RAIRO-Oper. Res. 49 (2015) 279–296
DOI: 10.1051/ro/2014037

RAIRO Operations Research
www.rairo-ro.org

ANALYSIS OF A SINGLE SERVER BATCH ARRIVAL RETRIAL QUEUEING SYSTEM WITH MODIFIED VACATIONS AND N-POLICY

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Abstract. In this paper, a batch arrival single server retrial queue with modified vacations under N-policy is considered. If an arriving batch of customers finds the server busy or on vacation, then the entire batch joins the orbit in order to seek the service again. Otherwise, one customer from the arriving batch receives the service, while the rest joins the orbit. The customers in the orbit will try for service one by one when the server is idle with a classical retrial policy with the retrial rate 'jv', where 'j' is the size of the orbit. At a service completion epoch, if the number of customers in the orbit is zero, then the server leaves for a secondary job (vacation) of random length. At a vacation completion epoch, if the orbit size is at least N, then the server remains in the system to render service for the primary customers or orbital customers. On the other hand, if the number of customers in the orbit is less than 'N' at a vacation completion epoch, the server avails multiple vacations subject to maximum 'M' repeated vacations. After availing 'M' consecutive vacations, the server returns to the system to render service irrespective of the orbit size. The model is studied using supplementary variable technique. For the proposed queueing system, the probability generating function of the steady state queue size distribution at an arbitrary time is obtained. Various performance measures are derived. A cost model for the queueing system is developed. Numerical illustration is provided.

Keywords. Batch arrival, retrial queue, modified vacations, N policy, classical retrial policy.

Mathematics Subject Classification. 60K25, 60K20, 90B22, 68M20.

Received December 13, 2012. Accepted June 25, 2014.

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Article published by EDP Sciences

1. Introduction

Retrial queueing system is characterized by the feature that the arriving customers, who encounter the server busy, join a virtual pool called orbit. An arbitrary customer in the orbit generates a stream of repeated requests that is independent of the rest of customers in the orbit. Such queueing systems play important roles in the analysis of many telephone switching systems, telecommunications networks and computer systems. The first result on $M/G/1$ retrial queues is due to Keilson *et al.* [16] who used the method of supplementary variable technique to investigate the join distribution of the channel state and the number of customers in orbit in the steady state. Later, Aleksandrov [2] considered the case of arbitrarily distributed service times. A variant of the $M/G/1$ retrial queue was considered by Neuts and Ramalhoto [21]. Artalejo [3] studied some results on $M/G/1$ queue with N-policy. The detailed overviews of the related references with retrial queues can be found in the book of Falin and Templeton [10] and survey papers of Artalejo [4, 5]. Gautam Choudhury [12] analyzed an $M/G/1$ queue with linear retrial policy with two phase service and Bernoulli vacation schedule. Modified vacation policy for $M/G/1$ retrial queue with balking and feedback was discussed by Ke and Chang [15]. Zaiming Liu *et al.* [23] analyzed an $M/G/1$ retrial G-queue with preemptive resume and feedback under N-policy subject to the server breakdowns and repairs.

Queueing systems with batch arrivals are common in many practical situations. In digital communication systems, messages which are transmitted could consist of a random number of packets. Falin [9] introduced the first batch arrival retrial queueing model who assumed the following rule: "*If the server is busy at the arrival epoch, then the whole batch joins the retrial group, whereas when the server is free, then one of the arriving units starts its service and the rest joins the retrial group*".

Krishnakumar and Pavai Madheswari [18] analyzed a bulk arrival retrial queue with multiple vacations and starting failures. Fu-Min Chang and Ke [11] analyzed a batch retrial queueing model with J vacations. Senthil Kumar and Arumuganathan [22] have analyzed the batch arrival single server retrial queue in which the server provides two phases of heterogeneous service and receives general vacation time under Bernoulli schedule. Choudhury *et al.* [7] analyzed a batch arrival retrial queueing system with two phases of service and service interruption. Aissani [1] analyzed an $M^X/G/1$ energetic retrial queue with vacations and it's control.

For a detailed survey on queueing system with server vacations, one can refer the references Lee *et al.* [20], Krisha Reddy *et al.* [17] and Arumuganathan *et al.* [6], *etc.* Lee *et al.* [20] analysed an $M^x/G/1$ queue with N-policy and multiple vacations. They have considered bulk arrival and single service. Bulk queue with N-policy, multiple vacations and setup times is analyzed by Krisha Reddy *et al.* [17] in which the arrivals occur in bulk and service process is done in bulk. Arumuganathan *et al.* [6] analyzed a bulk queue with multiple vacations, setup times with N-policy and closedown times. Ke [14] used supplementary variable technique to study an

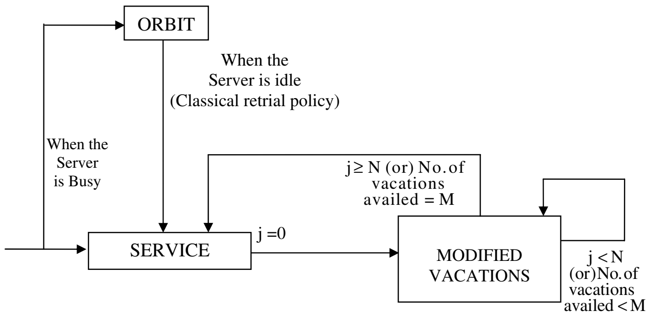

FIGURE 1. Schematic representation of the queueing model (j-orbit size).

$M^X/G/1$ queueing systems with balking under a variant vacation. Haridass and Arumuganathan [13] analyzed an $M^X/G(a,b)/1$ queueing system with vacation interruption. In all the above mentioned models, the authors discussed N-policy with vacations in classical queueing models. *But this paper focuses on N-policy with modified vacations in retrial queueing model.*

In earlier literature, very few authors have studied the comparable work on the variant vacations for the retrial queuing models which the server may take a sequence of finite vacations in his idle time. But as far as the authors' knowledge, there is no considerable amount of research work on retrial queueing system with N policy and modified vacations. This motivates us to develop a $M^x/G/1$ retrial queueing system with modified vacation policy and the threshold value 'N' for vacation policy. In which the server may take at most M vacations when the orbit size is less than 'N'.

In this paper, a batch arrival single server retrial queue with modified vacations under N-policy is considered. If an arriving batch of customers finds the server busy or on vacation, then the entire batch joins the orbit in order to seek the service again. Otherwise, one customer from the arriving batch receives the service, while the rest joins the orbit. The customers in the orbit will try for service one by one when the server is idle with a classical retrial policy with the retrial rate 'jv', where 'j' is the size of the orbit. At a service completion epoch, if the number of customers in the orbit is zero, then the server leaves for a secondary job (vacation) of random length. At a vacation completion epoch, if the orbit size is at least N, then the server remains in the system to render service for the primary customers or orbital customers. On the other hand, if the number of customers in the orbit is less than 'N' at a vacation completion epoch, the server avails multiple vacations subject to maximum 'M' repeated vacations. After availing 'M' consecutive vacations, the server returns to the system to render service irrespective of the orbit size. Analytical treatment of this model is obtained used supplementary variable technique. The model under study is schematically represented in Figure 1.

The following points are addressed in this paper. Constructed the mathematical model for the $M^x/G/1$ retrial queue under modified vacations with N-policy, where the server takes at most M vacations utilizing his idle time. Probability generating function of the steady state orbit size at an arbitrary time is obtained. Particular cases and some special cases are discussed. Various performance measures are derived. A cost model for the queueing system is developed. Numerical illustration is also presented.

1.1. Practical justification of the model

The motivation of the model comes from a situation observed in the mail system that uses simple main transfer protocol (SMTP) to deliver messages between mail servers. When a mail transfer program contacts a server on a remote machine, it forms a TCP connection over which it communicates. Once the connection is established, the two programs follow SMTP that allows the sender to identify it, specify a recipient, and transfer an e-mail message. After the sender deposits the e-mail in his/her own mail server, the mail server can repeatedly try to send the contact message to target server until it becomes operational. Typically, contacting messages arrive at the mail server following the Poisson stream. One message comprises collection of finite number of packets (*i.e.* Threshold Policy N). If all packets of a message are arrived to the mail server, the server starts to do service. When all the packets arrive at the mail server, one packet is selected to serve and the rest of the packets will join the buffer (*i.e.*, retrial group). In the buffer, each packet waits a certain amount of time and retries the service again. Various maintenance activities (*i.e.*, finite number of vacations) are needed to keep the mail server functioning well. For example, virus scan and spam filtering *etc.*, are the important maintenance activities for the mail server. It can be performed when the mail server is idle and be programmed to perform on a regular basis. In this scenario, the buffer, mail server service mechanism and maintenance activities are corresponding to orbit, the server, and modified vacations in queueing terminology, respectively. This can be modeled as a batch arrival single server retrial queue under modified vacations with N-policy.

2. Mathematical model

Let X be the group size random variable of the arrival, λ be the Poisson arrival rate. g_k be the probability that 'k' customers arrive in a batch. Let v be the retrial rate (classical retrial) of the customer from the orbit. Let $S(x)(s(x))\{\tilde{S}(\theta)\}[S^0(x)]$ be the cumulative distribution function (probability density function) {Laplace-Stieltjes transform} [remaining service time] of service. Let $V(x)(v(x))\{\tilde{V}(\theta)][V^0(x)]$ be the cumulative distribution function (probability density function) {Laplace-Stieltjes transform} [remaining vacation time] of vacation. Let N denotes the threshold value and M denotes the maximum number of vacations that a server can avail (modified vacations). $N(t)$ denotes the number of customers in the orbit at time t. The different states of the server at time 't' are defined as follows:

$$C(t) = \begin{cases} 0, & \text{if the server is busy with service} \\ 1, & \text{if the server is on vacation} \\ 2, & \text{if the server is idle.} \end{cases}$$

To obtain the system equations, the following state probabilities are defined:

$$P_{1,n}(x,t)\mathrm{d}t = \Pr\left\{N(t)=n, x \leqslant S^0(t) \leqslant x+\mathrm{d}t, C(t)=0\right\}, \ n \geqslant 0$$
$$Q_{l,n}(x,t)\mathrm{d}t = \Pr\left\{N(t)=n, x \leqslant V^0(t) \leqslant x+\mathrm{d}t, C(t)=1\right\}, \quad l=1,2,\dots M, \ n \geqslant 0$$
$$P_{0,n}(t) = \Pr\left\{N(t)=n, C(t)=2\right\}, \quad n \geqslant 0.$$

To obtain the probability generating function (PGF) of the number of customers in the orbit, the system equations are derived at various states using the supplementary variable technique and the above definitions. Cox [8] analyzed a Non-Markovian stochastic processes by the inclusion of Supplementary variables.

Now, the following system equations are obtained for the queueing system, using supplementary variable technique:

$$P_{0,0}(t+\Delta t) = P_{0,0}(t)(1-\lambda\Delta t) + Q_{M,0}(0,t)\Delta t \tag{2.1}$$

$$P_{0,j}(t+\Delta t) = P_{0,j}(t)(1-\lambda\Delta t - jv\Delta t) + P_{1,j}(0,t)\Delta t + Q_{M,j}(0,t)\Delta t; \quad 1 \leqslant j \leqslant N-1 \tag{2.2}$$

$$P_{0,j}(t+\Delta t) = P_{0,j}(t)(1-\lambda\Delta t - jv\Delta t) + P_{1,j}(0,t)\Delta t + \sum_{l=1}^{M} Q_{l,j}(0,t)\Delta t; \quad j \geqslant N \tag{2.3}$$

$$P_{1,j}(x-\Delta t, t+\Delta t) = P_{1,j}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j+1} \lambda g_k P_{0,j-k+1}(t)s(x)\Delta t + (j+1)vP_{0,j+1}(t)s(x)\Delta t + \sum_{k=1}^{j} \lambda g_k P_{1,j-k}(x,t)\Delta t; \ j \geqslant 0 \tag{2.4}$$

$$Q_{1,0}(x-\Delta t, t+\Delta t) = Q_{1,0}(x,t)(1-\lambda\Delta t) + P_{1,0}(0,t)v(x)\Delta t \tag{2.5}$$

$$Q_{1,j}(x-\Delta t, t+\Delta t) = Q_{1,j}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} Q_{1,j-k}(x,t)\lambda g_k \Delta t; \ j \geqslant 1 \tag{2.6}$$

$$Q_{l,0}(x-\Delta t, t+\Delta t) = Q_{l,0}(x,t)(1-\lambda\Delta t) + Q_{l-1,0}(0,t)v(x)\Delta t; \quad l=2,3,\dots M \tag{2.7}$$

$$Q_{l,j}(x-\Delta t, t+\Delta t) = Q_{l,j}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} Q_{l,j-k}(x,t)\lambda g_k \Delta t + Q_{l-1,j}(0,t)v(x)\Delta t \quad 1 \leqslant j \leqslant N-1; \quad l=2,3,\dots M \tag{2.8}$$

$$Q_{l,j}(x-\Delta t, t+\Delta t) = Q_{l,j}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} Q_{l,j-k}(x,t)\lambda g_k \Delta t; \quad j \geqslant N; \quad l=2,3,\dots M. \tag{2.9}$$

3. Probability generating function

Lee [19] developed a new technique to find the steady state probability generating function (PGF) of the number of customers in the queue at an arbitrary time epoch. To apply the technique, first the following probability generating functions are defined.

$$\begin{aligned}
\tilde{P}_1(z,\theta) &= \sum_{j=0}^{\infty} \tilde{P}_{1,j}(\theta) z^j \quad \text{and} \quad P_1(z,0) = \sum_{j=0}^{\infty} P_{1,j}(0) z^j; \\
\tilde{Q}_l(z,\theta) &= \sum_{j=0}^{\infty} \tilde{Q}_{l,j}(\theta) z^j \quad \text{and} \quad Q_l(z,0) = \sum_{j=0}^{\infty} Q_{l,j}(0) z^j \quad l = 1,2,3,\dots M \\
P_0(z) &= \sum_{j=0}^{\infty} P_{0,j} z^j.
\end{aligned} \tag{3.1}$$

The probability generating function $P(z)$ is obtained by the procedure followed in [13].

$$P(z) = P_0(z) + \tilde{P}_1(z,0) + \sum_{l=1}^{M} \tilde{Q}_l(z,0)$$

$$P(z) = \frac{(z-1)\left((\lambda X(z) - \lambda) P_0(z) + \left(\tilde{V}(\lambda - \lambda X(z)) - 1\right)\left(\sum_{j=0}^{N-1} q_j z^j + P_{1,0}(0)\right)\right)}{\left(z - \tilde{S}(\lambda - \lambda X(z))\right)(\lambda X(z) - \lambda)}. \tag{3.2}$$

Expression for $P_0(z)$ is

$$P_0(z) = \left(P_0(1) + \int_1^z \frac{f(t)}{k(t) v \left(t - \tilde{S}(\lambda - \lambda X(t))\right)} \mathrm{d}t\right) k(z) \tag{3.3}$$

where

$$k(z) = \exp\left(\frac{-\lambda}{v} \int_1^z \frac{\left(1 - \tilde{S}(\lambda - \lambda X(u)) \frac{X(u)}{u}\right)}{\left(u - \tilde{S}(\lambda - \lambda X(u))\right)} \mathrm{d}u\right)$$

and

$$f(z) = \left(\tilde{V}(\lambda - \lambda X(z)) - 1\right)\left(P_{1,0}(0) + \sum_{l=1}^{M-1} \sum_{j=0}^{N-1} Q_{l,j}(0) z^j\right).$$

The probability generating function $P(z)$ has to satisfy $P(1) = 1$. In order to satisfy the condition, applying L'Hospital's rule and evaluating $\lim_{z\to 1} P(z)$ and equating the expression to 1, $1 - \lambda E(X)E(S) > 0$ is obtained. Thus $\rho < 1$ is the condition to be satisfied for the existence of steady state for the model under consideration, where $\rho = \lambda E(X)E(S)$.

3.1. Computational aspects of unknown probabilities

Equation (3.2) gives the probability generating function $P(z)$ of the number of customers in the orbit at an arbitrary time, which involves $N+1$ unknown probabilities namely, $q_0^1, q_1^1, q_2^1, q_3^1, \ldots q_{N-1}^1$ and $P_{1,0}(0)$. Using Theorem 3.1, $q_0, q_1, q_2, \ldots q_{N-1}$ are expressed in terms of the single constant $P_{1,0}(0)$.

Theorem 3.1. *The unknown constants q_j involved in $P(z)$ are expressed in terms of $P_{1,0}(0)$ as,*

$$q_0 = \frac{1}{1-\alpha_0}\left(\alpha_0 P_{1,0}(0) - Q_{M,0}(0)\right)$$

and

$$q_j = \frac{1}{1-\alpha_0}\left(\alpha_j P_{1,0}(0) + \sum_{i=1}^{j} \alpha_i q_{j-i} - Q_{M,j}(0)\right), \quad j = 1, 2, 3, \ldots N-1,$$

where α_i is the probability that 'i' customers arrive during the vacation and

$$\begin{aligned} Q_{M,j}(0) = P_{1,0}(0) &\sum_{k_{m-1}=0}^{j} \alpha_{j-k_{m-1}} \sum_{k_{m-2}=0}^{k_{m-1}} \alpha_{k_{m-1}-k_{m-2}} \\ &\times \sum_{k_{m-3}=0}^{k_{m-2}} \alpha_{k_{m-2}-k_{m-3}} \cdots \sum_{k_1=0}^{k_2} \alpha_{k_2-k_1}\alpha_{k_1} \\ j = 0, 1, 2, 3, \ldots N-1.& \end{aligned} \tag{3.4}$$

As the procedure followed in [13], the proof of the above theorem is obtained.

4. Performance measures

In this section, some useful performance measures of the proposed model like, expected number of customers in the orbit, expected length of busy period and expected length of busy cycle are derived which are useful to find the total average cost of the system. Also, probability that the server is on vacation $P(V)$, probability that the server is idle $P(I)$ and probability that the server is busy $P(B)$ are derived.

4.1. Expected orbit length

The expected orbit length $E(Q)$ (*i.e.* mean number of customers waiting in the orbit) at an arbitrary time epoch, is obtained by differentiating $P(z)$ at $z = 1$ and is given by

$$\lim_{z\to 1}\left(\frac{\mathrm{d}}{\mathrm{d}z}P(z)\right) = E(Q)$$

$$E(Q) = \frac{\begin{pmatrix} 2\lambda^2E^2(X)(1-\rho)P_0'(1) + \lambda^2E^2(X)(S2)P_0(1) \\ +(\lambda E(X)(V2)(1-\rho)(S2) + \lambda E(X)(V1)(S2) \\ -\lambda E(X^2)(V1)(1-\rho))\left(\sum\limits_{j=0}^{N-1} q_j + P_{1,0}(0)\right) \\ +2\lambda E(X)(V1)(1-\rho)\sum\limits_{j=0}^{N-1} jq_j \end{pmatrix}}{2(1-\rho)^2\lambda^2E^2(X)}$$

$$= \frac{\begin{pmatrix} 2\lambda E(X)(1-\rho)P_0'(1) + \lambda E(X)P_0(1)S2 \\ + \big((1-\rho)\lambda^2E^2(X)E(V^2) + (V1)(S2)\big) \\ \times\left(\sum\limits_{j=0}^{N-1} q_j + P_{1,0}(0)\right) + 2(V1)(1-\rho)\sum\limits_{j=0}^{N-1} jq_j \end{pmatrix}}{2(1-\rho)^2\lambda E(X)}$$

$$E(Q) = \frac{\begin{pmatrix} \lambda E(X)\,(2(1-\rho)P_0'(1) + P_0(1)S2) \\ + ((V1)(S2) + (1-\rho)(V2)) \\ \times\left(\sum\limits_{j=0}^{N-1} q_j + P_{1,0}(0)\right) + 2(V1)(1-\rho)\sum\limits_{j=0}^{N-1} jq_j \end{pmatrix}}{2(1-\rho)^2\lambda E(X)} \tag{4.1}$$

where

$$S2 = \lambda E(S)X''(1) + \lambda^2E^2(X)E\left(S^2\right); V1 = \lambda E(X)E(V);$$
$$V2 = \lambda^2E^2(X)E\left(V^2\right); \rho = \lambda E(X)E(S)$$
$$P_0(1) = 1 - \rho - E(V)\left(\sum_{j=0}^{N-1} q_j + P_{1,0}(0)\right);$$
$$P_0'(1) = \frac{\left(\sum\limits_{j=0}^{N-1} q_j + P_{1,0}(0)\right)(V1) - \lambda P_0(1)(1-\rho-E(X))}{v(1-\rho)}\cdot \tag{4.2}$$

4.2. Expected length of busy period

If T_b is the length of busy period, then under the steady state conditions and by the argument of alternating renewal process, the expected length of busy period

$E(T_b)$ is obtained as:

$$E(T_b) = \frac{1}{\lambda E(X)} \left(\frac{1}{P_{0,0}} - 1 \right) \tag{4.3}$$

where $P_{0,0} = \frac{1}{\lambda} Q_{M,0}(0)$ and the expression for $Q_{M,0}(0)$ is given in (3.4).

4.3. Expected length of busy cycle

If T_c is the length of busy cycle, then under the steady state conditions and by the argument of alternating renewal process, the expected length of busy cycle $E(T_c)$ is obtained as

$$E(T_c) = \frac{1}{\lambda E(X)} \left(\frac{1}{P_{0,0}} \right) \tag{4.4}$$

where $P_{0,0} = \frac{1}{\lambda} Q_{M,0}(0)$ and the expression for $Q_{M,0}(0)$ is given in (3.4).

4.4. Probability that the server is on vacation

Let V be the random variable for modified vacations and $P(V)$ be the probability that the server is on modified vacations at time t. Then

$$P(V) = E(V) \left(\sum_{j=0}^{N-1} q_j + P_{1,0}(0) \right), \tag{4.5}$$

where

$$q_j = \frac{1}{1-\alpha_0} \left(\alpha_j P_{1,0}(0) + \sum_{i=1}^{j} \alpha_i q_{j-i} - Q_{M,j}(0) \right),$$

$E(V)$ is the expected vacation time and the expression for $Q_{M,j}(0)$ is given (3.4).

4.5. Probability that the server is busy

Let B be the busy period random variable and $P(B)$ be the probability that the server is busy at time t. Then

$$P(B) = E(S) \left(\lambda P_0(1) + v P_0'(1) \right). \tag{4.6}$$

The expressions for $P_0(1)$ and $P_0'(1)$ are given in (4.2).

4.6. Probability that the server is idle

Let I be the idle period random variable and let $P(I)$ be the probability that the server is idle at time t. Then

$$P(I) = 1 - \rho - E(V) \left(\sum_{j=0}^{N-1} q_j + P_{1,0}(0) \right). \tag{4.7}$$

5. Particular cases

In this section, some of the existing models are deduced as a particular case of the proposed model.

Case (i): If there is no N-policy and no modified vacations ($i.e.$, $\tilde{V}(0) = 1$), then equation (3.2) reduces to the following form:

$$P(z) = \frac{(z-1)\,P_0(z)}{z - \tilde{S}(\lambda - \lambda X(z))}$$

where

$$P_0(z) = (1 - \lambda E(X)E(S)) \exp\left(\frac{-\lambda}{v} \int_1^z \frac{\left(1 - \tilde{S}(\lambda - \lambda X(u)) \frac{X(u)}{u}\right)}{z - \tilde{S}(\lambda - \lambda X(u))} \mathrm{d}u \right).$$

This equation coincides with the result of orbit size distribution of $M^x/G/1$ retrial queueing system in [10].

Case (ii): If $M = 1$, the proposed model can be reduced to the $M^x/G/1$ retrial queueing system with N policy under single vacation. In this case equation (3.2) can be written in the following form:

$$\begin{aligned} P(z) = {} & \frac{(z-1)}{\left(z - \tilde{S}(\lambda - \lambda X(z))\right)(\lambda X(z) - \lambda)} \\ & \times \left((\lambda X(z) - \lambda)\, P_0(z) + \left(\tilde{V}(\lambda - \lambda X(z)) - 1\right) P_{1,0}(0) \right). \end{aligned}$$

Furthermore, when $M = \infty$, the proposed model can describe the $M^x/G/1$ retrial queueing system with multiple vacations and N policy.

5.1. Special cases

The model under study is general in nature as the service time is arbitrary. But for practical purposes, service time with particular distribution is required. In this section, some special cases of the proposed model by specifying service time random variable as exponential distribution, Erlangian distribution and vacation time random variable as exponential distribution are discussed by the procedure followed in [13].

Case (i): Single server batch arrival retrial queue with exponential service time, N-policy and modified vacations.

The PGF of the orbit size distribution of this special case of the queueing model is obtained as,

$$P(z) = \frac{(z-1)}{\left(z - \left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)\right)(\lambda X(z)-\lambda)} \times \left((\lambda X(z)-\lambda) P_0(z) + \left(\tilde{V}(\lambda-\lambda X(z))-1\right)\left(\sum_{j=0}^{N-1} q_j z^j + P_{1,0}(0)\right)\right)$$

where

$$P_0(z) = \left(P_0(1) + \int_1^z \frac{f(t)}{k(t)v\left(t - \left(\frac{\mu}{\mu+\lambda(1-X(t))}\right)\right)} \mathrm{d}t\right) k(z)$$

$$k(z) = \exp\left(\frac{-\lambda}{v}\int_1^z \frac{\left(1 - \left(\frac{\mu}{\mu+\lambda(1-X(u))}\right)\frac{X(u)}{u}\right)}{\left(u - \left(\frac{\mu}{\mu+\lambda(1-X(u))}\right)\right)}\mathrm{d}u\right),$$

$$f(z) = \left(\tilde{V}(\lambda-\lambda X(z))-1\right)\left(P_{1,0}(0) + \sum_{j=0}^{N-1} q_j z^j\right).$$

Case (ii): Single server batch arrival retrial queue with k-Erlangian service time, N-policy and modified vacations.

The PGF of the orbit size distribution of this special case of the queueing model is obtained as,

$$P(z) = \frac{(z-1)}{\left(z - \left(\frac{uk}{uk+\lambda(1-X(z))}\right)^k\right)(\lambda X(z)-\lambda)} \times \left((\lambda X(z)-\lambda) P_0(z) + \left(\tilde{V}(\lambda-\lambda X(z))-1\right)\left(\sum_{j=0}^{N-1} q_j z^j + P_{1,0}(0)\right)\right)$$

where

$$P_0(z) = k(z)P_0(1) + k(z)\int_1^z \frac{f(t)}{k(t)v\left(t - \left(\frac{uk}{uk+\lambda(1-X(t))}\right)^k\right)}\mathrm{d}t,$$

$$k(z) = \exp\left(\frac{-\lambda}{v}\int_1^z \left(\frac{\left(1 - \left(\frac{uk}{uk+\lambda(1-X(u))}\right)^k \frac{X(u)}{u}\right)}{u - \left(\frac{uk}{uk+\lambda(1-X(u))}\right)^k}\right)\mathrm{d}u\right)$$

and

$$f(z) = \left(\tilde{V}(\lambda-\lambda X(z))-1\right)\left(P_{1,0}(0) + \sum_{j=0}^{N-1} q_j z^j\right).$$

This result is suitable for a system in which 'k' servers with exponential service time.

Case (iii): Single server batch arrival retrial queue with exponential vacation time, N-policy and modified vacations.

The PGF of the orbit size distribution of this special case of the queueing model is obtained as,

$$P(z) = \frac{(z-1)}{\left(z - \tilde{S}(\lambda - \lambda X(z))\right)(\lambda X(z) - \lambda)} \times \left((\lambda X(z) - \lambda)\, P_0(z) + \left(\left(\frac{\gamma}{\gamma + \lambda(1 - X(z))} \right) - 1 \right) \left(\sum_{j=0}^{N-1} q_j z^j + P_{1,0}(0) \right) \right)$$

where

$$P_0(z) = \left(P_0(1) + \int_1^z \frac{f(t)}{k(t) v \left(t - \tilde{S}(\lambda - \lambda X(t)) \right)} \mathrm{d}t \right) k(z)$$

$$k(z) = \exp\left(\frac{-\lambda}{v} \int_1^z \frac{\left(1 - \tilde{S}(\lambda - \lambda X(u)) \frac{X(u)}{u}\right)}{\left(u - \tilde{S}(\lambda - \lambda X(u))\right)} \mathrm{d}u \right)$$

and

$$f(z) = \left(\left(\frac{\gamma}{\gamma + \lambda(1 - X(z))} \right) - 1 \right) \left(P_{1,0}(0) + \sum_{j=0}^{N-1} q_j z^j \right).$$

6. Cost model

Cost analysis is the most important fact in any practical situation at every stage. Cost involves startup cost, operating cost, holding cost and reward cost (if any). It is quite natural that the management of the system desires to minimize the total average cost. Addressing this, in this section, the cost model for the proposed queueing system is developed and the total average cost is obtained with the following assumptions:

C_{s} : Start up cost per cycle.
C_{h} : Holding cost per customer per unit time.
C_{o} : Operating cost per unit time.
C_{r} : Reward cost per cycle due to vacation.

Since the length of the cycle (T_{c}) is the sum of the idle period (T_{i}) and busy period (T_{b}), from equations (4.3) and (4.4), $E(T_{\mathrm{i}})$ is obtained as $\frac{1}{\lambda E(X)}$.

TABLE 1. Retrial rate (Vs) mean orbit size for various values of M (modified vacations) (for $\lambda = 2.0$, $\mu = 8$ and $\gamma = 5$).

Retrial rate	Mean orbit size — Threshold value $N = 6$ — Different values of M (modified vacations)			
	2	3	4	5
1	8.3578	8.6870	8.9514	9.1696
2	5.0081	5.2238	5.4008	5.5506
3	3.8916	4.0694	4.2172	4.3443
4	3.3333	3.4922	3.6254	3.7411
5	2.9984	3.1459	3.2704	3.3792
6	2.7751	2.9151	3.0337	3.1379
7	2.6156	2.7501	2.8646	2.9656

The total average cost (TAC) per unit time is given by

Total Average Cost = Start-up cost per cycle + holding cost of number of customers in the queue per unit time + Operating cost per unit time – reward cost due to vacation per cycle.

$$\mathrm{TAC} = C_{\mathrm{s}}\frac{1}{E(T_{\mathrm{c}})} + C_{\mathrm{h}}E(Q) + C_{\mathrm{o}}\frac{E(T_{\mathrm{b}})}{E(T_{\mathrm{c}})} - C_{\mathrm{r}}\frac{E(T_{\mathrm{i}})}{E(T_{\mathrm{c}})}$$

$$\mathrm{TAC} = C_{\mathrm{s}}\lambda E(X)P_{0,0} + C_{\mathrm{h}}E(Q) + C_{\mathrm{o}}(1 - P_{0,0}) - C_{\mathrm{r}}P_{0,0} \tag{6.1}$$

where $P_{0,0} = \frac{1}{\lambda}Q_{M,0}(0)$ and the expression for $Q_{M,0}(0)$ is given in (3.4).

7. Numerical illustration

In this section, the consistency of the theoretical results obtained in Sections 3 and 4 is justified numerically with the following assumptions and notations:

Service time distribution is 2-Erlang with parameter	μ.
Batch size distribution of the arrival is geometric with mean	2.
Retrial rate	v.
Vacation time is exponential with parameter	γ.
Threshold value	N.
Number of vacations (Modified vacations)	M.

7.1. Effects of various parameters on the expected orbit length

The effects of various parameters such as arrival rates, retrial rates, mean orbit size, threshold value and modified vacations are analyzed numerically and the results are reported in Tables 1–6 and represented in Figure 2. All numerical results are obtained using Mat Lab software.

The effects of different retrial rates 'v' on the mean orbit size for a fixed threshold value N with respect to various values of M (modified vacation) are presented in

TABLE 2. Retrial rate (Vs) mean orbit size for various values of N (threshold value) (for $\lambda = 2.0$, $\mu = 8$ and $\gamma = 5$).

Retrial rate	Mean orbit size Maximum value of $M = 4$ (Maximum number of vacations a server can avail) Different values of N (threshold value)			
	1	2	3	4
1	8.8242	8.9119	8.9644	8.9835
2	5.3066	5.3713	5.4128	5.4286
3	4.1341	4.1910	4.2289	4.2437
4	3.5478	3.6009	3.6370	3.6512
5	3.1960	3.2469	3.2818	3.2957
6	2.9616	3.0108	3.0451	3.0587
7	2.7941	2.8422	2.8759	2.8894

TABLE 3. Retrial rate (Vs) total average cost for various values of M (modified vacations) with arrival rate 4 (for $\lambda = 4$, $\mu = 20$ and $\gamma = 5$).

Retrial rate	Threshold value $N = 3$ Different values of M (modified vacations)							
	2		3		4		5	
	$E(Q)$	TAC	$E(Q)$	TAC	$E(Q)$	TAC	$E(Q)$	TAC
1	11.4414	15.7892	11.7675	15.9609	11.9810	16.0885	12.1196	16.1795
2	6.3369	10.6847	6.5534	10.7468	6.6969	10.8044	6.7910	10.8508
3	4.6354	8.9832	4.8154	9.0087	4.9355	9.0430	5.0148	9.0746
4	3.7847	8.1325	3.9464	8.1398	4.0548	8.1623	4.1267	8.1865
5	3.2743	7.6220	3.4249	7.6184	3.5264	7.6339	3.5938	7.6536
6	2.9339	7.2817	3.0773	7.2707	3.1741	7.2816	3.2386	7.2984
7	2.6909	7.0387	2.8290	7.0224	2.9225	7.0300	2.9849	7.0446

$E(Q)$ – expected orbit length; TAC – total average cost.

TABLE 4. Retrial rate (Vs) total average cost for various values of N (threshold value) with arrival rate 4 (for $\lambda = 4$, $\mu = 20$ and $\gamma = 5$).

Retrial rate	Maximum value of $M = 6$ (Maximum number of vacations a server can avail) Different values of N (threshold value)							
	2		3		4		5	
	$E(Q)$	TAC	$E(Q)$	TAC	$E(Q)$	TAC	$E(Q)$	TAC
1	12.0341	16.0437	12.2091	16.2187	12.3506	16.3602	12.4582	16.4678
2	6.7233	10.7329	6.8522	10.8619	6.9608	10.9705	7.0461	11.0557
3	4.9530	8.9627	5.0667	9.0763	5.1642	9.1739	5.2420	9.2516
4	4.0679	8.0775	4.1739	8.1835	4.2660	8.2756	4.3400	8.3496
5	3.5368	7.5464	3.6382	7.6478	3.7270	7.7366	3.7988	7.8084
6	3.1828	7.1924	3.2811	7.2907	3.3677	7.3773	3.4379	7.4476
7	2.9299	6.9395	3.0259	7.0356	3.1111	7.1207	3.1803	7.1899

E(Q) – expected orbit length; TAC – total average cost.

TABLE 5. Retrial rate (Vs) total average cost for various values of M (modified vacations) with arrival rate 5 (for $\lambda = 5$, $\mu = 20$ and $\gamma = 5$).

Retrial rate	Threshold value $N = 3$							
	Different values of M (modified vacations)							
	2		3		4		5	
	$E(Q)$	TAC	$E(Q)$	TAC	$E(Q)$	TAC	$E(Q)$	TAC
1	18.6455	23.0268	19.1801	23.3708	19.5041	23.5994	19.6979	23.7457
2	10.3447	14.7259	10.6995	14.8902	10.9170	15.0124	11.0484	15.0960
3	7.5777	11.9590	7.8727	12.0633	8.0547	12.1499	8.1652	12.2128
4	6.1943	10.5755	6.4592	10.6499	6.6235	10.7188	6.7236	10.7712
5	5.3642	9.7454	5.6112	9.8018	5.7648	9.8601	5.8586	9.9063
6	4.8108	9.1920	5.0458	9.2364	5.1923	9.2876	5.2819	9.3296
7	4.4155	8.7968	4.6419	8.8326	4.7834	8.8787	4.8701	8.9177

$E(Q)$ – expected orbit length; TAC – total average cost.

TABLE 6. Retrial rate (Vs) total average cost for various values of N (threshold value) with arrival rate 5 (for $\lambda = 5$, $\mu = 20$ and $\gamma = 5$).

Retrial rate	Maximum value of $M = 6$ (Maximum number of vacations a server can avail)							
	Different values of N (threshold value)							
	2		3		4		5	
	$E(Q)$	TAC	$E(Q)$	TAC	$E(Q)$	TAC	$E(Q)$	TAC
1	19.4875	23.4957	19.8126	23.8208	20.0952	24.1034	20.3328	24.3410
2	10.8918	14.9000	11.1266	15.1348	11.3385	15.3467	11.5221	15.5303
3	8.0265	12.0348	8.2313	12.2395	8.4196	12.4278	8.5853	12.5935
4	6.5939	10.6021	6.7834	10.7918	6.9602	10.9684	7.1168	11.1250
5	5.7344	9.7426	5.9150	9.9232	6.0845	10.0927	6.2357	10.2439
6	5.1613	9.1696	5.3359	9.3441	5.5008	9.5089	5.6484	9.6566
7	4.7520	8.7602	4.9223	8.9305	5.0838	9.0919	5.2288	9.2370

$E(Q)$ – expected orbit length; TAC – total average cost.

Tables 1, 3 and 5. A graphical representation is also shown in Figure 2. From the tables and the figure, the following points are observed:

- As retrial rate increases, the mean orbit size decreases.
- As number of modified vacation M increases, the mean orbit size increases.
- As arrival rate increases, the mean orbit size increases.

The effects of different retrial rates 'v' on the mean orbit size for a fixed value for M (*i.e.*, maximum number of vacations a server can avail) with respect to different threshold values N are presented in Tables 2, 4 and 6. From the tables, it is clear that,

- As retrial rate increases, the mean orbit size decreases.
- As threshold value N increases, the mean orbit size increases.
- As arrival rate increases, the mean orbit size increases.

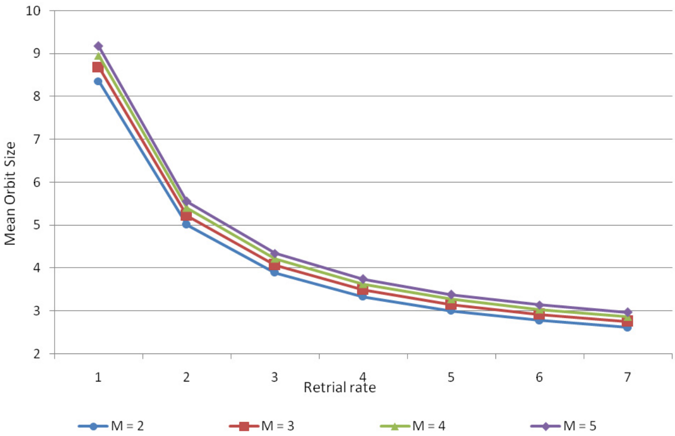

FIGURE 2. Retrial rate ($Vs.$) mean orbit size (for $\lambda = 2.0$, $\mu = 8$, $\gamma = 5$ and $N = 6$).

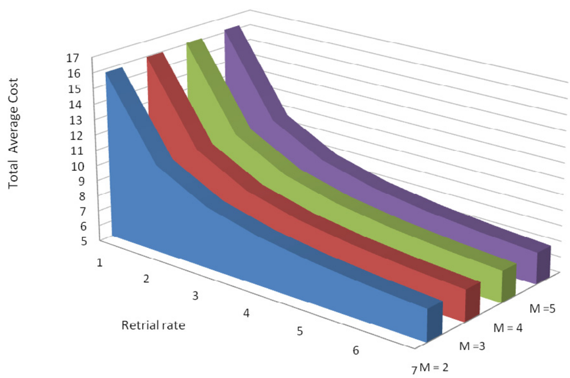

FIGURE 3. Retrial rate ($Vs.$) total average cost for various values of M (modified vacations) (for $\lambda = 4$, $\mu = 20$, $\gamma = 5$ and $N = 3$).

7.2. Effects of various parameters on the total average cost

The total average costs are obtained numerically with the following assumptions:

Start up cost	: Rs. 2.00.
Holding cost per customer	: Rs. 1.00.
Operating cost per unit time	: Rs. 4.00.
Reward cost per unit time due to vacation	: Rs. 1.00.

The effects of different retrial rate 'v' on the total average cost for a fixed threshold value N with respect to various values for M (modified vacation) are given in Tables 3 and 5. A graphical representation is also shown in Figure 3. From the tables and the figure, the following points are observed:

- As retrial rate increases, the total average cost decreases.

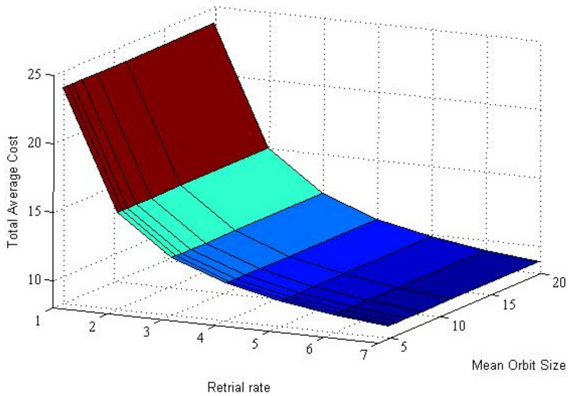

FIGURE 4. Retrial rate ($Vs.$) total average cost (for $\lambda = 5$, $\mu = 20$, $\gamma = 5$, $M = 6$ and $N = 3$).

- As number of modified vacation M increases, the total average cost increases.
- As arrival rate increases, the total average cost increases.

The effects of different retrial rates 'v' on the total average cost for a fixed value of M (*i.e.*, maximum number of vacations a server can avail) with respect to different threshold values N are given in Tables 4 and 6. A graphical representations is also shown in Figure 4. From the tables and the figure, the following points are observed:

- As retrial rate increases, the total average cost decreases.
- As threshold value N increases, the total average cost increases.
- As arrival rate increases, the total average cost increases.

Thus, the theoretical development of the model is justified with the numerical results which are consistent with the fact that when the retrial rate increases, the mean orbit size and the total average cost decrease.

8. Conclusion

In this chapter, "a batch arrival single server retrial queue with modified vacations under N-policy" is analyzed. Probability generating function of the steady state orbit size distribution at an arbitrary time is obtained. Various performance measures are derived. Some particular cases and special cases are discussed. A cost model for the queueing system is developed. The theoretical development of the model is justified with numerical results.

In the direction of future research, the model can be extended with service interruptions and bulk service concepts. An attempt may be made to derive the busy period distributions and idle period distributions. A discrete time model can also be developed.

Acknowledgements. The authors would be grateful to thank the referees and the editor for their insightful comments and suggestions, which helped in bringing the paper to its present form.

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