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NEW EFFECTIVE PROJECTION METHOD FOR VARIATIONAL INEQUALITIES PROBLEM

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Abstract. Among the most used methods to solve the variational inequalities problem (VIP), there exists an important class known as projection methods, these last are based primarily on the fixed point reformulation. The first proposed methods of projection suffered from major theoretical and algorithmic difficulties. Several studies were completed, in particular, those of Iusem, Solodov and Svaiter and that of Wang *et al.* with an aim to overcome these difficulties. Consequently, many developments were brought to improve the algorithmic behavior of this type of methods. In the same form of the algorithms of projection presented by the authors quoted above and under the same convergence hypotheses, we propose in this paper a new algorithm with a new displacement step which must satisfy a certain condition, this last ensures a faster convergence towards a solution. The algorithm is well defined and the theoretical results of convergence are suitably established. A comparative numerical study is carried out between the two algorithms (the algorithm of Solodov and Svaiter, the algorithm Wang *et al.*) and the new one. The results obtained by the new algorithm were very encouraging and show clearly the impact of our modifications.

Keywords. Variational inequalities problem, projection methods, pseudomonotone operators, fixed point.

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1. Introduction

Let C be a nonempty closed and convex set in $\mathbb{R}^n$ and F a continuous mapping from $\mathbb{R}^n$ to itself. The classical variational inequalities problem abbreviated $VIP\,(F,C)$ consists to find a point $\overline{x}$ such that:

$$\overline{x} \in C, \langle F\left(\overline{x}\right), \overline{x} - x\rangle \geq 0, \text{ for all } x \in C, \tag{1.1}$$

where $\langle .,.\rangle$ denotes the usual inner product in $\mathbb{R}^n$.

Variational inequalities problems play a significant role in economics, mathematical programming, transportation, regional science, *etc.*, and they have received a considerable attention these last years. The interested reader may consult the monographs by Glowinski [3], the survey paper of Harker and Pang [4] and the paper of Ferris and Pang [2].

Many methods have been proposed to solve $VIP\,(F,C)$. The oldest and the simplest one is the extragradient method which was proposed by Korpolevich [7] but its convergence requires the Lipschitz continuity of F. When F is not Lipschitz continuous or the Lipschitz constant is not known, the extragradient method and all its variants require an Armijo line-search procedure to compute a certain "candidate" stepsize [6]. We denote by x^k the current iterate and to calculate the following iterate, it is necessary first to evaluate $\mathrm{Proj}_C\left(x^k - \alpha_k F\left(x^k\right)\right)$ which is usually noted by y^k. But in order to determine the stepsize α_k, we need to use a line-search procedure which contains one projection. So at iteration k, if this procedure requires many steps to obtain the appropriate α_k and if we note this number of steps by m_k, then we need to evaluate m_k projections in addition to another projection to determine $x^{k+1} = \mathrm{Proj}_C\left(x^k - \lambda_k F\left(y^k\right)\right)$ where λ_k is given here by an explicit formula. All this numerous and necessary projections lead to an expensive computation.

To overcome this obstacle, Iusem and Svaiter [5] proposed a modified extragradient method for monotone variational inequalities which requires only two projections onto C at each iteration. Few years later, this method was improved by Solodov and Svaiter [13]. The method requires to compute a projection onto C and a projection onto $D_k \cap C$ at each iteration, where D_k is the halfspace associated to the current iterate and containing the solutions set. It can be seen that x^{k+1} thus computed belongs to C and the hyperplane H_k (the boundary of D_k). It's also important to note the two works of Wang *et al.* where in the first paper [14], they give a unified framework for the algorithm of Isuem [5] and that of Solodov [13], and in the second one [15] they provide another algorithm that presents a numerical improvement of the algorithm given in the first paper.

Inspired by these works, we thought to another direction where our idea is the following: why don't we go further and generate the sequence of iterates in the halfspace D_k to be more closer to solutions set and not only belong to H_k such as the algorithm of Solodov?

In order to clarify the geometric motivation behind this idea, we suppose that we have x^k a current approximation for the solution of $VIP\left(F, C\right)$. First, we compute $\mathrm{Proj}_C\left(x^k - \beta F\left(x^k\right)\right)$ where $0 < \beta < 1$. numerically, an inexpensive Armijo type procedure is used to find y^k.This point is determined such as the following hyperplane $H_k = \left\{x \in \mathbb{R}^n / \left\langle F\left(y^k\right), x - y^k\right\rangle = 0\right\}$ separates strictly x^k from any solution $\overline{x}$ of the problem. Once the hyperplane is constructed, the next iterate x^{k+1} is computed by projecting $x^k - \lambda_k F\left(x^k\right)$ onto C, but the stepsize λ_k must satisfies the following inequality: $\left\langle F\left(y^k\right), \mathrm{Proj}_C\left(x^k - \lambda_k F\left(y^k\right)\right) - y^k\right\rangle \leq 0$. This characterization allows to the iterate x^{k+1} to belong to the halfspace D_k and consequently x^{k+1} will be more closer to the solutions set than any other iterate computed by the other algorithms. In this paper, we establish a new version of projection method for variational inequalities which requires also only two projections onto C at each iteration. In Section 2, we summarize some basic definitions and properties to be used in this paper. In Section 3, we give a formally description of the new algorithm and we prove the global convergence results under the weaker hypothesis of continuity and pseudomonotocity of F. The above mentioned modifications seem to make a drastic difference in numerical performance when our algorithm will be compared to those of [13, 15]. Finally and in order to give a judgement about the behavior of this new algorithm, we have carried out a comparative numerical study between our algorithm, the algorithm of Solodov and that of Wang. Our preliminary computational experience using the new algorithm is quite encouraging and the results are reported in Section 4. The last section draws overall conclusion.

2. Preliminaries

Let C be a closed convex set in $\mathbb{R}^n$. Proj_C: $\mathbb{R}^n \longrightarrow C$ where

$\mathrm{Proj}_C\left(x\right) = \arg\min\left\{\left\|y - x\right\| / y \in C\right\}$, Proj_C is called the orthogonal projection operator onto C.

We summarize some well-known properties and results for projection operators in the lemmas below

Lemma 2.1 ([16]). *Let C be a nonempty closed and convex subset in $\mathbb{R}^n$. Then, for any $x, y \in \mathbb{R}^n$ and $z \in C$, the following statements hold:*

(i) $\left\langle \mathrm{Proj}_C\left(x\right) - x, z - \mathrm{Proj}_C\left(x\right)\right\rangle \geq 0$
(ii) $\left\|\mathrm{Proj}_C\left(x\right) - z\right\|^2 \leq \left\|x - z\right\|^2 - \left\|\mathrm{Proj}_C\left(x\right) - x\right\|^2$
(iii) $\left\langle z - x, z - \mathrm{Proj}_C\left(x\right)\right\rangle \geq \left\|z - \mathrm{Proj}_C\left(x\right)\right\|^2$.

Throughout this paper, we denote the solutions set of $VIP\left(F, C\right)$ by $\mathcal{T}$. Now, we state the assumptions which are necessary to our method:

(A1): $\mathcal{T}$ is nonempty
(A2): F is pseudomonotone, *i.e.*,

$$\left\langle F\left(x\right), y - x\right\rangle \geq 0 \Rightarrow \left\langle F\left(y\right), y - x\right\rangle \geq 0.$$

If (A2) holds, then for any $\overline{x} \in \mathcal{T}$:

$$\langle F(x), x - \overline{x} \rangle \geq 0, \text{ for all } x \in C.$$

Lemma 2.2 ([8]). *$\overline{x}$ is a solution of $VIP(F, C)$ if and only if*

$$\overline{x} = \mathrm{Proj}_C(x - \lambda F(x)), \lambda > 0. \tag{2.1}$$

The proof of this result is based on Lemma 2.1 (i). The first projection iterative formula is generated from this basic procedure and we note that the fixed point theorem is the fundamental tool in this procedure and in all projection methods developed later.

For $x \in C$ and $\lambda > 0$, we define the following projected residual function:

$$r(x, \lambda) = x - \mathrm{Proj}_C(x - \lambda F(x)), \ \lambda > 0 \ . \tag{2.2}$$

It's clear that the solutions of $VIP(F, C)$ coincide with zeros of this function. So, we have the following lemma:

Lemma 2.3 ([8]). *$\overline{x}$ is a solution of $VIP(F, C)$ if and only if*

$$r(\overline{x}, \lambda) = 0, \quad \lambda > 0. \tag{2.3}$$

Lemma 2.4 ([1, 15]). *Let Ω be a nonempty closed and convex subset in $\mathbb{R}^n$.*

1. *For $x, d \in \mathbb{R}^n$, and $\lambda \geq 0$, we define: $x(\lambda) = \mathrm{Proj}_\Omega(x - \lambda d)$, then, $\langle d, x - x(\lambda) \rangle$ is nondecreasing for $\lambda \geq 0$*
2. *For $x \in \Omega$, $d \in \mathbb{R}^n$, and $\lambda > 0$, we define*

$$\psi(\lambda) = \min\left\{ \|y - x + \lambda d\|^2 \ / \ y \in \Omega \right\},$$

then,

$$\psi'(\lambda) = 2\langle d, x(\lambda) - x + \lambda d \rangle .$$

3. Algorithm and its convergence

In this part, we will describe carefully our new algorithm for solving $VIP(F, C)$.

3.1. Description of the algorithm

Algorithm 3.1.

Begin algorithm

Initialization

Select any $\sigma, \gamma, \beta \in (0, 1)$ and let ε be a given tolerance

Let $x^0 \in C$, $k = 0$, compute $z^k = \mathrm{Proj}_C\left(x^k - \beta F\left(x^k\right)\right)$

- **While** $\left\| r\left(x^k,\beta\right)\right\| > \varepsilon$ **do**

$$y^k = (1-\alpha_k)\, x^k + \alpha_k z^k \tag{3.1}$$

where $\alpha_k = \gamma^j$ with j being the smallest nonnegative integer satisfying:

$$\left\langle F\left(x^k - \gamma^j r\left(x^k,\beta\right)\right), r\left(x^k,\beta\right)\right\rangle \geq \sigma \left\| r\left(x^k,\beta\right)\right\|^2. \tag{3.2}$$

Let $x^{k+1} = \mathrm{Proj}_C(x^k - \lambda_k F(y^k))$, where λ_k is chosen such that:

$$\left\langle F\left(y^k\right), \mathrm{Proj}_C\left(x^k - \lambda_k F\left(y^k\right)\right) - y^k\right\rangle \leq 0$$

$k = k+1$;

- **End While**

End algorithm

3.2. Analysis of the algorithm and its convergence

In what follows, we give the theoretical analysis about the convergence of the new algorithm under the assumptions (A1) and (A2).

In this algorithm, if $\left\| r\left(x^k,\beta\right)\right\| = 0$, so from Lemma 2.3, x^k is a solution of $VIP\,(F,C)$. Otherwise, for any $\overline{x} \in \mathcal{T}$, and according to the iterative schema of Algorithm 3.1, we have

$$\begin{aligned}\left\| x^{k+1} - \overline{x}\right\|^2 &= \left\| \mathrm{Proj}_C\left(x^k - \lambda_k F\left(y^k\right)\right) - \overline{x}\right\|^2 \\ &\leq \left\| x^k - \overline{x} - \lambda_k F\left(y^k\right)\right\|^2 - \left\| x^k - x^{k+1} - \lambda_k F\left(y^k\right)\right\|^2 \\ &\leq \left\| x^k - \overline{x}\right\|^2 + \lambda_k^2 \left\| F\left(y^k\right)\right\|^2 - 2\lambda_k \left\langle F\left(y^k\right), x^k - y^k\right\rangle \\ &\quad - \left\| x^k - x^{k+1} - \lambda_k F\left(y^k\right)\right\|^2.\end{aligned}$$

Where the first inequality follows from Lemma 2.1 (ii), and for the second inequality we use the assumption (A2).

Now, for any $\lambda \geq 0$, we define:

$$x^{k+1} = x\left(\lambda\right) = \mathrm{Proj}_C\left(x^k - \lambda F\left(y^k\right)\right)$$

and the function

$$\phi_k\left(\lambda\right) = 2\lambda \left\langle F\left(y^k\right), x^k - y^k\right\rangle + \left\| x^k - x^k\left(\lambda\right) - \lambda F\left(y^k\right)\right\|^2 - \lambda^2 \left\| F\left(y^k\right)\right\|^2 \tag{3.3}$$

where its derivative is

$$\phi_k'\left(\lambda\right) = 2\left\langle F\left(y^k\right), x^k\left(\lambda\right) - y^k\right\rangle. \tag{3.4}$$

Using the second result of Lemma 2.4, we can find easily the expression of ϕ_k'.

In what follows, we denote by:

λ_{k1} : the stepsize associated to the algorithm of Iusem which is given by the following explicit formula:

$$\lambda_{k1} = \frac{\left\langle F\left(y^k\right), x^k - y^k\right\rangle}{\left\|F\left(y^k\right)\right\|^2}$$

λ_{k2} : the stepsize associated to the algorithm of Wang where it is chosen as follows:

$$\lambda_{k2} \geq \lambda_{k1} \text{ and } \left\langle F\left(y^k\right), x^k\left(\lambda_{k2}\right) - y^k\right\rangle \geq 0$$

λ_{k3} : the stepsize associated to the algorithm of Solodov.

From [14], we recall that ϕ_k is a positive function for all value $\lambda \in [0, \lambda_{k3}]$, in particular, for λ_{k1} and λ_{k2} (λ_{k1}, $\lambda_{k2} \in]0, \lambda_{k3}]$). We can also see the same thing for the function ϕ'_k *i.e.*,

$$\phi'_k\left(\lambda_{k2}\right) = 2\left\langle F\left(y^k\right), x^k\left(\lambda_{k2}\right) - y^k\right\rangle \geq 0. \tag{3.5}$$

So

$$\left\langle F\left(y^k\right), x^k\left(\lambda_{k2}\right) - y^k\right\rangle \geq 0.$$

We can remark that the iterate x^{k+1} computed by the algorithm of Wang is: $x^{k+1} = \mathrm{Proj}_C\left(x^k - \lambda_{k2}F\left(y^k\right)\right) = x\left(\lambda_{k2}\right)$ belongs to the halfspace which does not contain the set of solutions $\mathcal{T}$ (the complement of D_k).

For λ_{k3}, the function ϕ_k reach its maximum at this value and for ϕ'_k we have:

$$\phi'_k\left(\lambda_{k3}\right) = 2\left\langle F\left(y^k\right), x^k\left(\lambda_{k3}\right) - y^k\right\rangle = 0.$$

We get

$$\left\langle F\left(y^k\right), x^k\left(\lambda_{k3}\right) - y^k\right\rangle = 0. \tag{3.6}$$

Geometrically, we see that the iterate x^{k+1} computed by the algorithm of Solodov is: $x^{k+1} = \mathrm{Proj}_C(x^k - \lambda_{k3}F(y^k)) = x(\lambda_{k3})$ is on the boundary of D_k (belongs to H_k).

For our new algorithm, the stepsize must satisfies the following inequality

$$\left\langle F\left(y^k\right), \mathrm{Proj}_C\left(x^k - \lambda_k F\left(y^k\right)\right) - y^k\right\rangle \leq 0. \tag{3.7}$$

This condition assures the properties below:

– The iterate x^{k+1} computed by Algorithm 3.1 is:

$$x^{k+1} = \mathrm{Proj}_C\left(x^k - \lambda_k F\left(y^k\right)\right) = x\left(\lambda_k\right)$$

– x^{k+1} belongs to D_k and

$$\phi'_k\left(\lambda_k\right) = 2\left\langle F\left(y^k\right), x\left(\lambda_k\right) - y^k\right\rangle \leq 0.$$

– The new stepsize $\lambda_k > \lambda_{k2}$ even $\lambda_k \geq \lambda_{k3}$ (since the function ϕ_k' is nonincreasing).

If such stepsize really exists, the corresponding algorithm converges quickly compared with other algorithms. But the question arises for this stepsize: what is the necessary condition that guarantees that the sequence $\left\{\left\|x^k - \overline{x}\right\|\right\}$, $(\forall \overline{x} \in \mathcal{T})$ is nonincreasing.

For this purpose, we give the following proposition.

Proposition 3.1. *Let $x^k(\lambda_{k3})$ and $x^k(\lambda_k)$ the following iterates corresponding to the iteration (k+1)) computed by the algorithms of Solodov and Algorithm* 3.1*, respectively.*

If $\left\|x^k - x^k(\lambda_k)\right\|^2 - \left\|x^k - x^k(\lambda_{k3})\right\|^2 \geq 2\lambda_k \left\langle F\left(y^k\right), y^k - x^k(\lambda_k)\right\rangle$, then:

(i) $\left\|x^k - x^k(\lambda_k)\right\|^2 - \left\|x^k - x^k(\lambda_{k3})\right\|^2 \geq 0.$

(ii) $\phi_k(\lambda_k) \geq \phi_k(\lambda_{k3})$.

Proof. We remark that the first point of the proposition results directly from the inequalities (3.6) and (3.7) satisfied by the stepsizes λ_{k3}, λ_k.

For the second point, we have from the definition of the function ϕ_k:

$$\begin{aligned}
\phi_k(\lambda_k) - \phi_k(\lambda_{k3}) &= 2\lambda_k \left\langle F\left(y^k\right), x^k - y^k\right\rangle + \left\|x^k - x^k(\lambda_k) - \lambda_k F\left(y^k\right)\right\|^2 \\
&\quad - \lambda_k^2 \left\|F\left(y^k\right)\right\|^2 - 2\lambda_{k3} \left\langle F\left(y^k\right), x^k - y^k\right\rangle \\
&\quad - \left\|x^k - x^k(\lambda_{k3}) - \lambda_{k3} F\left(y^k\right)\right\|^2 + \lambda_{k3}^2 \left\|F\left(y^k\right)\right\|^2 \\
&= 2\lambda_k \left\langle F\left(y^k\right), x^k - y^k\right\rangle + \left\|x^k - x^k(\lambda_k)\right\|^2 \\
&\quad + \lambda_k^2 \left\|F\left(y^k\right)\right\|^2 - 2\lambda_k \left\langle F\left(y^k\right), x^k - x^k(\lambda_k)\right\rangle \\
&\quad - \lambda_k^2 \left\|F\left(y^k\right)\right\|^2 - 2\lambda_{k3} \left\langle F\left(y^k\right), x^k - y^k\right\rangle \\
&\quad - \left\|x^k - x^k(\lambda_{k3})\right\|^2 - \lambda_{k3}^2 \left\|F\left(y^k\right)\right\|^2 \\
&\quad + 2\lambda_{k3} \left\langle F\left(y^k\right), x^k - x^k(\lambda_{k3})\right\rangle + \lambda_{k3}^2 \left\|F\left(y^k\right)\right\|^2 \\
&= \left\|x^k - x^k(\lambda_k)\right\|^2 - \left\|x^k - x^k(\lambda_{k3})\right\|^2 \\
&\quad + 2\lambda_k \left\langle F\left(y^k\right), x^k(\lambda_k) - y^k\right\rangle \\
&\quad + 2\lambda_{k3} \left\langle F\left(^k\right), y^k - x^k(\lambda_{k3})\right\rangle.
\end{aligned}$$

Using (i), we obtain the desired result. □

Now let us give the proposition and the theorem establishing the convergence of Algorithm 3.1.

Proposition 3.2. *Let $\left\{x^k\right\}$ the sequence generated by Algorithm* 3.1 *and suppose that the assumptions* (A1) *and* (A2) *are satisfied, so:*

(i) *The sequence $\{\left\|x^k - \overline{x}\right\|\}$ is nonincreasing for all $\overline{x} \in \mathcal{T}$.*

(ii) *The sequence* $\{x^k\}$ *is bounded.*
(iii) $\lim_{k\to\infty} \langle F(y^k), x^k - y^k \rangle = 0.$
(iv) *If a cluster point of the sequence* $\{x^k\}$ *belongs to* $\mathcal{T}$*, then* $\{x^k\}$ *converges to a solution in* $\mathcal{T}$*.*

Proof. From Proposition 3.1, we have: $\phi_k(\lambda_k) \geq \phi_k(\lambda_{k3})$ and consequently $\phi_k(\lambda_k) \geq \phi_k(\lambda_{k1})$, because the function ϕ_k reach its maximum on $[0, \lambda_{k3}]$ at λ_{k3} and we have λ_{k1}, $\lambda_{k2} \in]0, \lambda_{k3}]$.

(i) Furthermore, using the property (iii) of Lemma 2.1 and the assumption (A1) we get the following inequality:

$$\begin{aligned}
\left\|x^{k+1} - \overline{x}\right\|^2 &\leq \left\|x^k - \overline{x}\right\|^2 - \phi_k(\lambda_k) \\
&\leq \left\|x^k - \overline{x}\right\|^2 - \phi_k(\lambda_{k1}) \\
&= \left\|x^k - \overline{x}\right\|^2 - \lambda_{k1}^2 \left\|F(y^k)\right\|^2 - \left\|x^k - x^k(\lambda_{k1}) - \lambda_{k1} F(y^k)\right\|^2 \\
&\leq \left\|x^k - \overline{x}\right\|^2 - \lambda_{k1}^2 \left\|F(y^k)\right\|^2.
\end{aligned}$$

Then, the sequence $\{\|x^k - \overline{x}\|\}$ is nonincreasing, in addition it is positive so it converges.

(ii) We have this inequality: $\left\|x^k\right\| \leq \left\|x^k - \overline{x}\right\| + \|\overline{x}\|$. Using (i) we obtain: $\left\|x^k\right\| \leq \left\|x^0 - \overline{x}\right\| + \|\overline{x}\|$, which means that $\{x^k\}$ is bounded.

(iii) Using (i) an other time, we deduce that the sequence $\left\{\lambda_{k1}^2 \left\|F(y^k)\right\|^2\right\}$ converges to 0 when k tends toward (∞).
So, we can obtain:

$$\lim_{k\to\infty} \frac{\langle F(y^k), x^k - y^k \rangle}{\|F(y^k)\|} = \lim_{k\to\infty} \lambda_{k1} \left\|F(y^k)\right\| = 0.$$

Since $\{x^k\}$ is bounded, the same for $\{y^k\}$ and F continuous operator, so the sequence $\{F(y^k)\}$ is bounded also.
Using this result, we obtain: $\lim_{k\to\infty} \langle F(y^k), x^k - y^k \rangle = 0$.

(iiii) Let $\overline{x}$ a cluster point of the sequence $\{x^k\}$ belonging to $\mathcal{T}$ and $\{x^{i_k}\}$ a subsequence $\{x^k\}$ such that $\lim_{k\to\infty} x^{i_k} = \overline{x}$, then $\lim_{k\to\infty} \left\|x^{i_k} - \overline{x}\right\| = 0$.
On the other hand, we have $\overline{x} \in \mathcal{T}$ and the whole sequence $\left\{\left\|x^k - \overline{x}\right\|\right\}$ converges to some limit by (i), but since one a its subsequences converges to 0, so we get $\lim_{k\to\infty} \left\|x^k - \overline{x}\right\| = 0$, *i.e.*, $\lim_{k\to\infty} x^k = \overline{x}$.

□

Theorem 3.3. *Suppose that* F *is a continuous operator and the assumptions* (A1) *and* (A2) *are satisfied, then the sequence generated by Algorithm* 3.1 *converges to a solution of* $VIP(F, C)$*.*

Proof. It suffices to prove that some cluster point of $\left\{x^k\right\}$ belongs to $\mathcal{T}$. We note that existence of cluster point of $\left\{x^k\right\}$ follows from Proposition 3.2 (ii).

By Proposition 3.2 (iii) and the expression (3.1) we have

$$0 = \lim_{k \longrightarrow +\infty} \left\langle F\left(y^k\right), x^k - y^k \right\rangle = \lim_{k \longrightarrow +\infty} \left(\alpha_k \left\langle F\left(y^k\right), x^k - z^k \right\rangle\right) \tag{3.8}$$

we consider now tow cases:

Case 1: $\lim\limits_{k \longrightarrow +\infty} \alpha_k \neq 0$, *i.e.*, there exists a subsequence $\{\alpha_{i_k}\}$ of $\{\alpha_k\}$ and some $\overline{\alpha} > 0$ such that $\alpha_{i_k} > \overline{\alpha}$ for all k. In this case, it follows from (3.6) that

$$\lim_{k \longrightarrow +\infty} \left\langle F\left(y^{i_k}\right), x^{i_k} - z^{i_k} \right\rangle = 0. \tag{3.9}$$

Again, using (3.1) and (3.2),

$$\left\langle F\left(x^{i_k} - \gamma^j r\left(x^{i_k}, \beta\right)\right), r\left(x^{i_k}, \beta\right)\right\rangle \geq \sigma \left\|r\left(x^{i_k}, \beta\right)\right\|^2 \geqslant 0$$

that we can write:

$$\left\langle F\left(y^{i_k}\right), x^{i_k} - z^{i_k} \right\rangle \geq \sigma \left\|r\left(x^{i_k}, \beta\right)\right\|^2 \geqslant 0. \tag{3.10}$$

From (3.9) and (3.10), we conclude that

$$\lim_{k \longrightarrow +\infty} \left\|r\left(x^{i_k}, \beta\right)\right\| = 0.$$

Since $\left\{x^k\right\}$ is bounded by Proposition 3.2 (ii), without loss of generality (*i.e.*, refining the subsequence if needed), we assume that exists $\overline{x} \in \mathbb{R}^n$ such that $\lim\limits_{k \longrightarrow +\infty} \left\{x^{i_k}\right\} = \overline{x}$. Since F and Proj_C are continuous, taking limit in the last equality as k tends to ∞ we obtain

$$\left\|r\left(\overline{x}, \beta\right)\right\| = 0.$$

Therefore, $\overline{x} = \mathrm{Proj}_C\left(\overline{x} - \beta F\left(\overline{x}\right)\right)$ so that $\overline{x}$ belongs to $\mathcal{T}$ by Lemma 2.2. So, we have proved that in this case $\left\{x^k\right\}$ has a cluster point which solves $VIP\left(F, C\right)$.

Case 2: $\lim\limits_{k \longrightarrow +\infty} \alpha_k = 0$, in this case we have:

$$\lim_{k \longrightarrow +\infty} \frac{\alpha_k}{\gamma} = 0. \tag{3.11}$$

Let

$$\widehat{y}^k = \left(\frac{\alpha_k}{\gamma}\right) z^k + \left(1 - \frac{\alpha_k}{\gamma}\right) x^k. \tag{3.12}$$

Let $\overline{x}$ be a cluster point of $\left\{x^k\right\}$ and $\left\{x^{i_k}\right\}$ a subsequence of $\left\{x^k\right\}$ which converges to $\overline{x}$. Using the two last equalities, we get:

$$\lim_{k \longrightarrow +\infty} \widehat{y}^{i_k} = \overline{x}. \tag{3.13}$$

The procedure given by (3.2) implies that

$$\left\langle F\left(x^k - \left(\frac{\alpha_k}{\gamma}\right) r\left(x^k, \beta\right)\right), r\left(x^k, \beta\right)\right\rangle < \sigma \left\| r\left(x^k, \beta\right)\right\|^2 .$$

So

$$\left\langle F\left(\widehat{y}^k\right), r\left(x^k, \beta\right)\right\rangle < \sigma \left\| r\left(x^k, \beta\right)\right\|^2 . \tag{3.14}$$

Taking limits in (3.14) along this subsequence and using (3.13), we get

$$\left\langle F\left(\overline{x}\right), r\left(\overline{x}, \beta\right)\right\rangle \leq \sigma \left\| r\left(\overline{x}, \beta\right)\right\|^2 . \tag{3.15}$$

We write

$$u^k = x^k - \beta F\left(\left(x^k\right)\right). \tag{3.16}$$

Then we obtain

$$\overline{u} = \overline{x} - \beta F\left(\left(\overline{x}\right)\right). \tag{3.17}$$

Note that $\overline{x} \in C$, since $\left\{x^k\right\} \subset C$ and this last is closed. Thus using (3.17) and Lemma 2.1 (iii), we have

$$\beta \left\langle F\left(\overline{x}\right), r\left(\overline{x}, \beta\right)\right\rangle = \left\langle \overline{x} - \overline{u}, \overline{x} - \mathrm{Proj}_C\left(\overline{u}\right)\right\rangle \geq \left\| \overline{x} - \mathrm{Proj}_C\left(\overline{u}\right)\right\|^2 . \tag{3.18}$$

Combining (3.15) and (3.18), we obtain

$$\left\| \overline{x} - \mathrm{Proj}_C\left(\overline{u}\right)\right\|^2 \leq \sigma \left\| \overline{x} - \mathrm{Proj}_C\left(\overline{u}\right)\right\|^2 .$$

Since $\sigma \in (0, 1)$, it follows that $\left\| \overline{x} - \mathrm{Proj}_C\left(\overline{u}\right)\right\| = 0$, which means that

$$\overline{x} = \mathrm{Proj}_C\left(\overline{u}\right) = \mathrm{Proj}_C\left(\overline{x} - \beta F\left(\left(\overline{x}\right)\right)\right). \tag{3.19}$$

By (3.19) and Lemma 2.3, $\overline{x}$ belongs to $\mathcal{T}$, and we have proved that also in this case $\left\{x^k\right\}$ has a cluster point which solves $VIP(F, C)$. $\square$

3.3. How to compute the iterate x^{k+1}?

To calculate x^{k+1}, we can distinguish at least four cases, but some of these cases remain typically theoretical.

– As the first case, if C is a vectorial subspace, then the projection operator Proj_C will be a linear application.
  Then, we can choose for $\theta > 0$, and $\lambda_k > \lambda_{k3}$, the iterate x^{k+1} as follows:

$$x^{k+1} = x\left(\lambda_k\right) = x\left(\lambda_{k3} + \theta\right) = \mathrm{Proj}_C\left(x^k - \left(\lambda_{k3} + \theta\right) F\left(y^k\right)\right).$$

  We get:

$$x^{k+1} = \mathrm{Proj}_C\left(x^k - \left(\lambda_{k3}\right) F\left(y^k\right)\right) - \theta \mathrm{Proj}_C\left(F\left(y^k\right)\right).$$

  Unfortunately, in general it isn't the case for all C.

- The idea of the second case to determinate x^{k+1} is to start with some value of λ_k and this value should be update such as the condition (3.7) will be satisfied. But the problem that will arise in this case is the computation of many projections until we get the good choice of λ_k. Therefore, we will lose the ownership of such algorithms which is the computation of only two projections at each iteration.
- To avoid these difficulties, we thought to take x^{k+1} as a convex combination of $x(\lambda_{k3})$ and z^k:

$$x^{k+1} = \theta x(\lambda_{k3}) + (1-\theta) z^k, \qquad (\theta \in [0,1]).$$

From this form, we see that x^{k+1} belongs to C and satisfies the condition (3.7):

$$\begin{aligned}\left\langle F\left(y^k\right), \theta x(\lambda_{k3}) + (1-\theta) z^k - y^k\right\rangle &= \left\langle F\left(y^k\right), \theta x(\lambda_{k3}) + (1-\theta) z^k \right.\\ &\qquad \left. - (\theta + (1-\theta)) y^k\right\rangle \\ &= \theta \left\langle F\left(y^k\right), x(\lambda_{k3}) - y^k\right\rangle \\ &\qquad + (1-\theta)\left\langle F\left(y^k\right), z^k - y^k\right\rangle \le 0.\end{aligned}$$

The last inequality follows from $\left\langle F\left(y^k\right), x(\lambda_{k3}) - y^k\right\rangle = 0$ and

$$\left\langle F\left(y^k\right), z^k - y^k\right\rangle < 0.$$

We also note in this case, if $\theta = 1$, we find the iterate of the algorithm of Solodov.
- Also we can use a fixed stepsize ($\lambda_k = \lambda$, $\forall k$), but at each iteration we must check that this stepsize satisfied the condition (3.7). Until we find the good one, we will take some time, but once the stepsize is found the cost of computation will be less than in the case of a variable stepsize.

4. Computational experience

To give some insight into the behavior of the new projection algorithm (Algorithm 3.1) and to compare its effectiveness with the algorithm of Solodov *(Alg1)* [13] and the algorithm of Wang *(Alg2)* [15], we implemented them in Matlab and run them on a set of test problems which are described below. For our programs, we will use the two last cases presented in the previous paragraph, fixed stepsize *(Alg3)* and variable stepsize *(Alg3′)* to compute the iterate x^{k+1} and where the termination criterium is $\|r(x^k,\beta)\| \le \varepsilon = 10^{-6}$.

Example 4.1 ([12]). The Kojima–Shindo nonlinear complementarity problem (NCP) where the operator is defined by:

$$F:\mathbb{R}^4 \longrightarrow \mathbb{R}^4$$
$$F(x)=\begin{pmatrix} 3x_1^2+2x_1x_2+2x_2^2+x_3+3x_4-6 \\ 2x_1^2+x_1+2x_2^2+2x_3+2x_4-2 \\ 3x_1^2+x_1x_2+2x_2^2+9x_4-9 \\ x_1^2+3x_2^2+2x_3+3x_4-3 \end{pmatrix}$$

and the feasible set is the simplex $C=\{x\in\mathbb{R}^4_+ : \sum\limits_{i=1}^{4} x_i=4\}$.

Now, we give the computational results obtained from the implemented algorithms.

x^0	Iterations number				Computation time (s)			
	Alg1	*Alg2*	*Alg3*	*Alg3′*	*Alg1*	*Alg2*	*Alg3*	*Alg3′*
$(0,0,0,0)$	18	26	2	3	0.48	0.24	0.03	0.05
$(1,0,0,3)$	8	14	4	5	0.11	0.18	0.04	0.06
$(0,2,2,3)$	14	253	4	5	0.60	5.96	0.09	0.08
$(4,4,2,3)$	30	*	1	3	0.28	*	0.02	0.03
$(1,1,1,1)$	21	21	4	5	0.22	0.24	0.02	0.08
$(-1,4,2,-2)$	25	46	3	5	0.28	0.50	0.08	0.07
$(10,0,0,10)$	18	*	3	4	0.24	*	0.05	0.06
$(10,10,10,10)$	9	*	1	2	0.14	*	0.02	0.03

* Indicates that the algorithm does not provide any solution after 1000 iterations, therefore, we consider that the method did not converge.

Example 4.2 ([9]). The operator $F:\mathbb{R}^5 \longrightarrow \mathbb{R}^5$ is as follows

$$F(x)=\begin{pmatrix} 0.726 & -0.949 & 0.266 & -1.193 & -0.504 \\ 1.645 & 0.678 & 0.333 & -0.217 & -1.443 \\ -1.016 & -0.225 & 0.769 & 0.934 & 1.007 \\ 1.063 & 0.567 & -1.144 & 0.550 & -0.548 \\ -0.259 & 1.453 & -1.073 & 0.509 & 1.026 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$
$$+\rho\begin{pmatrix} \arctan(x_1-2) \\ \arctan(x_2-2) \\ \arctan(x_3-2) \\ \arctan(x_4-2) \\ \arctan(x_5-2) \end{pmatrix}+\begin{pmatrix} 5.308 \\ 0.008 \\ -0.938 \\ 1.024 \\ -1.312 \end{pmatrix}$$

and $C=\left\{x\in\mathbb{R}^5_+ : \sum_{i=1}^{5} x_i \geq 10\right\}$.

x^0	Iterations number				Computation time (s)			
	Alg1	*Alg2*	*Alg3*	*Alg3′*	*Alg1*	*Alg2*	*Alg3*	*Alg3′*
$(0,0,0,0,0)$	11	502	3	1	0.18	10.91	0.05	0.03
$(10,0,10,0,10)$	12	518	9	9	0.49	1.18	0.11	0.09
$(10,0,0,0,0)$	36	491	7	7	0.47	10.71	0.09	0.07
$(0,2.5,2.5,2.5,2.5)$	40	492	4	4	0.24	10.57	0.06	0.05
$(1,1,1,1,1)$	17	595	3	4	0.16	10.64	0.05	0.05
$(10,10,10,10,10)$	11	518	9	9	0.51	11.20	0.11	0.10
$(-1,-1,-1,-1,-1)$	41	418	4	4	0.17	10.40	0.06	0.05
$(-10,0,0,0)$	11	481	5	5	0.28	10.56	0.08	0.06
$(-2,0,0,0,0,0)$	19	519	5	5	0.27	11.18	0.08	0.06
$(25,0,0,0,0)$	50	492	14	14	0.61	10.67	0.16	0.13

Example 4.3 ([10])**.** $F:\mathbb{R}^{10}\longrightarrow\mathbb{R}^{10}$ and the data of this operator is given as presented in [10] and $C=\left\{x\in\mathbb{R}^{10}_{+}:\ x_i+x_{i+5}=\frac{i}{10},\ i=1,\ldots,5\right\}$.

For this problem, the results are presented using different starting points that it is difficult to write them on the same table because they require enough space.

$x^0=(0,0,0,0,0,0,0,0,0,0)^T$

Iterations number				Computation time (s)			
Alg1	*Alg2*	*Alg3*	*Alg3′*	*Alg1*	*Alg2*	*Alg3*	*Alg3′*
*	*	17	35	*	*	0.21	0.43

$x^0=\left(0,\frac{1}{10},\frac{1}{10}\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}\frac{2}{10},\frac{3}{10},\frac{4}{10}\right)^T$

Iterations number				Computation time (s)			
Alg1	*Alg2*	*Alg3*	*Alg3′*	*Alg1*	*Alg2*	*Alg3*	*Alg3′*
115	244	18	17	1.42	2.21	0.22	0.23

$x^0=(100,0,0,0,0,0,0,0,0,0)^T$

Iterations number				Computation time (s)			
Alg1	*Alg2*	*Alg3*	*Alg3′*	*Alg1*	*Alg2*	*Alg3*	*Alg3′*
165	162	20	18	6.26	1.60	0.25	0.28

$x^0=(10,0,10,0,10,0,10,0,10,0)^T$

Iterations number				Computation time (s)			
Alg1	*Alg2*	*Alg3*	*Alg3′*	*Alg1*	*Alg2*	*Alg3*	*Alg3′*
126	141	20	17	2.55	1.57	0.24	0.27

$x^0=(-10,0,0,0,0,0,0,0,0,0)^T$

Iterations number				Time calculation (s)			
Alg1	*Alg2*	*Alg3*	*Alg3′*	*Alg1*	*Alg2*	*Alg3*	*Alg3′*
*	*	18	33	*	*	0.21	0.43

Example 4.4 ([11])**.** $F(x) = Dx + c$ where D is the following $(n \times n)$ nonsymmetric matrix:

$$D = \begin{pmatrix} 4 & -1 & & & & & \\ -1 & 4 & -1 & & & & \\ & & 4 & -1 & & & \\ & & & \ddots & \ddots & & \\ & & & & 4 & -1 & \\ & & & & & 4 & -1 \\ & & & & & & 4 \end{pmatrix}, \ c = (-1, -1, \ldots, -1)^T \in \mathbb{R}^n$$

and $C = [0, 1]^n$.

The results are presented for two initial points and for different dimensions. We note that for the two cases $n = 2000$ and $n = 3000$, $\varepsilon = 10^{-4}$.

For $x^0 = (0, 0, \ldots, 0)$

Dim	Iterations number				Computation time (s)			
	Alg1	*Alg2*	*Alg3*	*Alg3′*	*Alg1*	*Alg2*	*Alg3*	*Alg3′*
100	26	14	5	11	1.71	0.74	0.23	1.34
200	28	14	5	11	5.95	2.85	0.97	2.14
500	27	14	5	11	81.92	42.05	14.77	31.94
1000	25	14	4	11	635.10	356.56	103.06	272.123
2000	21	11	3	8	5285.30	2667.03	453.65	1618.72
3000	22	11	3	8	22 101.43	15 983.54	2244.86	6256.40

For $x^0 = (1, 1, \ldots, 1)$

Dim	Iterations number				Computation time (s)			
	Alg1	*Alg2*	*Alg3*	*Alg3′*	*Alg1*	*Alg2*	*Alg3*	*Alg3′*
100	27	18	5	11	1.76	0.99	0.22	1.34
200	30	18	5	11	6.46	3.59	1.10	2.14
500	29	19	6	11	88.02	55.51	17.35	31.94
1000	31	19	6	11	777.31	437.64	148.30	272.12
2000	23	15	3	9	4964. 37	3139.49	644.45	1837.57
3000	23	15	3	9	30 719.21	19 413.00	2236.51	8551.91

4.1. Comments

Through the numerical tests and for different dimensions, we see that the results show the importance of the modifications introduced, expressed by the significant reduction in iterations number and computation time. We note also, that the algorithms *Alg1* and *Alg2* diverge in some examples, while our algorithm provides a solution after a reasonable number of iterations although the starting point is not feasible. Generally, we see that the two algorithms *Alg3* and *Alg3'* have almost similar behavior in the tested examples, but in comparison to the other algorithms (*Alg1* and *Alg2*) they seem more efficient. This can be justified by the fundamental result presented in this paper; that the stepsize λ_k associated to this new version of projection algorithm allows us to obtain a sequence of iterates more closer to the solutions set than the other algorithms. The second justification is the choice of stepsize β in the first projection (for the computation of z^k) which must be constant and different from 1, unlike the algorithms *Alg1* and *Alg2.* The line search procedure used is also a little different from those of *Alg1* and *Alg2.*

5. Conclusion

In this paper, we have proposed a new projection algorithm for solving variational inequalities problems on the basis of the works [13, 14]. The global convergence is proved under minimal assumptions of continuity and pseudomonotonocity of the underling operator. The new algorithm confirms the theoretical context and has some clear algorithmic advantages over most of the existing projection methods for different variational inequalities classes.

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