac{D}{E\left(Q-y\right)}\left[\left(\pi+\pi_{\circ}\left(1-\beta\right)\right)E\left(x-r\right)^{+}\right] \\
& + \frac{D}{1-E(p)}\delta, \quad L \epsilon \left[L_i, L_{i-1}\right],
\end{aligned} \tag{3.15}
$$

where

$$
E\left(Q-y\right) = Q - E\left(y\right) = Q\left(1-E\left(p\right)\right) \tag{3.16}
$$

$$
\begin{aligned}
E\left(Q-y\right)^2 & = \mathrm{Var}\left(Q-y\right) + \left(E\left(Q-y\right)\right)^2 \\
& = QE\left(p\left(1-p\right)\right) + Q^2\mathrm{Var}\left(p\right) + \left(Q\left(1-E\left(p\right)\right)\right)^2.
\end{aligned} \tag{3.17}
$$

The underlying assumption in the above model (16) is that ordering cost, A, is a fixed constant and not subject to control. In this study, we consider the ordering cost can be reduced through capital investment and it treats as a decision variable. Therefore, we seek to minimize the sum of capital investment cost of reducing ordering cost A and the inventory related costs by optimizing over Q, A, r and L constrained $0 \leqslant A \leqslant A_0$, that is, according to our model, the objective of our problem is to minimize the following expected total annual cost.

$$
EAC\left(Q, A, r, L\right) = \theta I\left(A\right) + EAC(QrL) \tag{3.18}
$$

Over $A \in [0, A_0)$, where θ is the fractional opportunity cost of capital per year and $I(A)$ follows the logarithmic investment function which is consistent with the japanese experince as reported in Hall [13], given by

$$
I\left(A\right) = \mathrm{bln}\left(\frac{A_0}{A}\right) \quad \text{for} \quad A \in \left(0, A_0\right]. \tag{3.19}
$$

where $\dfrac{1}{b}$ is the fraction of the reduction in A per dollar increase in investment.

Hence, expected annual cost can be expressed as:

$$
\begin{aligned}
EAC\left(Q,A,r,L\right) = \theta \mathrm{bln}\left(\frac{A_0}{A}\right) + \frac{D}{E\left(Q-y\right)} \\
\times \left[A + c_i\left(L_{i-1} - L\right) + \sum\nolimits_{j=1}^{i-1} c_j\left(b_j - a_j\right)\right] \\
+ h\frac{E\left(Q-y\right)^2}{2E\left(Q-y\right)} + h\left[r - E(x) + \left(1-\beta\right)E\left(x-r\right)^+\right] \\
+ \frac{D}{E\left(Q-y\right)}\left[\left(\pi + \pi_\circ\left(1-\beta\right)\right)E\left(x-r\right)^+\right] + \frac{D}{1-E\left(p\right)}\delta, \\
L\,\epsilon\,\left[L_i, L_{i-1}\right],\ A\epsilon\left(0, A_0\right].
\end{aligned}
\tag{3.20}
$$

In this paper, we consider a storage space limitation, which is dependent on the maximum inventory size. The restriction ensures that even if an item reaches to its maximum inventory position, the maximum available space is still enough for it. Therefore, the random storage space constraint for a partial backlogging policy is as follows:

$$
P\left\{f\left(Q - y + r - x + \left(1-\beta\right)\left(x-r\right)^+\right) \leqslant F\right\} \geqslant \gamma. \tag{3.21}
$$

The above constraint forces the probability that total used space is within maximum available storage space to be no smaller than γ. By using the chance-constrained programming technique which is originally developed by Charnes and Cooper [9] and considering Markov inequality, the random storage space constraint is transformed to a crisp constraint, as given below.

$$
\begin{aligned}
&\rightarrow \gamma \leqslant P\left\{f\left(x - \left(1-\beta\right)\left(x-r\right)^+ + y\right) + F \geqslant f\left(Q+r\right)\right\} \\
&\qquad \leqslant \frac{f\left(E\left(x\right) - \left(1-\beta\right)E\left(x-r\right)^+ + E\left(y\right)\right) + F}{f\left(Q+r\right)} \\
&\rightarrow \gamma \leqslant \frac{f\left(E\left(x\right) - \left(1-\beta\right)E\left(x-r\right)^+ + E\left(y\right)\right) + F}{f\left(Q+r\right)} \\
&\rightarrow \gamma f\left(Q+r\right) - F - f\left(E\left(x\right) + E(y)\right) + f\left(1-\beta\right)E(x-r)^+ \leqslant 0.
\end{aligned}
\tag{3.22}
$$

In this study, a restriction on maximum inventory investment has been considered. Warehousing inventory causes to lose the opportunity of investments in the other places and system managers would like to control it by considering this limitation on inventory system. In this paper, we assume that the purchasing cost is paid at the time an order is placed. Considering this assumption, the maximum inventory investment will occur at the time of order placing. For this reason, we establish a limitation on maximum inventory investment. Hence, if the purchasing cost payment is due at the time an order is placed then budget constraint is written as

$$P\left\{C_p\left(Q-y+r\right)\leqslant B\right\}\geqslant\varphi.$$

Hence, considering chance-constrained programming technique and Markov inequality, it can be written as:

$$\rightarrow\varphi\leqslant P\left(C_p y+B\geqslant C_p\left(Q+r\right)\right)\leqslant\frac{C_p E\left(y\right)+B}{C_p\left(Q+r\right)}$$

$$\rightarrow\varphi C_p\left(Q+r\right)-B-C_p E\left(y\right)\leqslant 0. \tag{3.23}$$

When the lead time demand X follows a normal probability density function (p.d.f) $f_X\left(x\right)$ with mean $E\left(x\right)=\mu_D L$ and standard deviation $\sigma_x=\sigma_D\sqrt{L}$. By giving the reorder point $r=E(x)+k\sigma_x$, the expected shortages quantity at the end of cycle $E\left(x-r\right)^+$ can be expressed by:

$$E(x-r)^+=\int_r^\infty\left(x-r\right)f\left(x\right)\mathrm{d}x,\qquad k=\frac{r-E(x)}{\sigma_x},\quad z=\frac{x-E(x)}{\sigma_x} \tag{3.24}$$

$$\rightarrow E(x-r)^+=\sigma_x\int_k^\infty\left(z-k\right)f\left(z\right)\mathrm{d}z$$

$$\rightarrow E(x-r)^+=\sigma_D\sqrt{L}\left[\int_k^\infty zf\left(z\right)\mathrm{d}z-k\overline{\Phi(k)}\right],$$

$$\left[\int_k^\infty zf\left(z\right)\mathrm{d}z-k\overline{\Phi(k)}\right]=\left[\phi\left(k\right)-k\overline{\Phi(k)}\right]=\psi(k)$$

$$\rightarrow E(x-r)^+=\sigma_D\sqrt{L}\psi(k)\geqslant 0. \tag{3.25}$$

Where ϕ is standard normal density function and $\bar{\Phi}$ is the standard normal complementary cumulative distribution function. Also it is noted that $\psi(k)$ is standard normal loss function and its amount can be found in Silver and Peterson (1985) (pp. 699–708).

Therefore, considering $r = \mu_D L + k\sigma_D\sqrt{L}$ and $E(x-r)^+ = \sigma_D\sqrt{L}\psi(k)$ the constrained model for discarding imperfect units is transformed as follows:

$$\begin{aligned} EAC^N(Q, A, k, L) =& \theta \text{bln}\left(\frac{A_0}{A}\right) + \frac{D}{Q(1-E(p))} \\ & \times \left[A + c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)\right] \\ & + \frac{h}{2}\left\{Q(1-E(p)) + \frac{Q\left[E\left(p^2\right) - E^2(p)\right]}{1-E(p)} + \frac{E(p(1-p))}{1-E(p)}\right\} \\ & + h\left[k\sigma_D\sqrt{L} + (1-\beta)\sigma_D\sqrt{L}\psi(k)\right] \\ & + \frac{D}{Q(1-E(p))}\left[(\pi + \pi_\circ(1-\beta))\sigma_D\sqrt{L}\psi(k)\right] \\ & + \frac{D}{1-E(p)}\delta. \end{aligned}$$

Subject to

$$\begin{aligned} & \gamma f\left(Q + \mu_D L + k\sigma_D\sqrt{L}\right) - F - f(\mu_D L + E(y)) + f(1-\beta)\sigma_D\sqrt{L}\psi(k) \leqslant 0 \\ & \varphi C_p\left(Q + \mu_D L + k\sigma_D\sqrt{L}\right) - B - C_p E(y) \leqslant 0 \\ & Q, k \geqslant 0,\ L \,\epsilon\, [L_i, L_{i-1}],\ A \epsilon (0, A_0]. \end{aligned} \tag{3.26}$$

The above model (3.26) can be solved with Lagrange multiplier method as given below:

$$\begin{aligned} & EAC^N(Q, A, k, L, \lambda_1, \lambda_2) = \theta \text{bln}\left(\frac{A_0}{A}\right) + \frac{D}{Q(1-E(p))} \\ & \times \left[A + c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)\right] + \frac{D}{1-E(p)}\delta \\ & + \frac{h}{2}\left\{Q(1-E(p)) + \frac{Q\left[E\left(p^2\right) - E^2(p)\right]}{1-E(p)} + \frac{E(p(1-p))}{1-E(p)}\right\} \\ & + h\left[k\sigma_D\sqrt{L} + (1-\beta)\sigma_D\sqrt{L}\psi(k)\right] + \frac{D}{Q(1-E(p))}\left[(\pi + \pi_\circ(1-\beta))\sigma_D\sqrt{L}\psi(k)\right] \\ & + \lambda_1\left[\gamma f\left(Q + \mu_D L + k\sigma_D\sqrt{L}\right) - F - f(\mu_D L + E(y)) + f(1-\beta)\sigma_D\sqrt{L}\psi(k)\right] \\ & + \lambda_2\left[\varphi C_p\left(Q + \mu_D L + k\sigma_D\sqrt{L}\right) - B - C_p E(y)\right], \end{aligned} \tag{3.27}$$

where λ_1 and λ_2 are nonnegative Lagrange multipliers.

Notice that for any given $(Q, A, k, \lambda_1, \lambda_2)$, $EAC^N(Q, A, k, L, \lambda_1, \lambda_2)$ is a concave function in $L \in [L_i, L_{i-1}]$, because

$$\begin{aligned}\frac{\partial^2 EAC^N(Q, A, k, L, \lambda_1, \lambda_2)}{\partial L^2} = &-\frac{1}{4}L^{-\frac{3}{2}}\left\{h\left[k\sigma_D + (1-\beta)\,\psi(k)\sigma_D\right]\right.\\ &+\left[\frac{D}{Q(1-E(p))}(\pi+\pi_0(1-\beta))\psi(k)\sigma_D\right]\\ &\left.+\lambda_1 f\left[\gamma k\sigma_D + (1-\beta)\,\psi(k)\sigma_D\right] + \lambda_2 C_p k\sigma_D\right\} \leqslant 0.\end{aligned} \tag{3.28}$$

Hence, for fixed $(Q, A, k, \lambda_1, \lambda_2)$, the minimum expected annual cost will occur at the end of point of the interval $L \in [L_i, L_{i-1}]$. It can be shown that for a given value of $L \in [L_i, L_{i-1}]\,, EAC^N(Q, A, k, L, \lambda_1, \lambda_2)$, is a convex function of $(Q, k, \lambda_1, \lambda_2)$. (See Appendix A for detailed proof). Therefore, for fixed $L \in [L_i, L_{i-1}]$, the Kuhn–Tucker necessary conditions for minimization of the function (3.27) are as follows:

$$\begin{aligned}&\frac{\partial EAC^N(Q, A, k, \lambda_1, \lambda_2)}{\partial Q}\\ &= -\frac{D}{Q^2(1-E(p))}\left[\left(A + c_i(L_{i-1}-L) + \sum\nolimits_{j=1}^{i-1} c_j(b_j - a_j)\right)\right.\\ &\quad\left.+(\pi+\pi_0(1-\beta))\,\sigma_D\sqrt{L}\psi(k)\right]\\ &\quad+\frac{h}{2}\left((1-E(p)) + \frac{\left[E(p^2) - E^2(p)\right]}{1-E(p)}\right)\\ &\quad+\lambda_1\gamma f + \lambda_2\varphi C_p = 0\end{aligned} \tag{3.29}$$

$$\begin{aligned}&\frac{\partial EAC^N(Q, A, k, \lambda_1, \lambda_2)}{\partial k}\\ &= h\sigma_D\sqrt{L} - \left[h(1-\beta)\,\sigma_D\sqrt{L}\,\overline{\Phi(k)}\right]\\ &\quad-\left[\frac{D}{Q(1-E(p))}(\pi+\pi_0(1-\beta))\,\overline{\Phi(k)}\sigma_D\sqrt{L}\right]\\ &\quad+\lambda_1 f\left[\gamma\sigma_D\sqrt{L} - (1-\beta)\,\sigma_D\sqrt{L}\,\overline{\Phi(k)}\right] + \lambda_2 C_p\sigma_D\sqrt{L} = 0\end{aligned} \tag{3.30}$$

$$\frac{\partial EAC^N(Q, A, k, \lambda_1, \lambda_2)}{\partial A} = -\frac{\theta b}{A} + \frac{D}{Q(1-E(p))} = 0 \tag{3.31}$$

$$\lambda_1\left[\gamma f\left(Q + \mu_D L + k\sigma_D\sqrt{L}\right) - F - f(\mu_D L + E(y)) + f(1-\beta)\,\sigma_D\sqrt{L}\psi(k)\right] = 0 \tag{3.32}$$

$$\lambda_2\left[\varphi C_p\left(Q+\mu_D L+k\sigma_D\sqrt{L}\right)-B-C_pE\left(y\right)\right]=0. \tag{3.33}$$

Solving equation (3.29), Q is obtained as follows:

$$Q=\left[\frac{D\left\{\left(A+c_i(L_{i-1}-L)+\sum_{j=1}^{i-1}c_j(b_j-a_j)\right)+\left[(\pi+\pi_0\left(1-\beta\right))\sigma_D\sqrt{L}\psi(k)\right]\right\}}{\frac{h}{2}(1-2E\left(p\right)+E\left(p^2\right))+\lambda_1\gamma f+\lambda_2\varphi C_p}\right]^{\frac{1}{2}}. \tag{3.34}$$

As a result, the following solution procedure can be established to find the optimal (Q, A, r, L).

Step 1. For each L_i, $i=0,1,2,\ldots,n$ perform Step 1-1 to Step 1-6.

Step 1-1. Input the values of $D, h, k, \pi\pi_0, B, F, \delta, C_p$ and f.

Step 1-2. Put Q in equations (3.31), (3.32) and (3.33).

Step 1-3. Obtain $A^k, \lambda_1^k, \lambda_2^k$ by simultaneously solving equations (3.31), (3.32) and (3.33).

Step 1-4. Find Q^k by putting $A^k, \lambda_1^k, \lambda_2^k$ into equation (3.34).

Step 1-5 Put $Q^k, A^k, \lambda_1^k, \lambda_2^k$ in equation (3.27) and find $EAC^N\left(Q^k, A^k, \lambda_1^k, \lambda_2^k\right)$

Step 1-6. Compute $EAC\left(Q^k, A^k, \lambda_1^k, \lambda_2^k\right)$ in terms of different k and $\psi(k)$ and stop when you get to $\left(\left(Q^{k_i}, A^{k_i}, \lambda_1^{k_i}, \lambda_2^{k_i}\right)=(Q_i, A_i, k_i, L_i, \lambda_{1_i}, \lambda_{2_i})\right)$:

$$\begin{gathered}EAC^N\left(Q^{k_i}, A^{k_i}, \lambda_1^{k_i}, \lambda_2^{k_i}\right)<\operatorname{Min}\left\{EAC^N\left(Q^{k_i-\xi}, A^{k_i-\xi}, \lambda_1^{k_i-\xi}, \lambda_2^{k_i-\xi}\right),\right.\\ \left.EAC^N\left(Q^{k_i+\xi}, A^{k_i+\xi}, \lambda_1^{k_i+\xi}, \lambda_2^{k_i+\xi}\right)\right\}\\ \left|EAC^N\left(Q^{k_i}, A^{k_i}, \lambda_1^{k_i}, \lambda_2^{k_i}\right)-EAC^N\left(Q^{k_i-\xi}, A^{k_i-\xi}, \lambda_1^{k_i-\xi}, \lambda_2^{k_i-\xi}\right)\right|\leqslant\varepsilon\\ \left|EAC^N\left(Q^{k_i}, A^{k_i}, \lambda_1^{k_i}, \lambda_2^{k_i}\right)-EAC^N\left(Q^{k_i+\xi}, A^{k_i+\xi}, \lambda_1^{k_i+\xi}, \lambda_2^{k_i+\xi}\right)\right|\leqslant\varepsilon.\end{gathered}$$

Step 2. Compare A_i and A_0:

Step 2-1. If $A_i<A_0$, then the solution found in Step 1 is the optimal solution for L_i, and go to Step 3.

Step 2-2. If $A_i\geqslant A_0$, then set $A_i=A_0$ and find optimal solution by simultaneously solving equations (3.30), (3.32), (3.33) and (3.34) by a procedure similar to Step 1.

Step 3. For each $(Q_i, A_i, k_iL_i, \lambda_{1_i}\lambda_{2_i})$ compute $EAC^N\left(Q_i, A_i, k_i, L_i, \lambda_{1_i}, \lambda_{2_i}\right)$, $i=0,1,\ldots,n$.

Step 4. Find $\mathrm{MIN}_{i=0,1,\ldots,n}EAC^N\left(Q_i, A_i, k_i, L_i, \lambda_{1_i}, \lambda_{2_i}\right)=EAC^N\left(Q^*, A^*, k^*, L^*, \lambda_1, \lambda_2\right)$ and hence the optimal reorder point is $r^*=\mu_D L^*+k^*\sigma\sqrt{L^*}$.

4. Model formulation for distribution free approach

In many practical situations, the probability distributional information of the lead time demand is often limited in practices. Therefore, in this section, we relax the restriction about the normal distribution demand by only assuming that the lead time demand X has given finite first two moment (and hence, mean and variance are also given) (*i.e.*, the p.d.f. f_X of X belongs to the class F of the p.d.f. with mean $E\,(x) = \mu_D L$ and variance $\mathrm{Var}\,(x) = \sigma_x^2 = L\sigma_D^2$). Hence, using minimax distribution free procedure the problem can be expressed as follows:

$$\left\{\mathrm{Max}\, EAC\,(Q, A, r, L) = EAC^U\,(Q, A, r, L)\right\}.$$

Subject to

$$\gamma f\,(Q + r) - F - f\,(\mu_D L + E(y)) + f\,(1 - \beta)\, E(x - r)^+ \leqslant 0$$
$$\varphi C_p\,(Q + r) - B - C_p E\,(y) \leqslant 0$$

$$Q \geqslant 0,\ r \geqslant \mu_D L,\ L\,\epsilon\,[L_i, L_{i-1}]\,,\ A \in (0, A_0]\,. \tag{4.1}$$

For this purpose, we need the following proposition, which was asserted by Gallego and Moon [11]:

$$E\,(x-r)^+ \leqslant \frac{1}{2}\left[\sqrt{\mathrm{Var}\,(x) + (r-E(x))^2} - (r-E\,(x))\right] = \frac{1}{2}\left(\sqrt{1+k^2} - k\right)\sigma_D\sqrt{L}. \tag{4.2}$$

Considering $r = E\,(x) + k\sigma_x$ and $E(x - r)^+ \leqslant \frac{1}{2}\left(\sqrt{1 + k^2} - k\right)\sigma_D\sqrt{L}$, the expected annual cost of distribution free approach is changed as follows:

$$\begin{aligned}
EAC^U\,(Q, A, k, L) = {} & \theta b\ln\left(\frac{A_0}{A}\right) + \frac{D}{Q\,(1 - E\,(p))} \\
& \times\left[A + c_i\,(L_{i-1} - L) + \sum\nolimits_{j=1}^{i-1} c_j\,(b_j - a_j)\right] \\
& + \frac{h}{2}\left\{Q\,(1 - E\,(p)) + \frac{Q\left[E\left(p^2\right) - E^2\,(p)\right]}{1 - E\,(p)} + \frac{E\,(p\,(1 - p))}{1 - E\,(p)}\right\} \\
& + h\left[k\sigma_D\sqrt{L} + (1 - \beta)\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\sqrt{1 + k^2} - k\right)\right] \\
& + \frac{D}{Q\,(1 - E\,(p))}\left[(\pi + \pi_\circ\,(1 - \beta))\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\sqrt{1 + k^2} - k\right)\right] \\
& + \frac{D}{1 - E(p)}\delta.
\end{aligned}$$

Subject to

$$\begin{aligned}
\gamma f\left(Q + \mu_D L + k\sigma_D\sqrt{L}\right) - F - f\,(\mu_D L + E\,(y)) & \\
+ f\,(1 - \beta)\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\sqrt{1 + k^2} - k\right) & \leqslant 0
\end{aligned}$$

$$\varphi C_p \left(Q + \mu_D L + k\sigma_D \sqrt{L}\right) - B - C_p E\left(y\right) \leqslant 0$$

$$Q,\, k \geqslant 0,\, L\,\epsilon\, \left[L_i, L_{i-1}\right],\, A\epsilon\left(0, A_0\right], \tag{4.3}$$

where $EAC^U\left(Q, A, k, L\right)$ is the least upper bound of $EAC\left(Q, A, k, L\right)$.

In this section, model (4.3) is solved using Lagrange method and its function can be expressed as below:

$$\begin{aligned}
EAC^U\ \left(Q, A, k, L, \lambda_1, \lambda_2\right) &= \theta \text{bln}\left(\frac{A_0}{A}\right) + \frac{D}{Q\left(1 - E\left(p\right)\right)} \\
&\times \left[A + c_i\left(L_{i-1} - L\right) + \sum\nolimits_{j=1}^{i-1} c_j\left(b_j - a_j\right)\right] \\
&+ h\left\{Q\left(1 - E\left(p\right)\right) + \frac{Q\left[E\left(p^2\right) - E^2\left(p\right)\right]}{1 - E(p)} + \frac{E\left(p\left(1 - p\right)\right)}{1 - E\left(p\right)}\right\} \\
&+ h\left[k\sigma_D\sqrt{L} + \left(1 - \beta\right)\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\sqrt{1 + k^2} - k\right)\right] \\
&+ \frac{D}{Q\left(1 - E\left(p\right)\right)}\left[\left(\pi + \pi_\circ\left(1 - \beta\right)\right)\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\sqrt{1 + k^2} - k\right)\right] \\
&+ \frac{D}{1 - E(p)}\delta + \lambda_1\left[\gamma f\left(Q + \mu_D L + k\sigma_D\sqrt{L}\right) - F - f\left(\mu_D L + E(y)\right)\right. \\
&\left. + f\left(1 - \beta\right)\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\sqrt{1 + k^2} - k\right)\right] \\
&+ \lambda_2\left[\varphi C_p\left(Q + \mu_D L + k\sigma_D\sqrt{L}\right) - B - C_p E\left(y\right)\right].
\end{aligned} \tag{4.4}$$

Notice that for any given $(Q, A, k, \lambda_1, \lambda_2)$, $EAC^U\left(Q, A, k, L, \lambda_1, \lambda_2\right)$ is a concave function in $L \in [L_i, L_{i-1}]$, because

$$\begin{aligned}
&\frac{\partial^2 EAC^U\left(Q, A, k, L, \lambda_1, \lambda_2\right)}{\partial L^2} = \\
&-\frac{1}{2}L^{-\frac{3}{2}}\left\{h\left[k\left(\frac{\sigma_D}{2}\right) + (1 - \beta)\left(\sqrt{1 + k^2} - k\right)\left(\frac{\sigma_D}{4}\right)\right]\right. \\
&+ \left[\frac{D}{Q\left(1 - E\left(p\right)\right)}\left(\pi + \pi_0(1 - \beta)\right)\left(\sqrt{1 + k^2} - k\right)\left(\frac{\sigma_D}{4}\right)\right] + \lambda_1 f\left[\gamma k\left(\frac{\sigma_D}{2}\right)\right. \\
&\left.\left. + \left(1 - \beta\right)\left(\sqrt{1 + k^2} - k\right)\left(\frac{\sigma_D}{4}\right)\right] + \lambda_2 C_p \varphi k\left(\frac{\sigma_D}{2}\right)\right\} \leqslant 0.
\end{aligned} \tag{4.5}$$

For this reason, similar to previous section, for fixed $(Q, A, k, \lambda_1, \lambda_2)$, the minimum expected annual cost will occur at the end of point of the interval $L \in [L_i, L_{i-1}]$. It can be shown that for a given value of $L \in [L_i, L_{i-1}]$ $EAC^N(Q, A, k, L, \lambda_1, \lambda_2)$, is a convex function of $(Q, A, k, \lambda_1, \lambda_2)$.

(See Appendix B for detailed proof). Therefore, for fixed $L \in [L_i, L_{i-1}]$, the Kuhn-Tucker necessary conditions for minimization of the function (4.4) are as follows:

$$\begin{aligned}\frac{\partial EAC^U(Q,A,k,\lambda_1,\lambda_2)}{\partial Q} = &-\frac{D}{Q^2(1-E(p))}\\ &\times\left[\left(A + c_i(L_{i-1}-L) + \sum\nolimits_{j=1}^{i-1} c_j(b_j-a_j)\right)\right.\\ &\left.+(\pi+\pi_0(1-\beta))\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\sqrt{1+k^2}-k\right)\right]\\ &+\frac{h}{2}\left((1-E(p)) + \frac{\left[E(p^2)-E^2(p)\right]}{1-E(p)}\right)\\ &+\lambda_1\gamma f + \lambda_2\varphi C_p = 0 \end{aligned} \tag{4.6}$$

$$\begin{aligned}\frac{\partial EAC^U(Q,A,k,\lambda_1,\lambda_2)}{\partial k} = &\ h\sigma_D\sqrt{L} + \left[h(1-\beta)\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\frac{k}{\sqrt{1+k^2}}-1\right)\right]\\ &+\left[\frac{D}{Q(1-E(p))}(\pi+\pi_0(1-\beta))\left(\frac{k}{\sqrt{1+k^2}}-1\right)\left(\frac{\sigma_D\sqrt{L}}{2}\right)\right]\\ &+\lambda_1 f\left[\gamma\sigma_D\sqrt{L} + (1-\beta)\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\frac{k}{\sqrt{1+k^2}}-1\right)\right]\\ &+\lambda_2\varphi C_p\sigma_D\sqrt{L} = 0 \end{aligned} \tag{4.7}$$

$$\frac{\partial EAC^U(Q,A,k,\lambda_1,\lambda_2)}{\partial A} = -\frac{\theta b}{A} + \frac{D}{Q(1-E(p))} = 0 \tag{4.8}$$

$$\begin{aligned}\lambda_1\Big[\gamma f\left(Q+\mu_D L + k\sigma_D\sqrt{L}\right) - F - f(\mu_D L + E(y))\\ +f(1-\beta)\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\sqrt{1+k^2}-k\right)\Big] = 0\end{aligned} \tag{4.9}$$

$$\lambda_2\left[\varphi C_p\left(Q+\mu_D L + k\sigma_D\sqrt{L}\right) - B - C_p E(y)\right] = 0. \tag{4.10}$$

Solving equation (4.6), Q is obtained as follows:

$$Q = \left[\frac{D\left\{\left(A+c_i(L_{i-1}-L)+\sum_{j=1}^{i-1}c_j(b_j-a_j)\right)+\left[(\pi+\pi_0(1-\beta))\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\sqrt{1+k^2}-k\right)\right]\right\}}{\frac{h}{2}(1-2E(p)+E(p^2)) + \lambda_1\gamma f + \lambda_2\varphi C_p}\right]^{\frac{1}{2}}. \tag{4.11}$$

Consequently, we can establish the following algorithm to find the optimal (Q, A, r, L).

Step 1. For each L_i, $i = 0, 1, 2, \ldots, n$ perform Step 1-1 to Step 1-4.
Step 1-1. Input the values of $D, h, \pi, \pi_0, B, F, \delta, C_p$ and f
Step 1-2. Put Q in equations (4.7), (4.8), (4.9) and (4.10).
Step 1-3. Obtain $A_i, k_i, \lambda_{1_i}, \lambda_{2_i}$ by simultaneously solving equations (4.7), (4.8), (4.9) and (4.10).
Step 1-4. Find Q_i by putting $A_i, k_i, \lambda_{1_i}, \lambda_{2_i}$ into equation (4.11).
Step 2. Compare A_i and A_0:
Step 2-1. If $A_i < A_0$, then the solution found in Step 1 is the optimal solution, and go to Step 3.
Step 2-2. If $A_i \geqslant A_0$, then set $A_i = A_0$ and find optimal solution by simultaneously solving equations (4.7), (4.9), (4.10) and (4.11) by a procedure similar to Step 1.
Step 3. For each $(Q_i, A_i, k_i, L_i, \lambda_{1_i}\lambda_{2_i})$ compute $EAC^U(Q_i, A_i, k_i, L_i, \lambda_{1_i}, \lambda_{2_i})$, $i = 0, 1, \ldots, n$
Step 4. Find $\mathrm{MIN}_{i=0,1,\ldots,n} EAC^U(Q_i, A_i, k_i, L_i, \lambda_{1_i}, \lambda_{2_i}) = EAC^U(Q^*, A^*, k^*, L^*, \lambda_1, \lambda_2)$ and hence the optimal reorder point is $r^* = \mu_D L^* + k^* \sigma \sqrt{L^*}$.

5. Numerical example

Example 1.

In this section, we solve some examples using the procedure outlined in the previous section. The purpose is to illustrate the solution procedure, conduct a sensitivity analysis for important model parameters and highlight important features of the developed models.

$$D = 600\,\text{unit/ year},\ A_0 = 200\$, h = 20\$,\ \mu_D = 13\,\text{unit/week}, \sigma_D = 4\,\text{unit/week}$$

$$\begin{aligned}&\pi = 50\$, \pi_0 = 100\$, C_p = 60\$\,, f = 1.5M^2,\\ &\gamma = 0.95\,,\ \varphi = 0.95,\ F = 170M^2, B = 11\,000\,\$\end{aligned}$$

$$\delta = 1.5\$,\ \varepsilon = 0.01.$$

Defective rate p has a Beta distribution function with parameters $\alpha = 1$ and $\beta = 4$; that is, the p.d.f. of p is given by:

$$g(p) = 4(1-p)^3\,,\ \ 0 < p < 1$$

Therefore, we have:

$$E(p) = \frac{\alpha}{\alpha+\beta} = 0.2 \ \text{ and } \ E\left(p^2\right) = \frac{\alpha(\alpha+\beta)}{(\alpha+\beta)(\alpha+\beta+1)} = 0.066.$$

TABLE 1. Lead time data.

Lead time component i	Normal duration b_i (days)	Minimum duration a_i (days)	Unit crashing cost c_i ($/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

TABLE 2. Results of solution procedure for perfect demand (L_i in weeks).

β	L_i	$C(L_i)$	Q_i	$r_i(k_i)$	A^*	EAC^N (Q_i, r_i, L_i)	λ_1, λ_2
0.	8	0	83.04	127.41(2.07)	110.72	3979.45	0, 0.11
	6	5.6	120.69	97.69(2.01)	160.93	3844.71*	0, 0.02
	4	22.4	134.41	67.84(1.98)	179.22	3847.78	0.49, 0
	3	57.4	136.59	52.44(1.94)	182.12	3988.68	1.58, 0
0.5	8	0	85.62	125.38(1.89)	114.16	3933.90	0, 0.10
	6	5.6	123.05	95.83(1.82)	164.07	3812.21*	0, 0.01
	4	22.4	136.36	66.32(1.79)	181.81	3821.48	0.39, 0
	3	57.4	138.27	51.12(1.75)	184.37	3963.88	1.47, 0
0.8	8	0	88.06	123.45(1.72)	117.41	3892.05	0, 0.09
	6	5.6	125.16	94.16(1.65)	166.88	3781.99*	0, 0.01
	4	22.4	138.21	64.88(1.61)	184.28	3796.95	0.31, 0
	3	57.4	139.88	49.87(1.57)	186.51	3940.71	1.37, 0
1	8	0	65.98	121.53(1.55)	120.66	3850.64	0, 0.09
	6	5.6	127.27	92.50(1.48)	169.70	3751.75*	0, 0.01
	4	22.4	140.08	63.44(1.43)	186.78	3772.32	0.23, 0
	3	57.4	141.51	48.63(1.39)	188.68	3917.39	1.28, 0

Besides, for the ordering cost reduction we take $\theta = 0.1$, $b = 10\,000$.

Suppose that the demand during lead time is normally distributed. The lead time has three components with data shown in Table 1. Applying the proposed solution procedure yields the results shown in Table 2 for $\beta = 0.0$, 0.5, 0.8, and 1. From Table 2, the optimal inventory policy can be founded by comparing $(Q_i, k_i, L_i, \lambda_{1_i}, \lambda_{2_i})$ for $i = 0, 1, 2, 3$ and the summary is presented in Table 3 and to see the effects of constrained problem, we tabulate optimal values of unconstrained problem in the same table.

Example 2.

In this example, we assume that the probability distribution of lead time demand is unknown. Applying the proposed solution procedure yields the results shown in Table 4 for $\beta = 0.0, 0.5, 0.8$, and 1, and the summary of results are tabulated in Table 5.

Also, some results of the various backorder rates against optimal values are as follows:

TABLE 3. Summary of results for perfect demand (L_i in weeks).

Constrained model					
β	Q^*	$r^*(k^*)$	A^*	L^*	EAC^N (Q^*, A^*, r^*, L^*)
0.0	120.69	97.69(2.01)	160.93	6	3844.71
0.5	123.05	95.83(1.82)	164.07	6	3812.21
0.8	125.16	94.16(1.65)	166.88	6	3781.99
1	127.27	92.50(1.48)	169.70	6	3751.75

Unconstrained model					
β	Q^*	$r^*(k^*)$	A^*	L^*	EAC^N (Q^*, A^*, r^*, L^*)
0.0	133.58	97.49(1.99)	178.11	6	3839.00
0.5	134.09	95.73(1.81)	178.79	6	3807.99
0.8	134.83	93.97(1.63)	179.77	6	3778.93
1	135.36	92.30(1.46)	180.48	6	3749.61

TABLE 4. Results of solution procedure for partial demand (L_i in weeks).

β	L_i	$C(L_i)$	Q_i	$r_i(k_i)$	A^*	EAC^U (Q_i, r_i, L_i)	λ_1, λ_2
0.	8	0	76.55	132.54(2.52)	102.21	5609.45	0, 0.56
	6	5.6	111.68	104.80(2.73)	148.91	4806.71	0, 0.19
	4	22.4	125.48	74.23(2.77)	167.32	4557.62*	5.12, 0
	3	57.4	129.15	57.73(2.70)	172.32	4601.54	5.55, 0
0.5	8	0	81.67	128.50(2.16)	108.87	5050.99	0, 0.39
	6	5.6	117.68	100.07(2.25)	156.97	4488.65	0, 0.13
	4	22.4	131.20	70.00(2.25)	174.91	4323.98*	3.62, 0
	3	57.4	134.05	54.12(2.18)	178.80	4398.28	4.22, 0
0.8	8	0	86.28	124.86(1.84)	115.05	4686.50	0, 0.29
	6	5.6	122.50	96.26(1.86)	163.35	4271.04	0, 0.09
	4	22.4	135.63	66.74(1.84)	181.21	4161.43*	2.60, 0
	3	57.4	137.86	51.31(1.77)	183.85	4255.52	3.34, 0
1	8	0	90.55	121.49(1.54)	120.73	4418.68	0, 0.23
	6	5.6	126.74	92.91(1.52)	169.83	4104.92	0, 0.07
	4	22.4	139.43	63.95(1.49)	186.52	4035.72*	1.90, 0
	3	57.4	141.11	48.94(1.43)	188.16	4144.30	2.70, 0

1. For unconstrained problem ($\lambda_1 = \lambda_2 = 0$), if $\beta = 1$, equation (4.4) reduces to the expected annual cost of backorder case and hence equation (4.11) becomes

$$Q = \left[\frac{D\left\{ \left(A + c_i\left(L_{i-1} - L\right) + \sum_{j=1}^{i-1} c_j\left(b_j - a_j\right) \right) + \left[\frac{\pi}{2}\left(\sqrt{1+k^2} - k\right)\sigma_D \sqrt{L} \right] \right\}}{\frac{h}{2}\left(1 - 2E(p) + E(p^2)\right)} \right]^{\frac{1}{2}}. \tag{5.1}$$

TABLE 5. Summary of results for partial demand (L_i in weeks).

Constrained model					
β	Q^*	$r^*(k^*)$	A^*	L^*	EAC^* (Q_i, r_i, L_i)
0.0	125.48	74.23(2.77)	167.32	4	4557.62
0.5	131.20	70.00(2.25)	174.91	4	4323.98
0.8	135.63	66.74(1.84)	181.21	4	4161.43
1.0	139.43	63.95(1.49)	186.52	4	4035.72

Unconstrained model					
β	Q^*	$r^*(k^*)$	A^*	L^*	EAC^* (Q_i, r_i, L_i)
0.0	172.43	74.14(2.76)	200	4	4430.09
0.5	167.18	69.87(2.23)	200	4	4252.54
0.8	163.40	66.60(1.82)	200	4	4120.24
1	160.46	63.85(1.48)	200	4	4012.54

When $\beta = 0$, equation (4.4) reduces to that of the lost sale case, and thus equation (4.11) becomes

$$Q = \left[\frac{D\left\{\left(A + c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)\right) + \left[\frac{(\pi+\pi_0)}{2}\left(\sqrt{1+k^2} - k\right)\sigma_D\sqrt{L}\right]\right\}}{\frac{h}{2}(1 - 2E(p) + E(p^2))}\right]^{\frac{1}{2}}. \tag{5.2}$$

Hence, for fixed L and k, comparing equations (5.1) and (5.2), we get $Q_{\beta=0} > Q_\beta > Q_{\beta=1}$, that is, the order quantity per cycle in the lost sale case for unconstrained problem is greater than backorder case.

However, for the constrained problem, if $\beta = 1$, equation (4.4) reduces to the expected annual cost of backorder case and hence equation (4.11) becomes

$$Q = \left[\frac{D\left\{\left(A + c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)\right) + \left[\frac{\pi}{2}\left(\sqrt{1+k^2} - k\right)\sigma_D\sqrt{L}\right]\right\}}{\frac{h}{2}(1 - 2E(p) + E(p^2)) + \lambda_1\gamma f + \lambda_2\varphi C_p}\right]^{\frac{1}{2}} \tag{5.3}$$

Also storage space constrained for backorder case is transformed as follows:

$$g_{\beta=1} = \gamma f\left(Q + \mu_D L + k\sigma_D\sqrt{L}\right) - F - f(\mu_D L + E(y)) \tag{5.4}$$

For $\beta = 0$, equation (4.4) reduces to the lost sale case, and thus equation (4.11) becomes

$$Q = \left[\frac{D\left\{\left(A + c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)\right) + \left[\frac{(\pi+\pi_0)}{2}\left(\sqrt{1+k^2} - k\right)\sigma_D\sqrt{L}\right]\right\}}{\frac{h}{2}(1 - 2E(p) + E(p^2)) + \lambda_1\gamma f + \lambda_2\varphi C_p}\right]^{\frac{1}{2}} \tag{5.5}$$

TABLE 6. Comparison of two procedure.

β	EAC^U (Q^*, A^*, r^*, L^*)	EAC^N (Q^*, A^*, r^*, L^*)	$EVAI$	Cost penalty
0.	4557.62	3844.71	712.91	1.185
0.5	4323.98	3812.21	511.77	1.134
0.8	4161.43	3781.99	379.44	1.100
1.0	4035.72	3751.75	283.97	1.075

Furthermore, storage space constraint for lost sale case is as follows:

$$\begin{aligned} g_{\beta=0} =& \gamma f\left(Q + \mu_D L + k\sigma_D\sqrt{L}\right) - F - f\left(\mu_D L + E(y)\right) \\ &+ f\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\sqrt{1+k^2} - k\right). \end{aligned} \tag{5.6}$$

Therefore, for fixed L and k storage space constraint for lost sale case is larger than backorder case ($g_{\beta=0} > g_{\beta=1}$). Hence,$\lambda_{\beta=0} > \lambda_{\beta=1}$. So, in the case of constrained problem, $Q_{\beta=0} < Q_\beta < Q_{\beta=1}$. For this reason, the order quantity per cycle in the backorder case is greater than lost sale case. Similarly, for normally distributed demand case parallel results can be obtained.

2. The effect of β on the minimum of the expected annual cost, say EAC^*_β, may be examined. It has the minimum value when $\beta = 1$ (backorder case) and the maximum value when $\beta = 0$ (lost sale case). Hence, for $0 < \beta < 1$, $EAC^*_{\beta=1} < EAC^*_\beta < EAC^*_{\beta=0}$ for both normally distributed and distribution free demand problems.

3. Increasing the value of backorder rate (β), will result in a decrease in safety factor and reorder point for both partial and perfect lead time demand problems.

5. for a fixed β, if lead time increases then the order quantity decreases.

Moreover, The expected value of additional information, EVAI, is the largest amount that one is willing to pay for the knowledge of the form of the lead time demand distribution and in equal to $EAC^U\,(Q^*, A^*, r^*, L^*) - EAC^N\,(Q^*, A^*, r^*, L^*)$. Also, the cost penalty is the ratio of the approximate expected annual cost (partial demand information) to the expected one (perfect demand information). From Table 6, it is observed that the cost performance of distribution free approach is improving as β gets larger.

In Table 7, we obtain optimal values in terms of different maximum available space and inventory investment for partial and perfect lead time demand and minimum cost is shown for $\beta = 0.5$. In this case, with an increase in maximum permissible (F) and keeping the remaining parameters fixed, the optimal order quantity (Q^*) and optimal reorder point (r^*) increase, however the optimal reorder point (r^*) and expected annual cost (EAC^*) decreases. From an economic view point, it implies that when maximum permissible inventory investment arises, the

Table 7. Effect of changes in maximum available space (F) and inventory investment (B).

	Partial demand information		Perfect demand information	
	$(Q^*, A^*, L^*, r^*, k^*, \lambda_1, \lambda_2)$	EAC^U	$(Q^*, A^*, L^*, r^*, k^*, \lambda_1, \lambda_2)$	EAC^N
F				
130	(95.39, 127.30, 4, 70.21, 2.27, 9.77, 0)	4574.22	(97.46, 129.95, 6, 96.32, 1.87, 2.99, 0)	3861.58
150	(113.18, 150.92, 4, 70.20, 2.27, 6.04, 0)	4418.79	(115.61, 154.15, 6, 96.02, 1.84, 1.25, 0)	3820.21
190	(149.27, 199.02, 4, 69.77, 2.22, 1.85, 0)	4270.54	(122.93, 163.91, 6, 95.93, 1.83, 0, 0.01)	3812.21
210	(156.03, 200.00, 4, 69.80, 2.22, 0, 0.02)	4259.24	(122.93, 163.91, 6, 95.93, 1.83, 0, 0.01)	3812.12
B				
8000	(89.06, 118.60, 4, 70.03, 2.25, 0, 0.29)	4652.88	(55.14, 73.53, 6, 96.81, 1.92, 0, 0.3)	4173.99
9000	(111.16, 148.22, 4, 70.12, 2.26, 0, 0.16)	4433.54	(77.74, 103.65, 6, 96.51, 1.89, 0, 0.14)	3956.73
10 000	(131.17, 174.91, 4, 70.03, 2.25, 3.58, 0)	4323.98	(100.33, 133.78, 6, 96.22, 1.86, 0, 0.06)	3852.81
11 000	(131.17, 174.91, 4, 70.03, 2.25, 3.58, 0)	4323.98	(122.93, 163.91, 6, 95.93, 1.83, 0, 0.01)	3812.21

maximum permissible inventory level increases, then order quantity (Q^*) should be increased to diminish expected annual cost (EAC^*). The same results can be obtained in increasing maximum inventory investment (B).

6. Conclusion

This paper studied the effect budget and storage space constraints for the continuous review inventory model when the amount received is uncertain due to rejection during inspection. Two models with objective of minimizing the expected annual cost are formulated and analyzed. The first model considers lead time demand follows a normal distribution. The second model relaxes the assumption about form of probability distribution of lead time demand and applies a minimax distribution free procedure to solve the problem. Also, in each case, we show that the constrained model is jointly convex on the decision variable. Moreover, two numerical examples are presented to illustrate the important issues related to the proposed models. The proposed model can be extended in several directions. For instance we may consider inflationary condition in the models or consider sampling inspection to the model.

Acknowledgements. The authors acknowledge the constructive comments made by the anonymous referees on an earlier version of paper.

Appendix A.

For a given value of L, we obtain the Hessian matrix H for objective function as follows:

$$\begin{bmatrix} \frac{\partial^2 EAC^N(Q,A,k)}{\partial Q^2} & \frac{\partial^2 EAC^N(Q,A,k)}{\partial Q\partial k} & \frac{\partial^2 EAC^N(Q,A,k)}{\partial Q\partial A} \\ \frac{\partial^2 EAC^N(Q,A,k)}{\partial k\partial Q} & \frac{\partial^2 EAC^N(Q,A,k)}{\partial k^2} & \frac{\partial^2 EAC^N(Q,A,k)}{\partial k\partial A} \\ \frac{\partial^2 EAC^N(Q,A,k)}{\partial A\partial Q} & \frac{\partial^2 EAC^N(Q,A,k)}{\partial A\partial k} & \frac{\partial^2 EAC^N(Q,A,k)}{\partial A^2} \end{bmatrix}$$

where

$$\frac{\partial^2 EAC^N(Q,A,k)}{\partial Q^2} = \frac{2D}{Q^3(1-E(p))}\left\{(A+C(L)) + \left[\pi_1\psi(k)\sigma_D\sqrt{L}\right]\right\} \tag{A.1}$$

$$\frac{\partial^2 EAC^N(Q,A,k)}{\partial k^2} = \left[\phi(k)h(1-\beta)\sigma_D\sqrt{L}\right] + \left[\frac{D}{Q(1-E(p))}\pi_1\sigma_D\sqrt{L}\phi(k)\right] \tag{A.2}$$

$$\frac{\partial^2 EAC^N(Q,A,k)}{\partial A^2} = \frac{\theta b}{A^2} \tag{A.3}$$

$$\frac{\partial^2 EAC^N(Q,A,k)}{\partial Q\partial k} = \frac{\partial^2 EAC^N(Q,A,k)}{\partial k\partial Q} = \frac{D}{Q^2(1-E(p))}\pi_1\sigma_D\sqrt{L}\,\overline{\Phi(k)} \tag{A.4}$$

$$\frac{\partial^2 EAC^N(Q,A,k)}{\partial Q\partial A} = \frac{\partial^2 EAC^N(Q,A,k)}{\partial A\partial Q} = -\frac{D}{Q^2(1-E(p))} \tag{A.5}$$

$$\frac{\partial^2 EAC^N(Q,A,k)}{\partial A\partial k} = \frac{\partial^2 EAC^N(Q,A,k)}{\partial k\partial A} = 0 \tag{A.6}$$

where

$$\pi_1 = (\pi + \pi_0(1-\beta)) \tag{A.7}$$

$$C(L) = c_i(L_{i-1} - L) + \sum\nolimits_{j=1}^{i-1} c_j(b_j - a_j). \tag{A.8}$$

The first principal minor of H for objective function is

$$\begin{aligned} |H_{11}| &= \frac{\partial^2 EAC^N(Q,A,k)}{\partial Q^2} \\ &= \frac{2D}{Q^3(1-E(p))}\left\{(A+C(L)) + \left[\pi_1\psi(k)\sigma_D\sqrt{L}\right]\right\} > 0. \end{aligned} \tag{A.9}$$

The second principal minor of H for objective function is

$$\begin{aligned}|H_{22}| &= \frac{\partial^2 EAC^N(Q,A,k)}{\partial Q^2} \times \frac{\partial^2 EAC^N(Q,A,k)}{\partial k^2} - \left(\frac{\partial^2 EAC^N(Q,A,k)}{\partial k \partial Q}\right)^2 \\ &= \frac{2D^2}{Q^4(1-E(p))^2}\left(A + C(L) + \pi_1 \psi(k)\sigma_D\sqrt{L}\right)\pi_1\sigma_D\sqrt{L}\phi(k) \\ &+ \frac{2D}{Q^3(1-E(p))}\left(A + C(L) + \pi_1\sigma_D\sqrt{L}\psi(k)\right)h\sigma_D\sqrt{L}(1-\beta)\phi(k) \\ &- \frac{D^2}{Q^4(1-E(p))^2}\pi_1^2\left(\sigma_D\sqrt{L}\right)^2\left(\varPhi\bar{(k)}\right)^2 \\ &= \frac{2D^2}{Q^4(1-E(p))^2}(A + C(L))\pi_1\sigma_D\sqrt{L}\phi(k) \\ &+ \frac{2D}{Q^3(1-E(p))}\left(A + C(L) + \pi_1\sigma_D\sqrt{L}\psi(k)\right)h\sigma_D\sqrt{L}(1-\beta)\phi(k) \\ &+ \frac{D^2}{Q^4(1-E(p))^2}\pi_1^2\left(\sigma_D\sqrt{L}\right)^2\left[2\phi(k)\psi(k) - \left(\varPhi\bar{(k)}\right)^2\right] \end{aligned} \tag{A.10}$$

Let $\zeta(k) = 2\phi(k)\psi(k) - \left(\varPhi\bar{(k)}\right)^2$. From (A.10), we can find that to verify $|H_{22}| > 0$ only need to prove that $\zeta(k) > 0$ since all other terms are positive. We know that $\lim_{k\to 0^+}\zeta(k) = 1$ and $\lim_{k\to +\infty}\zeta(k) = 0$. Also we know $d\zeta(k)/dk < 0$. Hence, $\zeta(k)$ is a decreasing function of k and $\zeta(k)$ is positive.

The third principal minor of H for objective function is:

$$\begin{aligned}|H_{33}| = \frac{\theta b}{A^2}|H_{22}| + \left(\frac{D}{Q^2(1-E(p))}\right)^2 \Big\{ &\left[\phi(k) h(1-\beta)\sigma_D\sqrt{L}\right] \\ &+ \left[\frac{D}{Q(1-E(p))}\pi_1\sigma_D\sqrt{L}\phi(k)\right]\Big\} > 0. \end{aligned} \tag{A.11}$$

It follows from $|H_{11}| > 0$, $|H_{22}| > 0$ and $|H_{33}| > 0$ that H is positive finite. That is to say, objective function is a convex function in (Q,k) for any $L \in [L_i, L_{i-1}]$

Moreover, for a given value of $L \in [L_i, L_{i-1}]$, storage space constraint $(g(Q,k,L))$ is a convex function of (Q,k), since

$$g(Q,k) = \gamma f\left(Q + \mu_D L + k\sigma_D\sqrt{L}\right) - F - f(\mu_D L + E(y)) + f(1-\beta)\sigma_D\sqrt{L}\psi(k) \tag{A.12}$$

$$\frac{\partial^2 g(Q,k)}{\partial k^2} = f(1-\beta)\sigma_D\sqrt{L}\phi(k) > 0. \tag{A.13}$$

Because, Also for a fixed $L \in [L_i, L_{i-1}]$, budget constraint is a linear function of (Q,k). Consequently, the necessary KKT conditions for the constrained inventory problem are also sufficient. So in this case, for a fixed value of $L \in [L_i, L_{i-1}]$, if there exists a solution (Q^*, A^*, k^*) that satisfies the KKT necessary conditions, then (Q^*, A^*, k^*) is an optimal solution of our constrained inventory problem. In fact, (Q^*, A^*, k^*) is the global minimum.

Appendix B.

For a given value of L, we obtain the Hessian matrix H for objective function as follows:

$$\begin{bmatrix} \frac{\partial^2 EAC^U(Q,A,k)}{\partial Q^2} & \frac{\partial^2 EAC^U(Q,A,k)}{\partial Q \partial k} & \frac{\partial^2 EAC^U(Q,A,k)}{\partial Q \partial A} \\ \frac{\partial^2 EAC^U(Q,A,k)}{\partial k \partial Q} & \frac{\partial^2 EAC^U(Q,A,k)}{\partial k^2} & \frac{\partial^2 EAC^U(Q,A,k)}{\partial k \partial A} \\ \frac{\partial^2 EAC^U(Q,A,k)}{\partial A \partial Q} & \frac{\partial^2 EAC^U(Q,A,k)}{\partial A \partial k} & \frac{\partial^2 EAC^U(Q,A,k)}{\partial A^2} \end{bmatrix}$$

where

$$\frac{\partial^2 EAC^U(Q,A,k)}{\partial Q^2} = \frac{2D}{Q^3(1-E(p))}\left\{(A + C(L)) + \left[\pi_1 \rho(k) \frac{\sigma_D \sqrt{L}}{2}\right]\right\} \tag{B.1}$$

$$\frac{\partial^2 EAC^U(Q,A,k)}{\partial k^2} = \left[\tau(k) h (1-\beta)\left(\frac{\sigma_D \sqrt{L}}{2}\right)\right] + \left[\frac{D}{Q(1-E(p))}\pi_1 \left(\frac{\sigma_D \sqrt{L}}{2}\right)\tau(k)\right] \tag{B.2}$$

$$\frac{\partial^2 EAC^U(Q,A,k)}{\partial A^2} = \frac{\theta b}{A^2} \tag{B.3}$$

$$\frac{\partial^2 EAC^U(Q,A,k)}{\partial Q \partial k} = \frac{\partial^2 EAC^U(Q,A,k)}{\partial k \partial Q} = -\frac{D}{Q^2(1-E(p))}\pi_1 \left(\frac{\sigma_D \sqrt{L}}{2}\right)\eta(k) \tag{B.4}$$

$$\frac{\partial^2 EAC^U(Q,A,k)}{\partial Q \partial A} = \frac{\partial^2 EAC^U(Q,A,k)}{\partial A \partial Q} = -\frac{D}{Q^2(1-E(p))} \tag{B.5}$$

$$\frac{\partial^2 EAC^U(Q,A,k)}{\partial A \partial k} = \frac{\partial^2 EAC^U(Q,A,k)}{\partial k \partial A} = 0 \tag{B.6}$$

where

$$\rho(k) = \left(\sqrt{1+k^2} - k\right) > 0 \tag{B.7}$$

$$\pi_1 = (\pi + \pi_0(1-\beta)) \tag{B.8}$$

$$\tau(k) = \left(\frac{1}{\sqrt{1+k^2}} - \frac{k^2}{(1+k^2)^{\frac{3}{2}}}\right) > 0 \tag{B.9}$$

$$\eta(k) = \left(\frac{k}{\sqrt{1+k^2}} - 1\right) < 0 \tag{B.10}$$

$$C(L) = c_i(L_{i-1} - L) + \sum\nolimits_{j=1}^{i-1} c_j(b_j - a_j). \tag{B.11}$$

The first principal minor of H for objective function is:

$$|H_{11}| = \frac{\partial^2 EAC^U(Q,A,k)}{\partial Q^2} = \frac{2D}{Q^3(1-E(p))}\left\{(A+C(L))+\left[\pi_1 \frac{\sigma_D\sqrt{L}}{2}\rho(k)\right]\right\} > 0. \tag{B.12}$$

The second principal minor of H for objective function is:

$$\begin{aligned}
|H_{22}| =& \frac{\partial^2 EAC^U(Q,A,k)}{\partial Q^2} \times \frac{\partial^2 EAC^U(Q,A,k)}{\partial k^2} - \left(\frac{\partial^2 EAC^U(Q,A,k)}{\partial k \partial Q}\right)^2 \\
=& \frac{2D^2}{Q^4(1-E(p))^2}\left(A+C(L)+\pi_1\left(\frac{\sigma_D\sqrt{L}}{2}\right)\rho(k)\right)\pi_1 \\
&\times \left(\frac{\sigma_D\sqrt{L}}{2}\right)\tau(k) + \frac{2D}{Q^3(1-E(p))} \\
&\times \left(A+C(L)+\pi_1\left(\frac{\sigma_D\sqrt{L}}{2}\right)\rho(k)\right)h\left(\frac{\sigma_D\sqrt{L}}{2}\right)(1-\beta)\tau(k) \\
&- \frac{D^2}{Q^4(1-E(p))^2}\pi_1^2\left(\frac{\sigma_D\sqrt{L}}{2}\right)^2\eta^2(k) \\
=& \frac{2D^2}{Q^4(1-E(p))^2}(A+C(L))\pi_1\left(\frac{\sigma_D\sqrt{L}}{2}\right)\tau(k) \\
&+ \frac{2D}{Q^3(1-E(p))}\left(A+C(L)+\pi_1\left(\frac{\sigma_D\sqrt{L}}{2}\right)\rho(k)\right)h \\
&\times \left(\frac{\sigma_D\sqrt{L}}{2}\right)(1-\beta)\tau(k) \\
&+ \frac{D^2}{Q^4(1-E(p))^2}(\pi_1^2\left(\frac{\sigma_D\sqrt{L}}{2}\right)^2\left[2\tau(k)\rho(k)-\eta^2(k)\right].
\end{aligned} \tag{B.13}$$

Let $\xi(k) = 2\tau(k)\rho(k) - \eta^2(k)$. From (B.13), we can find that to verify $|H_{22}| > 0$ only need to prove that $\xi(k) > 0$ since all other terms are positive. We know that $\lim_{k\to 0^+}\xi(k) = 1$ and $\lim_{k\to+\infty}\xi(k) = 0$. Also we know $\mathrm{d}\xi(k)/\mathrm{d}k < 0$. Hence, $\xi(k)$ is a decreasing function of k and $\xi(k)$ is positive.

The third principal minor of H for objective function is:

$$\begin{aligned}
|H_{33}| = \frac{\theta b}{A^2}|H_{22}| + \left(\frac{D}{Q^2(1-E(p))}\right)^2 \Bigg\{&\left[\tau(k)h(1-\beta)\left(\frac{\sigma_D\sqrt{L}}{2}\right)\right] \\
&+ \left[\frac{D}{Q(1-E(p))}\pi_1\left(\frac{\sigma_D\sqrt{L}}{2}\right)\tau(k)\right]\Bigg\} > 0.
\end{aligned} \tag{B.14}$$

It follows from $|H_{11}| > 0$ and $|H_{22}| > 0$ that H is positive finite. That is to say, objective function is a convex function in (Q, k) for any $L \in [L_i, L_{i-1}]$.

Also for a given value of $L \in [L_i, L_{i-1}]$, storage space constraint $(g(Q, k, L))$ is a convex function of (Q, k), since

$$\begin{aligned} g(Q,k) = {} & \gamma f\left(Q + \mu_D L + k\sigma_D\sqrt{L}\right) - F - f\left(\mu_D L + E(y)\right) \\ & + f(1-\beta)\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\sqrt{1+k^2} - k\right) \end{aligned} \tag{B.15}$$

$$\frac{\partial^2 g(Q,k)}{\partial k^2} = f(1-\beta)\left(\frac{\sigma_D\sqrt{L}}{2}\right)\left(\frac{1}{\sqrt{1+k^2}} - \frac{k^2}{(1+k^2)^{\frac{3}{2}}}\right) > 0. \tag{B.16}$$

Because

$$\left(\frac{1}{\sqrt{1+k^2}} - \frac{k^2}{(1+k^2)^{\frac{3}{2}}}\right) > 0. \tag{B.17}$$

In addition, for a fixed $L \in [L_i, L_{i-1}]$, budget constraint is a linear function of (Q, k). Consequently, the necessary KKT conditions for the constrained inventory problem are also sufficient. So in this case, for a fixed value of $L \in [L_i, L_{i-1}]$, if there exists a solution (Q^*, A^*, k^*) that satisfies the KKT necessary conditions, then (Q^*, A^*, k^*) is an optimal solution of our constrained inventory problem. In fact, (Q^*, A^*, k^*) is the global minimum.

References

[1] M. Ben-Daya and M. Hariga, Lead-time reduction in a stochastic inventory system with learning consideration. *Int. J. Prod. Res.* **41** (2003) 571–579.

[2] M. Ben-Daya and S.M. Noman, Integrated inventory and inspection policies for stochastic demand. *Euro. J. Oper. Res.* **185** (2008) 159–169.

[3] P. Billington, The classic economic production quantity model with set up cost as a function of capital expenditure. *Decision Sci.* **18** (1987) 25-42.

[4] R.G. Brown and G. Gerson, Decision rules for inventory management, Holt, Rinehart and Winston, New York (1967).

[5] U.K. Bera, M. Rong, N.K. Mahaparta and M. Maiti, A multi-item mixture inventory model involving random lead time and demand with budget constraint and surprise function. *Appl. Math. Model.* **33** (2009) 4337–4344.

[6] M. Ben-Daya and A. Rauf, Inventory models involving lead time as decision variable. *J. Oper. Res. Soc.* **45** (1994) 579–582.

[7] C. Chandra and J. Grabis, Inventory management with variable lead-time dependent procurement cost. *Omega* **36** (2008) 877–887.

[8] H.C. Chang, An application of fuzzy sets theory to the EOQ model with imperfect quality items. *Comput. Oper. Res.* **31** (2004) 2079–2092.

[9] A. Charnes and W.W. Cooper, Chance constrained programming. *Manag. Sci.* **6** (1959) 73–79.

[10] B.R. Chaung, L.Y. Ouyang and K.W. Chaung, A note on periodic review inventory model with controllable setup cost and lead time. *Comput. Oper. Res.* **31** (2004) 549–561.

[11] G. Gallego and I. Moon, The distribution of free news boy problem: review and extension. *J. Oper. Res. Soc.* **44** (1993) 825–834.

[12] C.H. Glock, Lead time reduction strategies in a single-vendor-single-buyer integrated inventory model with lot size-dependent lead times and stochastic demand. *Int. J. Prod. Econ.* **136** (2012) 37–44.

[13] R.W. Hall, Zero inventories, Dow Jones Irwin, Homewood, Illinois (1983).

[14] M.A. Hariga, A single-item continuous review inventory problem with space restriction. *Int. J. Prod. Econ.* **128** (2010) 153–158.

[15] M. Hariga and M. Ben-Daya, Some stochastic inventory models with deterministic variable lead time. *Eur. J. Oper. Res.* **113** (1999) 42–51.

[16] C.J. Liao and C.H. Shyu, An analytical determination of lead time with normal demand. *Int. J. Prod. Manage.* **11** (1991) 72–78.

[17] H.J. Lin, Reducing lost-sales rate on the stochastic inventory model with defective goods for the mixture of distributions. *Appl. Math. Model.* **37** (2013) 3296–3306.

[18] S.W. Lin, Y.W. Wou and P. Julian, Note on minimax distribution free procedure for integrated inventory model with defective goods and stochastic lead time. *Appl. Math. Model.* **35** (2011) 2087–2093.

[19] I. Moon and G. Gallego, Distribution free procedures for some inventory models. *J. Oper. Res. Soc.* **45** (1994) 651–658.

[20] I. Moon, B.H. Ha and J. Kim, Inventory systems with variable capacity. *Eur. J. Industr. Engrg.* **6** (2012) 68–86.

[21] L.Y. Ouyang, N.C. Yen and K.S. Wu, Mixture inventory models with backorders and lost sales for variable lead time. *J. Oper. Res. Soc.* **47** (1996) 829–832.

[22] E.L. Porteus, Investing in reduced setups in the EOQ model. *Management Sci.* **31** (1985) 998–1010.

[23] M.J. Paknejad, F. Nasri and J.F. Affiso, Defective units in a continuous review (s,Q) system. *Int. J. Prod. Res.* **33** (1995) 2767–2777.

[24] K.L. Kim and J.C. Haya, Setup reduction in economic quantity production model. *Decision Sci.* **23** (1992) 500–508.

[25] K. Xu and M.T. Leung, Stocking policy in a two-party vendor managed channel with space restrictions. *Int. J. Prod. Econ.* **117** (2009) 271–285.

[26] B.R. Sarker, E.R. Coates, Manufacturing setup cost reduction under variable lead times and finite opportunities for investment. *Int. J. Prod. Econ.* **49** (1997) 237–247.

[27] K.S. Wu, I.C. Lin, Extend (Q,r) inventory model under lead time and ordering cost reductions when receiving quantity is different from the ordered quantity. *Quality and Quantity* **38** (2004) 771–786.

[28] J.W. Wu, W.C. Lee and H.Y. Tsai, Computational algorithmic procedure of optimal inventory policy involving a negative exponential crashing cost and varuable lead time. *Appl. Math. Comput.* **184** (2007) 798–808.

[29] S.M. Ross, Stochastic processes. 2nd edition, Wiley, New York (1996).

[30] M.K. Salameh and M.Y. Jaber, Economic production quantity model for items with imperfect quality. *Int. J. Prod. Econ.* **64** (2000) 59–64.

[31] D.A. Schrady and U.C. Choe, Models for multi-item inventory continuous review inventory policies subject to constraints. *Nav. Rese. Logist.* **18** (1971) 451–464.

[32] J. Taheri-Tolgari, A. Mirzazadeh and F. Jolai, An inventory model for imperfect items under inflationary conditions with considering inspection errors. *Comput. Math. Appl.* **63** (2012) 1007–1019.

[33] R.J. Tersine, Principle of Inventory and Material Management. 4th edition, Prentice-Hall, USA (1994).