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OPTIMALITY AND DUALITY IN MULTIOBJECTIVE PROGRAMMING INVOLVING SUPPORT FUNCTIONS*

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Abstract. In this paper a vector optimization problem (VOP) is considered where each component of objective and constraint function involves a term containing support function of a compact convex set. Weak and strong Kuhn−Tucker necessary optimality conditions for the problem are obtained under suitable constraint qualifications. Necessary and sufficient conditions are proved for a critical point to be a weak efficient or an efficient solution of the problem (VOP) assuming that the functions belong to different classes of pseudoinvex functions. Two Mond Weir type dual problems are considered for (VOP) and duality results are established.

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1. Introduction

Multiobjective programming problems have a very wide range of applications in fields like operations research, economics, finance, product and process design, aircraft, automobile design and many more. For references see [19, 20]. The study of solutions of a multiobjective programming in literature has been done mainly using two aspects: by locating conditions which are easier to deal with computationally and which guarantee efficiency as well as through the study of dual problems. These conditions are widely known as optimality conditions which are mainly of two types: Fritz−John (FJ) and Kuhn−Tucker (KT) necessary and sufficient optimality conditions. The convexity concept plays an important role as a fundamental condition in obtaining the desired results.

In the past few years, attempts have been made to weaken the convexity hypothesis and to explore the extent of these optimality conditions applicability. One of the most useful generalizations is invexity which was introduced by Hanson [9] and Craven [7] for differentiable functions. Craven and Glover [8] established a characterization of invex function by proving the fact that a function from $\mathbb{R}^n$ to $\mathbb{R}$ is invex iff each of its stationary point is its global minimum [5,8] whereas invexity is only sufficient for a critical point to be a solution of constrained scalar problem.

In this respect Martin [14] defined a weaker invexity notion called KT-invexity and shown that it is necessary and sufficient for a Kuhn−Tucker critical point of a constrained scalar problem to become its optimal

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solution. Further efforts have been made by many authors to extend this concept to constrained multiobjective differentiable programming problems by generalizing KT-invexity introduced by Martin [14]. For references see [1, 2, 16, 17].

In this regard recently Arana−Jiménez *et al.* [3] have extended the concept of KT-invexity to multiobjective programming involving locally Lipschitz functions by introducing KT-pseudoinvex II and FJ-pseudoinvex II functions and established some characterization results.

Motivated by the above work, in this paper a vector optimization problem (VOP) containing support function of a compact convex set in both objective and constraint functions is considered. The popularity of this kind of problem seems to originate from the fact that even though the objective and constraint functions are nonsmooth, a simple representation of the dual problem may be found. For references see [10,11,15,18]. Weak and strong KT necessary conditions for (VOP) are obtained by using suitable constraint qualifications. Weak KT conditions are the usual KT conditions where the Lagrange multiplier corresponding to at least one component of objective function is non zero which indicates the active role of that component in determining the optimal solution. In contrast to this, strong KT conditions are those where all the components of objective function are active in determining the optimal solution *i.e.* Lagrange multiplier corresponding to each component of objective function is non zero.

This paper is organized as follows:

Section 2 presents some notations and definitions which will be used throughout the paper. In Section 3, weak and strong KT necessary optimality conditions are obtained for weak efficient solution of (VOP). In Section 4, we introduce the notions of KT-pseudoinvex I, KT-pseudoinvex II, FJ-pseudoinvex I and FJ-pseudoinvex II functions. Necessary and sufficient conditions are derived for FJ or KT vector critical point to be weak efficient or efficient solution of (VOP) assuming that the functions involved belong to the above newly introduced classes of functions. In Section 5, two Mond Weir type duals of (VOP) are considered and weak, strong duality results are established using the above defined classes of functions.

2. Preliminaries and definitions

The following convention for equalities and inequalities involving the vectors in $\mathbb{R}^n$ will be used throughout the paper. For any $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$,
(a) $x = y$ iff $x_i = y_i$, $\forall\, i = 1, \ldots, n$;
(b) $x < y$ iff $x_i < y_i$, $\forall\, i = 1, \ldots, n$;
(c) $x \leqq y$ iff $x_i \leq y_i$, $\forall\, i = 1, \ldots, n$;
(d) $x \leq y$ iff $x \leqq y$ and $x \neq y$;
(e) $x \nleq y$ is the negation of $x \leq y$.
If $x, y \in \mathbb{R}$, then $x < y$ and $x \leq y$ have usual meanings.

Definition 2.1 ([4])**.** Let $\phi : \mathbb{R}^n \to \mathbb{R}$. Then the directional derivative $\phi^{'}(\bar{x}, d)$ of ϕ at $\bar{x} \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is given by

$$\phi^{'}(\bar{x}, d) = \lim_{t \to 0+} \frac{\phi(\bar{x} + td) - \phi(\bar{x})}{t},$$

provided the limit exists.

Now we give the following definition and concepts from [4, 6].

Definition 2.2. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. Then the Clarke's generalized directional derivative of ϕ at $\bar{x} \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is denoted by $\phi^o(\bar{x}; d)$ and is defined as

$$\phi^o(\bar{x}; d) = \limsup_{\substack{y \to \bar{x} \\ t \downarrow 0}} \frac{\phi(y + td) - \phi(y)}{t}.$$

The Clarke's generalized subdifferential of ϕ at $\bar{x} \in \mathbb{R}^n$ is the set

$$\partial^c \phi(\bar{x}) = \{\xi \in \mathbb{R}^n : \phi^o(\bar{x}; d) \geq \langle \xi, d \rangle \,,\ \forall\, d \in \mathbb{R}^n\}.$$

Let $h = (h_1, h_2, \ldots, h_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a vector valued locally Lipschitz function. Then the Clarke's generalized directional derivative of h at $\bar{x} \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ and Clarke's generalized subdifferential of h at $\bar{x}$ are given respectively by

$$h^o(\bar{x}; d) = ({h_1}^o(\bar{x}; d), {h_2}^o(\bar{x}; d), \ldots, {h_m}^o(\bar{x}; d)),$$

$$\partial^c h(\bar{x}) = \partial^c h_1(\bar{x}) \times \partial^c h_2(\bar{x}) \times \ldots \times \partial^c h_m(\bar{x}).$$

For any $z \in \partial^c h(\bar{x})$, we denote $z = (z_1, \ldots, z_m)^T$ where each $z_i \in \partial^c h_i(\bar{x})$.

Let $k_1 : \mathbb{R}^n \to \mathbb{R}$ be a convex function, then for any $\bar{x} \in \mathbb{R}^n$, k_1 is locally Lipschitz at $\bar{x}$ and

$$\partial^c k_1(\bar{x}) = \partial k_1(\bar{x}) = \{\xi \in \mathbb{R}^n : k_1(y) - k_1(\bar{x}) \geq \ \langle \xi, y - \bar{x} \rangle, \forall\, y \in \mathbb{R}^n\},$$

where $\partial k_1(\bar{x})$ denotes the convex subdifferential of k_1 at $\bar{x}$.

Let $k_2 : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable at $\bar{x}$, then k_2 is locally Lipschitz at $\bar{x}$ and $\partial^c k_2(\bar{x}) = \{\triangledown k_2(\bar{x})\}$.

Definition 2.3 ([4])**.** Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The function ϕ is said to be regular if for all $x \in \mathbb{R}^n$ and for every direction $d \in \mathbb{R}^n$

(i) $\phi^{'}(x, d)$ exists,

(ii) $\phi^{'}(x, d) = \phi^o(x; d)$.

If ϕ and ψ are two regular functions, then $\phi + \psi$ is also a regular function.

Definition 2.4 ([4])**.** Let $C \subseteq \mathbb{R}^n$ be a non empty set. The support function of C is a function $s(\cdot\,|C) : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ given by

$$s(\bar{x}|C) = \sup_{x \in C}\ x^T \bar{x}.$$

Let $C \subseteq \mathbb{R}^n$ be a compact convex set. The support function of a compact convex set, being convex and finite everywhere, has a subgradient at every $\bar{x} \in \mathbb{R}^n$ and the set of all subgradients at $\bar{x}$ called the subdifferential is given by [18] as follows:

$$\partial s(\bar{x}|C) = \{z \in C : \bar{x}^T z = s(\bar{x}|C)\}.$$

3. Necessary optimality conditions

Consider the vector optimization problem:

(VOP) Minimize $F(x) = (f_1(x) + s(x|C_1), \ldots, f_p(x) + s(x|C_p))^T$
subject to

$$G(x) = (g_1(x) + s(x|D_1), \ldots, g_m(x) + s(x|D_m))^T \leqq 0,$$

where $F(x) = (F_1(x), \ldots, F_p(x))^T : \mathbb{R}^n \to \mathbb{R}^p$, $F_i(x) = f_i(x) + s(x|C_i)$, $i \in I = \{1, 2, \ldots, p\}$, $G(x) = (G_1(x), \ldots, G_m(x))^T : \mathbb{R}^n \to \mathbb{R}^m$, $G_j(x) = g_j(x) + s(x|D_j)$, $j \in J = \{1, 2, \ldots, m\}$, f_i, g_j are continuously differentiable functions for $i \in I$ and $j \in J$ respectively, $C_i, i \in I$ and $D_j, j \in J$ are non empty compact convex sets in $\mathbb{R}^n$. Let $X_0 = \{x \in \mathbb{R}^n : G(x) \leqq 0\}$ be the feasible set of (VOP) and $J(\bar{x}) = \{j \in J : G_j(\bar{x}) = 0\}$.

Definition 3.1. A point $\bar{x} \in X_0$ is said to be a weak efficient solution of (VOP) if there does not exist any $x \in X_0$ such that

$$F(x) < F(\bar{x}).$$

Definition 3.2. A point $\bar{x} \in X_0$ is said to be an efficient solution of (VOP) if there does not exist any $x \in X_0$ such that

$$F(x) \leq F(\bar{x}).$$

On the basis of results given by Kaniappan [12], we obtain the following Fritz−John necessary optimality conditions for (VOP).

Theorem 3.3 (FJ conditions)**.** *Let $\bar{x} \in X_0$ be a weak efficient solution of* (VOP)*. Then there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $(\bar{\lambda}, \bar{\mu}) \geq 0$ such that*

$$0 \in \bar{\lambda}^T \partial^c F(\bar{x}) + \bar{\mu}^T \partial^c G(\bar{x}), \tag{3.1}$$

$$\bar{\mu}^T G(\bar{x}) = 0. \tag{3.2}$$

Proof. Since $f_i, i \in I$ and $g_j, j \in J$ are continuously differentiable functions therefore they are locally Lipschitz functions on $\mathbb{R}^n$. Also as support functions $s(\cdot\,|C_i), i \in I$ and $s(\cdot\,|D_j), j \in J$ are convex functions, therefore they are also locally Lipschitz and hence for each $i \in I$ and $j \in J$

$$F_i(x) = f_i(x) + s(x|C_i) \quad \text{and} \quad G_j(x) = g_j(x) + s(x|D_j)$$

are locally Lipschitz functions on $\mathbb{R}^n$ showing that F and G are locally Lipschitz functions. Therefore by Kaniappan [12] we get that there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $(\bar{\lambda}, \bar{\mu}) \geq 0$ such that (3.1–3.2) hold. □

The condition (3.1) is equivalent to saying that there exist $(\bar{\lambda}, \bar{\mu}_{J(\bar{x})}) \geq 0$ such that

$$0 \in \bar{\lambda}^T \partial^c F(\bar{x}) + \bar{\mu}^T_{J(\bar{x})} \partial^c G_{J(\bar{x})}(\bar{x}).$$

Remark 3.4. Since F is a vector valued locally Lipschitz function, therefore Clarke's generalized subdifferential $\partial^c F(x)$ of F at any $x \in \mathbb{R}^n$ is given by

$$\begin{aligned} \partial^c F(x) &= \partial^c(f_1(x) + s(x|C_1)) \times \ldots \times \partial^c(f_p(x) + s(x|C_p)) \\ &= (\partial^c f_1(x) + \partial^c s(x|C_1)) \times \ldots \times (\partial^c f_p(x) + \partial^c s(x|C_p)), \end{aligned}$$

as for each $i \in I$, f_i and $s(x|C_i)$ are regular functions. Since for each $i \in I$, f_i is continuously differentiable therefore we obtain

$$\partial^c F(x) = (\nabla f_1(x) + \partial^c s(x|C_1)) \times \ldots \times (\nabla f_p(x) + \partial^c s(x|C_p)).$$

Similarly Clarke's generalized subdifferential of G can be obtained.

On the lines of Mangasarian [13], we give the following two constraint qualifications (CQ) to prove KT conditions.

Definition 3.5. (CQ1) (VOP) is said to satisfy (CQ1) at $x \in \mathbb{R}^n$ if the set $\{\nabla g_1(x) + \zeta_1, \ldots, \nabla g_m(x) + \zeta_m\}$ is linearly independent for any $\zeta_j \in \partial^c s(x|D_j), j \in J$.

Definition 3.6. (CQ2) (VOP) is said to satisfy (CQ2) at $x \in \mathbb{R}^n$ if for each $i \in I$, the set $M_i = \{\nabla f_1(x) + \xi_1, \ldots, \nabla f_{i-1}(x) + \xi_{i-1}, \nabla f_{i+1}(x) + \xi_{i+1}, \ldots, \nabla f_p(x) + \xi_p, \nabla g_1(x) + \zeta_1, \ldots, \nabla g_m(x) + \zeta_m\}$ is linearly independent for any $\xi_i \in \partial^c s(x|C_i), i \in \{1, \ldots, i-1, i+1, \ldots, p\}$ and $\zeta_j \in \partial^c s(x|D_j), j \in J$.

Theorem 3.7. (KT conditions) *Let $\bar{x} \in X_0$ be a weak efficient solution of* (VOP) *at which* (CQ1) *holds. Then there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $\bar{\lambda} \geq 0$, $\bar{\mu} \geqq 0$ such that* (3.1–3.2) *hold.*

Proof. Since $\bar{x}$ is a weak efficient solution of (VOP), therefore by Theorem 3.3, there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $(\bar{\lambda}, \bar{\mu}) \geq 0$ such that (3.1–3.2) hold. Now it remains to show that $\bar{\lambda} \neq 0$. So, let us suppose that $\bar{\lambda} = 0$. Then $\bar{\mu} \geq 0$ and (3.1) becomes

$$0 \in \bar{\mu}^T \partial^c G(\bar{x})$$

$$\Rightarrow \qquad 0 \in \sum_{j \in J} \bar{\mu}_j \partial^c G_j(\bar{x})$$

$$\Rightarrow \qquad 0 \in \sum_{j \in J} \bar{\mu}_j (\nabla g_j(\bar{x}) + \partial^c s(\bar{x}|D_j))$$

$$\Rightarrow \qquad 0 = \sum_{j \in J} \bar{\mu}_j (\nabla g_j(\bar{x}) + \bar{\zeta}_j),$$

for some $\bar{\zeta}_j \in \partial^c s(\bar{x}|D_j), j \in J$. This shows that $\{\nabla g_j(\bar{x}) + \bar{\zeta}_j\}_{j \in J}$ is linearly dependent which contradicts (CQ1). Hence $\bar{\lambda} \neq 0$. □

Here in KT conditions the condition (3.1) is equivalent to saying that there exist $\bar{\lambda} \geq 0$, $\bar{\mu}_{J(\bar{x})} \geqq 0$ such that

$$0 \in \bar{\lambda}^T \partial^c F(\bar{x}) + \bar{\mu}^T_{J(\bar{x})} \partial^c G_{J(\bar{x})}(\bar{x}).$$

Theorem 3.8 (Strong KT conditions)**.** *Let $\bar{x} \in X_0$ be a weak efficient solution of* (VOP) *at which* (CQ2) *holds. Then there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $\bar{\lambda} > 0$, $\bar{\mu} \geqq 0$ such that* (3.1–3.2) *hold.*

Proof. Since $\bar{x}$ is a weak efficient solution of (VOP), therefore by Theorem 3.3, there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $(\bar{\lambda}, \bar{\mu}) \geq 0$ such that (3.1–3.2) hold. Now to show that $\bar{\lambda}_i > 0$ for each $i \in I$. Let if possible, $\bar{\lambda}_i = 0$ for $i = 1$. Then $(\bar{\lambda}_2, \ldots, \bar{\lambda}_p, \bar{\mu}_1, \ldots, \bar{\mu}_m) \geq 0$ and (3.1) becomes

$$0 \in \sum_{i=2}^{p} \bar{\lambda}_i \partial^c F_i(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial^c G_j(\bar{x})$$

$$\Rightarrow \qquad 0 \in \sum_{i=2}^{p} \bar{\lambda}_i (\nabla f_i(\bar{x}) + \partial^c s(\bar{x}|C_i)) + \sum_{j=1}^{m} \bar{\mu}_j (\nabla g_j(\bar{x}) + \partial^c s(\bar{x}|D_j))$$

$$\Rightarrow \qquad 0 = \sum_{i=2}^{p} \bar{\lambda}_i (\nabla f_i(\bar{x}) + \bar{\xi}_i) + \sum_{j=1}^{m} \bar{\mu}_j (\nabla g_j(\bar{x}) + \bar{\zeta}_j),$$

for some $\bar{\xi}_i \in \partial^c s(\bar{x}|C_i), i \in I \setminus \{1\}$ and $\bar{\zeta}_j \in \partial^c s(\bar{x}|D_j), j \in J$ which shows that the set M_1 is linearly dependent. This contradicts (CQ2). Hence $\bar{\lambda}_i > 0$ for each $i \in I$. □

We now give the following definitions of vector critical point:

Definition 3.9. A point $\bar{x} \in X_0$ is said to be a Fritz−John vector critical point (FJVCP) for (VOP) if there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $(\bar{\lambda}, \bar{\mu}) \geq 0$ such that (3.1–3.2) hold.

Definition 3.10. A point $\bar{x} \in X_0$ is said to be a Kuhn−Tucker vector critical point (KTVCP) for (VOP) if there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $\bar{\lambda} \geq 0$, $\bar{\mu} \geqq 0$ such that (3.1–3.2) hold.

Definition 3.11. A point $\bar{x} \in X_0$ is said to be a strong Kuhn−Tucker vector critical point (SKTVCP) for (VOP) if there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $\bar{\lambda} > 0$, $\bar{\mu} \geqq 0$ such that (3.1–3.2) hold.

4. Characterization of (weak) efficient solutions through FJ/KT-pseudoinvexity

In previous section, we have given necessary conditions for (VOP) which are not sufficient. In this section, we define a class of functions on the lines of [1,14,16] and prove that they are both necessary and sufficient for every FJ or KT vector critical point to be weak efficient or efficient solution of (VOP).

Definition 4.1. Let there exist a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{p\times n} \times \mathbb{R}^{m\times n} \to \mathbb{R}^n$. Then the Problem (VOP) is said to be

(1) FJ-pseudoinvex I if

$$F(x_1) < F(x_2) \quad \Rightarrow \quad (\nabla f_i(x_2) + \xi_i)^T \eta(x_1, x_2, \xi, \zeta) < 0,\ \ \forall\, i \in I,$$
$$(\nabla g_j(x_2) + \zeta_j)^T \eta(x_1, x_2, \xi, \zeta) < 0,\ \ \forall\, j \in J(x_2),$$

(2) FJ-pseudoinvex II if

$$F(x_1) \leq F(x_2) \quad \Rightarrow \quad (\nabla f_i(x_2) + \xi_i)^T \eta(x_1, x_2, \xi, \zeta) < 0,\ \ \forall\, i \in I,$$
$$(\nabla g_j(x_2) + \zeta_j)^T \eta(x_1, x_2, \xi, \zeta) < 0,\ \ \forall\, j \in J(x_2),$$

(3) KT-pseudoinvex I if

$$F(x_1) < F(x_2) \quad \Rightarrow \quad (\nabla f_i(x_2) + \xi_i)^T \eta(x_1, x_2, \xi, \zeta) < 0,\ \ \forall\, i \in I,$$
$$(\nabla g_j(x_2) + \zeta_j)^T \eta(x_1, x_2, \xi, \zeta) \leq 0,\ \ \forall\, j \in J(x_2),$$

(4) KT-pseudoinvex II if

$$F(x_1) \leq F(x_2) \quad \Rightarrow \quad (\nabla f_i(x_2) + \xi_i)^T \eta(x_1, x_2, \xi, \zeta) < 0,\ \ \forall\, i \in I,$$
$$(\nabla g_j(x_2) + \zeta_j)^T \eta(x_1, x_2, \xi, \zeta) \leq 0,\ \ \forall\, j \in J(x_2),$$

for all feasible points x_1, x_2 and for all $\xi = (\xi_1, \ldots, \xi_p)$, $\xi_i \in \partial^c s(x_2|C_i)$, $\zeta = (\zeta_1, \ldots, \zeta_m)$, $\zeta_j \in \partial^c s(x_2|D_j)$.

Remark 4.2.

(a) FJ-pseudoinvex II $\Rightarrow$ FJ-pseudoinvex I $\Rightarrow$ KT-pseudoinvex I.

(b) FJ-pseudoinvex II $\Rightarrow$ KT-pseudoinvex II $\Rightarrow$ KT-pseudoinvex I.

We give the following example to illustrate the fact that converse of above implications may not hold.

Example 4.3. Consider the multiobjective programming problem

$$\begin{array}{ll} \text{Minimize} & F(x) = (f_1(x) + s(x|C_1), f_2(x) + s(x|C_2)) \\ \text{subject to} & G(x) = (g_1(x) + s(x|D_1), g_2(x) + s(x|D_2)) \leqq 0, \end{array}$$

where $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are given by

$$f_1(x) = x + \mathrm{e}^x,\ \ f_2(x) = 2x,\ \ g_1(x) = -x,\ \ g_2(x)) = -(x+1)^2.$$

Let $\eta : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2\times 1} \times \mathbb{R}^{2\times 1} \to \mathbb{R}$ be given by,

$$\eta(x_1, x_2, \xi, \zeta) = \begin{cases} x_1 - x_2, & x_2 > 0, \\ \quad 1, & x_2 \leq 0, \end{cases}$$

$$C_1 = [-\tfrac{3}{2}, -1],\ C_2 = [-3, -1],\ D_1 = [0, 1],\ D_2 = [-1, 0].$$

Then,

$$s(x|D_1) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}, \qquad s(x|D_2) = \begin{cases} 0, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Similarly, support functions can be found for the sets C_1 and C_2. Here, feasible set is $X_0 = [0, \infty)$. Of all the feasible points x_1, x_2, we have

$$F(x_1) < F(x_2) \text{ whenever } x_1 < x_2.$$

And for all those feasible points x_1, x_2 for which $x_1 < x_2$, we obtain $x_2 > 0$ and

$$(\triangledown f_i(x_2) + \xi_i)\ \eta(x_1, x_2, \xi, \zeta) < 0, \text{ for } i = 1, 2$$

where $\xi = (\xi_1, \xi_2)^T = (-1, -1)^T$.
Further G_1 is active at all the feasible points and G_2 is active at none of them. And we have

$$(\triangledown g_1(x_2) + \zeta_1)\ \eta(x_1, x_2, \xi, \zeta) = 0 \text{ when } x_2 > 0,$$

$$(\triangledown g_1(x_2) + \zeta_1)\ \eta(x_1, x_2, \xi, \zeta) \leq 0 \text{ when } x_2 = 0,$$

as for $x_2 > 0$, $\zeta = (\zeta_1, \zeta_2)^T = (1, 0)^T$ and for $x_2 = 0$, $\zeta_1 = [0, 1]$ and $\zeta_2 = [-1, 0]$.

Hence, the given problem is KT-pseudoinvex I. Also it can be seen that it is KT-pseudoinvex II but it is neither FJ-pseudoinvex I nor FJ-pseudoinvex II.

Next we give an Example of a problem which is FJ-pseudoinvex I but not KT-pseudoinvex II.

Example 4.4. Consider the multiobjective programming problem

$$\begin{array}{ll} \text{Minimize} & F(x) = (f_1(x) + s(x|C_1), f_2(x) + s(x|C_2)) \\ \text{subject to} & G(x) = (g_1(x) + s(x|D_1), g_2(x) + s(x|D_2)) \leqq 0, \end{array}$$

where $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are given by

$$f_1(x) = x^2,\ f_2(x) = x^3,\ g_1(x) = x - 1,\ g_2(x)) = -x^2$$

and

$$C_1 = [-1, 0],\ C_2 = [1, 2],\ D_1 = \{-1\},\ D_2 = [-2, -1].$$

Then,

$$s(x|C_1) = \begin{cases} 0, & x \geq 0 \\ -x, & x < 0 \end{cases}, \qquad s(x|C_2) = \begin{cases} 2x, & x \geq 0 \\ x, & x < 0. \end{cases}$$

Similarly, support functions can be found for the sets D_1 and D_2. Then the feasible set becomes $X_0 = (-\infty, -2] \cup [0, \infty)$. Let $\eta : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2\times 1} \times \mathbb{R}^{2\times 1} \to \mathbb{R}$ be given by,

$$\eta(x_1, x_2, \xi, \zeta) = \begin{cases} -1, & x_2 > 0, x_1 \geq 0, \\ 1, & x_2 = 0, \\ -2, & x_2 < 0, \\ 0, & x_1 < 0, x_2 > 0, {x_1}^2 - x_1 = {x_2}^2, \\ -1.5, & x_1 < 0, x_2 > 0, {x_1}^2 - x_1 \neq {x_2}^2. \end{cases}$$

Of all the feasible points x_1, x_2, we have $F(x_1) < F(x_2)$ if

(i) $x_1 \geq 0$, $x_2 \geq 0$, $x_1 < x_2$,
(ii) $x_1 < 0$, $x_2 > 0$, ${x_1}^2 - x_1 < {x_2}^2$.

And for both of the above cases

$$(\nabla f_i(x_2) + \xi_i)\, \eta(x_1, x_2, \xi, \zeta) < 0, \text{ for } i = 1, 2$$

as for $x_2 > 0$, $\xi = (\xi_1, \xi_2)^T = (0, 2)^T$ and for $x_2 = 0$, $\xi_1 = [-1, 0]$ and $\xi_2 = [1, 2]$.
But for $x_1 = -2$ and $x_2 = 2\sqrt{6}$, we have

$$F(x_1) \leq F(x_2) \text{ and } (\nabla f_1(x_2) + \xi_1)\, \eta(x_1, x_2, \xi, \zeta) = 0.$$

Further G_1 is active at none of the feasible points and G_2 is active at $x_2 = 0, -2$. And we have

$$(\nabla g_2(0) + \zeta_2)\, \eta(x_1, x_2, \xi, \zeta) < 0,$$

$$(\nabla g_2(-2) + \zeta_2)\, \eta(x_1, x_2, \xi, \zeta) < 0,$$

as for $x_2 < 0$, $\zeta = (\zeta_1, \zeta_2)^T = (-1, -2)^T$ and for $x_2 = 0$, $\zeta_1 = \{-1\}$ and $\zeta_2 = [-2, -1]$.
Hence, the above problem is FJ-pseudoinvex I but it is not KT-pseudoinvex II. Also it can be verified that this problem is KT-pseudoinvex I but it is not FJ-pseudoinvex II.

Theorem 4.5. *Every* (KTVCP) *is a weak efficient solution of* (VOP) *iff the problem* (VOP) *is* KT-*pseudoinvex* I.

Proof. Let $\bar{x}$ be a (KTVCP) and (VOP) is KT-pseudoinvex I. Then to show that $\bar{x}$ is a weak efficient solution of (VOP). Let if possible, that $\bar{x}$ is not a weak efficient solution of (VOP). Then there exists some $x \in X_0$ such that

$$F(x) < F(\bar{x}).$$

Since (VOP) is KT-pseudoinvex I, therefore there exists $\eta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{p\times n} \times \mathbb{R}^{m\times n} \to \mathbb{R}^n$ such that

$$(\nabla f_i(\bar{x}) + \xi_i)^T \eta(x, \bar{x}, \xi, \zeta) < 0, \ \ \forall\, i \in I, \tag{4.1}$$

$$(\nabla g_j(\bar{x}) + \zeta_j)^T \eta(x, \bar{x}, \xi, \zeta) \leq 0, \ \ \forall\, j \in J(\bar{x}), \tag{4.2}$$

for all $\xi = (\xi_1, \ldots, \xi_p)$, $\xi_i \in \partial^c s(\bar{x}|C_i)$, $\zeta = (\zeta_1, \ldots, \zeta_m)$, $\zeta_j \in \partial^c s(\bar{x}|D_j)$. As $\bar{x}$ is a (KTVCP), therefore there exist $\bar{\lambda} \geq 0$, $\bar{\mu}_{J(\bar{x})} \geqq 0$, $\bar{\xi}_i \in \partial^c s(\bar{x}|C_i)$, $i \in I$, $\bar{\zeta}_j \in \partial^c s(\bar{x}|D_j), j \in J(\bar{x})$ such that

$$0 = \sum_{i\in I} \bar{\lambda}_i(\nabla f_i(\bar{x}) + \bar{\xi}_i) + \sum_{j\in J(\bar{x})} \bar{\mu}_j(\nabla g_j(\bar{x}) + \bar{\zeta}_j). \tag{4.3}$$

Since $\bar{\lambda} \geq 0$, $\bar{\mu}_{J(\bar{x})} \geqq 0$, therefore by using (4.1), (4.2), we get that for all $\xi = (\xi_1, \ldots, \xi_p), \xi_i \in \partial^c s(\bar{x}|C_i)$, $\zeta = (\zeta_1, \ldots, \zeta_m)$, $\zeta_j \in \partial^c s(\bar{x}|D_j)$,

$$\left(\sum_{i\in I} \bar{\lambda}_i(\nabla f_i(\bar{x}) + \xi_i) + \sum_{j\in J(\bar{x})} \bar{\mu}_j(\nabla g_j(\bar{x}) + \zeta_j)\right)^T \eta(x, \bar{x}, \xi, \zeta) < 0,$$

which contradicts (4.3). Hence $\bar{x}$ is a weak efficient solution of (VOP).

Conversely, let us suppose that every (KTVCP) is a weak efficient solution of (VOP). Then to show that (VOP) is KT-pseudoinvex I. Let there exist two feasible points $x_1, x_2 \in X_0$ such that

$$F(x_1) < F(x_2).$$

This shows that x_2 is not a weak efficient point and hence by given hypothesis, it is not (KTVCP). This means that given any $\xi = (\xi_1, \ldots, \xi_p),\ \xi_i \in \partial^c s(x_2|C_i),\ \zeta = (\zeta_1, \ldots, \zeta_m),\ \zeta_j \in \partial^c s(x_2|D_j)$,

$$0 = \sum_{i \in I} \lambda_i(\nabla f_i(x_2) + \xi_i) + \sum_{j \in J(x_2)} \mu_j(\nabla g_j(x_2) + \zeta_j)$$

has no solution of the form $\lambda \geq 0,\ \mu_{J(x_2)} \geqq 0$. Therefore by Motzkin's alternative theorem, there exists $u \in \mathbb{R}^n$ such that

$$(\nabla f_i(x_2) + \xi_i)^T u < 0, \ \ \forall\, i \in I,$$
$$(\nabla g_j(x_2) + \zeta_j)^T u \leq 0, \ \ \forall\, j \in J(x_2).$$

Taking $u = \eta(x_1, x_2, \xi, \zeta)$, we get that (VOP) is KT-pseudoinvex I. □

The following three theorems can be proved on the lines of Theorem 4.5. We give only the statements.

Theorem 4.6. *Every* (KTVCP) *is an efficient solution of* (VOP) *iff* (VOP) *is* KT-*pseudoinvex* II.

Theorem 4.7. *Every* (FJVCP) *is a weak efficient solution of* (VOP) *iff* (VOP) *is* FJ-*pseudoinvex* I.

Theorem 4.8. *Every* (FJVCP) *is an efficient solution of* (VOP) *iff* (VOP) *is* FJ-*pseudoinvex* II.

Remark 4.9. If we consider the problem taken in Example 4.3, then it can be seen that $x = 0$ is the only KTVCP. And it is a weak efficient as well as efficient solution of the considered problem. Therefore Example 3.1 illustrate Theorems 4.5 and 4.6.

Remark 4.10. (KTVCP) in Theorems 4.5 and 4.6 can be replaced by (SKTVCP) which means KT-pseudoinvex I and KT-pseudoinvex II functions also characterize every strong Kuhn Tucker vector critical point as weak efficient and efficient solution respectively of (VOP) and *vice versa*.

5. Duality

In this section, we will establish duality results between the multiobjective problem (VOP) and the two associated dual problems of Mond-Weir type. First we give the following definitions on the lines of Osuna-Gómez and Beato-Moreno [16]. Let $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable functions.

Definition 5.1. Let there exists a vector valued function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Then the pair of functions (f, g) is said to be

(1) FJ-pseudoinvex I at $u \in \mathbb{R}^n$ on a subset $D \subseteq \mathbb{R}^n$ if

$$f(x) < f(u) \quad \Rightarrow \quad \nabla f(u)\eta(x, u) < 0,$$
$$\nabla g_{J(u)}(u)\eta(x, u) < 0,$$

(2) FJ-pseudoinvex II at $u \in \mathbb{R}^n$ on a subset $D \subseteq \mathbb{R}^n$ if

$$f(x) \leq f(u) \quad \Rightarrow \quad \nabla f(u)\eta(x, u) < 0,$$
$$\nabla g_{J(u)}(u)\eta(x, u) < 0,$$

(3) KT-pseudoinvex I at $u \in \mathbb{R}^n$ on a subset $D \subseteq \mathbb{R}^n$ if

$$f(x) < f(u) \quad \Rightarrow \quad \nabla f(u)\eta(x,u) < 0,$$

$$\nabla g_{J(u)}(u)\eta(x,u) \leqq 0,$$

(4) KT-pseudoinvex II at $u \in \mathbb{R}^n$ on a subset $D \subseteq \mathbb{R}^n$ if

$$f(x) \leq f(u) \quad \Rightarrow \quad \nabla f(u)\eta(x,u) < 0,$$

$$\nabla g_{J(u)}(u)\eta(x,u) \leqq 0,$$

for all $x \in D$ where $J(u) = \{j = 1, \ldots, m : g_j(u) = 0\}$.

Let us begin with the first problem (MWD1) formulated as follows:

(MWD1) Maximize $f(u) + u^T z$
subject to

$$\lambda^T(\nabla f(u) + z) + \mu^T(\nabla g(u) + w) = 0, \tag{5.1}$$

$$\mu^T(g(u) + u^T w) = 0,$$

$$u \in \mathbb{R}^n, \lambda \in \mathbb{R}^p, \mu \in R^m, \lambda \geq 0, \mu \geqq 0,$$

$$z = (z_1, \ldots, z_p)^T, z_i \in C_i, u^T z = (u^T z_1, \ldots, u^T z_p)^T,$$

$$w = (w_1, \ldots, w_m)^T, w_j \in D_j, u^T w = (u^T w_1, \ldots, u^T w_m)^T.$$

Let X_1 be the feasible set of (MWD1).

First we give the duality results between (VOP) and (MWD1) for weak efficient solutions using KT-pseudoinvexity I. Let us begin with the weak duality.

Theorem 5.2 (Weak Duality). *Let $x \in X_0$ and $(u, z, w, \lambda, \mu) \in X_1$. Assume that $(f(\cdot) + (\cdot)^T z, g(\cdot) + (\cdot)^T w)$ is* KT*-pseudoinvex* I *at u on X_0. Then*

$$F(x) \nless f(u) + u^T z.$$

Proof. Since $(u, z, w, \lambda, \mu) \in X_1$, therefore (5.1) holds which is equivalent to saying that there exist $\lambda \geq 0$, $\mu_{J(u)} \geqq 0$ such that

$$\lambda^T(\nabla f(u) + z) + \mu_{J(u)}^T(\nabla g_{J(u)}(u) + w_{J(u)}) = 0, \tag{5.2}$$

where $J(u) = \{j = 1, \ldots, m : g_j(u) + u^T w_j = 0\}$. Now let us suppose that

$$F(x) < f(u) + u^T z.$$

$$\Rightarrow \qquad f_i(x) + s(x|C_i) < f_i(u) + u^T z_i, \quad \forall i \in I. \tag{5.3}$$

Since $z_i \in C_i$, we have $s(x|C_i) \geq x^T z_i, \forall i \in I$, therefore (5.3) gives

$$f_i(x) + x^T z_i < f_i(u) + u^T z_i, \quad \forall i \in I$$

or

$$f(x) + x^T z - f(u) - u^T z < 0.$$

Since $(f(\cdot) + (\cdot)^T z, g(\cdot) + (\cdot)^T w)$ is KT-pseudoinvex I at u on X_0, therefore there exists $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(\nabla f(u) + z)\eta(x, u) < 0,$$

$$(\nabla g_{J(u)}(u) + w_{J(u)})\eta(x, u) \leqq 0.$$

Since $\lambda \geq 0$, $\mu_{J(u)} \geqq 0$, therefore we obtain

$$(\lambda^T(\nabla f(u) + z) + \mu^T_{J(u)}(\nabla g_{J(u)}(u) + w_{J(u)}))\eta(x, u) < 0,$$

which contradicts (5.2). Hence

$$F(x) \not< f(u) + u^T z.$$

□

Next we prove the strong duality result.

Theorem 5.3 (Strong duality). *Let $\bar{x}$ be a weak efficient solution of* (VOP) *at which* (CQ1) *holds. Then there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $\bar{\lambda} \geq 0$, $\bar{\mu} \geqq 0$, $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_p)^T, \bar{z}_i \in C_i$, $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_m)^T, \bar{w}_j \in D_j$ such that $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ is feasible for* (MWD1) *and two objective function values are equal. Further if the conditions of weak duality Theorem* 5.2 *hold for all the feasible solutions of* (VOP) *and* (MWD1)*, then $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ is a weak efficient solution of* (MWD1).

Proof. Since $\bar{x}$ is a weak efficient solution of (VOP) at which (CQ1) holds, therefore by Theorem 3.7, there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $\bar{\lambda} \geq 0$, $\bar{\mu} \geqq 0$ such that (3.1–3.2) hold. Now (3.1) gives

$$0 \in \sum_{i \in I} \bar{\lambda}_i \partial^c F_i(\bar{x}) + \sum_{j \in J} \bar{\mu}_j \partial^c G_j(\bar{x}),$$

which implies that there exist $\bar{z}_i \in \partial^c s(\bar{x}|C_i), i \in I$ and $\bar{w}_j \in \partial^c s(\bar{x}|D_j), j \in J$ such that

$$0 = \sum_{i \in I} \bar{\lambda}_i(\nabla f_i(\bar{x}) + \bar{z}_i) + \sum_{j \in J} \bar{\mu}_j(\nabla g_j(\bar{x}) + \bar{w}_j).$$

Since $\partial^c s(\bar{x}|C_i) = \partial s(\bar{x}|C_i) \subseteq C_i$ and $\partial^c s(\bar{x}|D_j) = \partial s(\bar{x}|D_j) \subseteq D_j$, we get that $\bar{z}_i \in C_i, i \in I$, $\bar{w}_j \in D_j, j \in J$ and

$$0 = \bar{\lambda}^T(\nabla f(\bar{x}) + \bar{z}) + \bar{\mu}^T(\nabla g(\bar{x}) + \bar{w}), \tag{5.4}$$

where $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_p)^T$ and $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_m)^T$.

Now (3.2) gives that

$$\sum_{j \in J} \bar{\mu}_j(g_j(\bar{x}) + s(\bar{x}|D_j)) = 0.$$

Since $\bar{w}_j \in \partial s(\bar{x}|D_j)$, we have $\bar{x}^T \bar{w}_j = s(\bar{x}|D_j), j \in J$ and hence

$$\sum_{j \in J} \bar{\mu}_j(g_j(\bar{x}) + \bar{x}^T \bar{w}_j) = 0.$$

That is,

$$\bar{\mu}^T(g(\bar{x}) + \bar{x}^T \bar{w}) = 0. \tag{5.5}$$

(5.4) and (5.5) together imply that $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ is a feasible solution of (MWD1). Now as $\bar{z}_i \in \partial s(\bar{x}|C_i)$, therefore $\bar{x}^T \bar{z}_i = s(\bar{x}|C_i), i \in I$ which gives that objective function values of (VOP) and (MWD1) are equal at $\bar{x}$ and $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ respectively.

Now let us suppose that $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ is not a weak efficient solution of (MWD1). Then there exists $(x, z, w, \lambda, \mu) \in X_1$ such that

$$f(x) + x^T z > f(\bar{x}) + \bar{x}^T \bar{z},$$

which implies that

$$f_i(x) + x^T z_i > f_i(\bar{x}) + \bar{x}^T \bar{z}_i = f_i(\bar{x}) + s(\bar{x}|C_i), \quad \forall i \in I.$$

Hence

$$f(x) + x^T z > F(\bar{x}),$$

which contradicts weak duality theorem. Hence $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ is a weak efficient solution of (MWD1). □

Now by using KT-pseudoinvexity II, duality results can be established on the lines of above theorems between (VOP) and (MWD1) for efficient solutions as follows:

Theorem 5.4 (Weak duality)**.** *Let $x \in X_0$ and $(u, z, w, \lambda, \mu) \in X_1$. Assume that $(f(\cdot) + (\cdot)^T z, g(\cdot) + (\cdot)^T w)$ is* KT*-pseudoinvex* II *at u on X_0. Then*

$$F(x) \nleq f(u) + u^T z.$$

Proof. Since $(u, z, w, \lambda, \mu) \in X_1$, therefore (5.1) holds which is equivalent to saying that there exist $\lambda \geq 0$, $\mu_{J(u)} \geqq 0$ such that (5.2) holds. Suppose that

$$F(x) \leq f(u) + u^T z.$$

$$\Rightarrow \qquad f_j(x) + s(x|C_j) < f_j(u) + u^T z_j, \quad \text{for some } j \in I \tag{5.6}$$

$$\text{and} \qquad f_i(x) + s(x|C_i) \leq f_i(u) + u^T z_i, \quad \forall i \in I, i \neq j. \tag{5.7}$$

Because $z_i \in C_i$, we have $s(x|C_i) \geq x^T z_i, \forall i \in I$, therefore (5.6) and (5.7) give

$$f_j(x) + x^T z_j < f_j(u) + u^T z_j, \quad \text{for some } j \in I$$

$$\text{and} \qquad f_i(x) + x^T z_i \leq f_i(u) + u^T z_i, \quad \forall i \in I, i \neq j$$

which implies that

$$f(x) + x^T z - f(u) - u^T z \leq 0.$$

Rest of the proof follows on the lines of Theorem 5.2 by using the condition of KT-pseudoinvexity II in place of KT-pseudoinvexity I. □

Theorem 5.5 (Strong Duality)**.** *Let $\bar{x}$ be an efficient solution of* (VOP) *at which* (CQ1) *holds. Then there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $\bar{\lambda} \geq 0$, $\bar{\mu} \geqq 0$, $\bar{z} = (\bar{z}_1, \dots, \bar{z}_p)^T, \bar{z}_i \in C_i$, $\bar{w} = (\bar{w}_1, \dots, \bar{w}_m)^T, \bar{w}_j \in D_j$ such that $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ is feasible for* (MWD1) *and two objective function values are equal. Further if the conditions of weak duality Theorem* 5.4 *hold for all the feasible solutions of* (VOP) *and* (MWD1)*, then $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ is an efficient solution of* (MWD1).

Proof. The feasibility of $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ for (MWD1) and the equality of objective function values of (VOP) at $\bar{x}$ and (MWD1) at $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ can be proved on the lines of Theorem 5.3. Further suppose that $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ is not an efficient solution of (MWD1). Then there exists $(x, z, w, \lambda, \mu) \in X_1$ such that

$$f(x) + x^T z \geq f(\bar{x}) + \bar{x}^T \bar{z},$$

which gives that

$$f_j(x) + x^T z_j > f_j(\bar{x}) + \bar{x}^T \bar{z}_j, \text{ for some } j \in I$$

and

$$f_i(x) + x^T z_i \geq f_i(\bar{x}) + \bar{x}^T \bar{z}_i, \ \forall i \in I, i \neq j.$$

Since $s(\bar{x}|C_i) = \bar{x}^T \bar{z}_i \ \forall \ i \in I$, therefore we get

$$f_j(x) + x^T z_j > f_j(\bar{x}) + s(\bar{x}|C_j), \text{ for some } j \in I$$

and

$$f_i(x) + x^T z_i \geq f_i(\bar{x}) + s(\bar{x}|C_i), \ \forall i \in I, i \neq j.$$

Hence

$$f(x) + x^T z \geq F(\bar{x}),$$

which contradicts weak duality Theorem 5.4. Hence $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ is an efficient solution of (MWD1). □

Similarly duality results can be obtained for weak efficient and efficient solutions between (VOP) and a dual problem by using FJ-pseudoinvex I and FJ-pseudoinvex II functions respectively. For this, consider the second dual problem (MWD2) as follows:

(MWD2) Maximize $f(u) + u^T z$
subject to

$$\begin{aligned}
&\lambda^T(\nabla f(u) + z) + \mu^T(\nabla g(u) + w) = 0,\\
&\mu^T(g(u) + u^T w) = 0,\\
&u \in \mathbb{R}^n, \lambda \in \mathbb{R}^p, \mu \in R^m, (\lambda, \mu) \geq 0,\\
&z = (z_1, \ldots, z_p)^T, z_i \in C_i, u^T z = (u^T z_1, \ldots, u^T z_p)^T,\\
&w = (w_1, \ldots, w_m)^T, w_j \in D_j, u^T w = (u^T w_1, \ldots, u^T w_m)^T.
\end{aligned}$$

Let X_2 be the feasible set of (MWD2).

Now we give only the statements for duality results between (VOP) and (MWD2) for weak efficient solutions as the proofs can be done on the lines of Theorems 5.2 and 5.3.

Theorem 5.6 (Weak Duality). *Let $x \in X_0$ and $(u, z, w, \lambda, \mu) \in X_2$. Assume that $(f(\cdot) + (\cdot)^T z, g(\cdot) + (\cdot)^T w)$ is* FJ-*pseudoinvex* I *at u on X_0. Then*

$$F(x) \not< f(u) + u^T z.$$

Theorem 5.7 (Strong Duality). *Let $\bar{x}$ be a weak efficient solution of* (VOP). *Then there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $(\bar{\lambda}, \bar{\mu}) \geq 0$, $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_p)^T, \bar{z}_i \in C_i$, $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_m)^T, \bar{w}_j \in D_j$ such that $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ is feasible for* (MWD2) *and two objective function values are equal. Further if the conditions of weak duality Theorem* 5.6 *hold for all the feasible solutions of* (VOP) *and* (MWD2), *then $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ is a weak efficient solution of* (MWD2).

Similarly following are only the statements for duality results between (VOP) and (MWD2) for efficient solutions using FJ-pseudoinvexity II as the proofs can be done on the lines of Theorems 5.4 and 5.5.

Theorem 5.8 (Weak Duality). *Let $x \in X_0$ and $(u, z, w, \lambda, \mu) \in X_2$. Assume that $(f(\cdot) + (\cdot)^T z, g(\cdot) + (\cdot)^T w)$ is* FJ-*pseudoinvex* II *at u on X_0. Then*

$$F(x) \nleq f(u) + u^T z.$$

Theorem 5.9 (Strong Duality). *Let $\bar{x}$ be an efficient solution of* (VOP). *Then there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$, $(\bar{\lambda}, \bar{\mu}) \geq 0$, $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_p)^T, \bar{z}_i \in C_i$, $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_m)^T$, $\bar{w}_j \in D_j$ such that $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ is feasible for* (MWD2) *and two objective function values are equal. Further if the conditions of weak duality Theorem* 5.8 *hold for all the feasible solutions of* (VOP) *and* (MWD2), *then $(\bar{x}, \bar{z}, \bar{w}, \bar{\lambda}, \bar{\mu})$ is an efficient solution of* (MWD2).

Note: In Theorems 5.7 and 5.9, the feasibility of a point for the dual problem can be proved using Theorem 3.3.

References

[1] M. Arana-Jiménez, A. Rufián-Lizana, R. Osuna-Gómez and G. Ruiz-Garzón, Pseudoinvexity, optimality conditions and efficiency in multiobjective problems; duality. *Nonlin. Anal.* **68** (2008) 24–34.

[2] M. Arana-Jiménez, A. Rufián-Lizana, R. Osuna-Gómez and G. Ruiz-Garzón, A characterization of pseudoinvexity in multiobjective programming. *Math. Comput. Model.* **48** (2008) 1719–1723.

[3] M. Arana-Jiménez, G. Ruiz-Garzón, A. Rufián-Lizana and B. Hernández-Jiménez, A characterization of pseudoinvexity for the efficiency in non-differentiable multiobjective problems. Duality. *Nonlin. Anal.* **73** (2010) 1109–1117.

[4] C.R. Bector, S. Chandra and J. Dutta, *Principles of optimization theory*. Narosa Publishing House, New Delhi (2005)

[5] A. Ben-Israel and B. Mond, What is invexity? *J. Aust. Math. Soc. Ser. B* **28** (1986) 1–9.

[6] F.H. Clarke, Optimization and nonsmooth analysis. *Interscience Publication*. Wiley, New York (1983).

[7] B.D. Craven, Invex functions and constrained local minima. *J. Aust. Math. Soc. Ser. B* **24** (1983) 357–366.

[8] B.D. Craven and B.M. Glover, Invex functions and duality. *J. Aust. Math. Soc. Ser. A* **39** (1985) 1–20.

[9] M.A. Hanson, On sufficiency of the Kuhn Tucker conditions. *J. Math. Anal. Appl.* **80** (1981) 545–550.

[10] I. Husain and Z. Abha, Jabeen, On nonlinear programing with support functions. *J. Appl. Math. Comput.* **10** (2002) 83–99.

[11] I. Husain and Z. Jabeen, On fractional programming containing support functions. *J. Appl. Math. Comput.* **18** (2005) 361–376.

[12] P. Kanniappan, Necessary conditions for optimality of nondifferentiable convex multiobjective programming. *J. Optim. Theory Appl.* **40** (1983) 167–174.

[13] O.L. Mangasarian, *Nonlinear Programming*. McGraw Hill, New York (1969)

[14] D.H. Martin, The essence of invexity. *J. Optim. Theory Appl.* **47** (1985) 65–76.

[15] B. Mond and M. Schechter, A duality theorem for a homogeneous fractional programming problem. *J. Optim. Theory Appl.* **25** 349–359 (1978)

[16] R. Osuna-Gómez, A. Beato-Moreno and A. Rufián-Lizana, Generalized convexity in multiobjective programming. *J. Math. Anal. Appl.* **233** (1999) 205–220.

[17] R. Osuna-Gómez, A. Rufián-Lizana and P. Ruíz-Canales, Invex functions and generalized convexity in multiobjective programming. *J. Optim. Theory Appl.* **98** (1998) 651–661.

[18] M. Schechter, A subgradient duality theorem. *J. Math. Anal. Appl.* **61** (1977) 850–855.

[19] W. Stadler (ed.), Multicriteria optimization in mechanics: a survey. *Appl. Mech. Rev.* **37** (1984) 277–286.

[20] W. Stadler, *Multicriteria Optimization in Engineering and in the Sciences*. Plenum Press, New York (1988).