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PERFORMANCE EVALUATION OF GENERAL NETWORK PRODUCTION PROCESSES WITH UNDESIRABLE OUTPUTS: A DEA APPROACH

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Abstract. Performance evaluation of production systems with network structure has been widely studied in recent data envelopment analysis (DEA) literature. Production systems in which outputs of some stages are consumed as the inputs of some other stages in producing final outputs. In real world applications, production processes are often complex and may produce not only desirable but also undesirable intermediate or final outputs. In this paper modelling a general network DEA is considered in the presence of undesirable outputs. A weak disposable production set consistent with undesirable outputs is introduced and some network DEA models are also proposed for performance evaluation of the production units. The proposed method is illustrated by some numerical example, including an empirical application.

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1. Introduction

Data envelopment analysis (DEA) is a non-parametric approach to evaluate the performance of decision making units (DMUs), which was first introduced by Charnes *et al.* [1]. In assessing DMUs by usual DEA models, better performance means producing the most possible outputs for the least consumption of inputs. However, in special situations, when there are undesirable outputs in the production system, an increase in the production of this type of outputs is not desirable. In such cases, one of the best solutions is to use weak disposability assumption for outputs in constructing the underlying productivity possibility set (PPS). Weak disposability means that in order to decrease the amount of undesirable outputs, the desirable outputs have to decrease proportionally, too. The question of how to abate outputs leads to two different viewpoints, and consequently two different PPSs.

While Färe and Grosskopf [2] used one abatement factor for all DMUs, Kuosmanen [3] applied a distinct abatement factor for every single DMU. Kuosmanen and Podinovski [4] illustrated that using an identical abatement factor may result in non-convex PPS. They also proved that their recommended PPS is the correct minimum extrapolation technology that satisfies the strong disposability of desirable outputs and inputs; weak disposability of all outputs and convexity axioms.

In traditional DEA models, a DMU is considered as "black box" and the effects of internal structure of the unit on its performance are totally ignored. In these models, availability of inputs and outputs value is sufficient

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to evaluate their efficiency. In recent years, however, scholars have focused on the performance valuation of the production units by taking into account internal structure and relations of sub-processes. Publishing hundreds of papers on this topic has led to the emergence of a new field in DEA called Network Data Envelopment Analysis (NDEA). It could be said that the first study in this area by Charnes *et al.* [5] dates back to 1986. They proposed that the process of Army recruitment consists of two phases including informing through advertisement and creating contract. They showed that separating these phases could contribute to the recognition of the real effect of inputs factors. Then, Färe and Whittaker [6] used a non-parametric technique in 1995 which included intermediate products to measure efficiency scores. Färe and Grosskopf [7, 8] developed a network model for different layouts. A relatively complete list of papers on NDEA along with the applied models is available in Kao [9].

Based on structure of internal operational components of DMUs, NDEA models can be categorized into series, parallel and mixed or general. In series structure, each process can consume inputs and intermediate products produced by the previous process to produce other outputs and intermediate products. Two-stage DEA, to which a lot of papers have been devoted, is a special case of series structure. A complete list and classification of papers on two-stage structure in DEA literature are presented in Halkos *et al.* [10]. In parallel structure, no intermediate product is exchanged between processes and each process operates independently. General network DEA is a combination of series and parallel modes. In other words, each process can consume inputs and intermediate products produced by other processes and produce other outputs and intermediate products.

The existence of undesirable outputs in NDEA has been dealt with in several studies. Kordrostami and Amirteimoori [11] considered a two-stage system in which the intermediate product could be undesirable. They put forward a multiplier model to evaluate performance of DMUs in this mode. Fukuyama and Weber [12] used a two-stage network model to examine DMU's performance with undesirable outputs. They used slack-based inefficiency (SBI) and directional distance function (DDF) models to compute an efficiency score in general structure of NDEA. Akther *et al.* [13] also used SBI and DDF models in two-stage network DEA and evaluated the case in which the undesirable output from the first process was used as the undesirable input to the second process. Lozano *et al.* [14] surveyed general network DEA with undesirable outputs. Taking into consideration weak disposability axiom for outputs and using an identical abatement factor for all units, they introduced a production technology to this structure and also introduced a DDF model for performance evaluation. Maghbouli *et al.* [15] examined two-stage network while the undesirable output from the first process was used as the undesirable input to the second process. They used weak disposability axiom for outputs and distinct abatement factors to build that technology. Bian *et al.* [16] used a two-stage model that included undesirable factors. They used slack-based measure (SBM) to calculate efficiency scores and divided it into production efficiency and abatement efficiency. Liu *et al.* [17] examined two-stage model in which either the input, intermediate product or output could be undesirable, by using strong disposability axiom. Wu *et al.* [18] used an additive model for two-stage structure considering undesirable intermediate product and then introduced efficiency decomposition. Lozano [19] also introduced an SBI measure in general structure with undesirable outputs, using weak disposability axiom for outputs and distinct abatement factors. He used this measure to introduce some targets for inputs, outputs and intermediate products.

In general structure of NDEA with undesirable outputs, using distinct abatement factor to define the PPS, similar to what Kuosmanen [3] done in "black box" state, is not taken into consideration in detail up to now. To develop and complete this issue, therefore, we first introduce a technology by considering distinct abatement factor to each process of DMUs. Then, we illustrate the advantage of this new modeling of undesirable outputs in NDEA in comparison with previously introduced sets. The present study is organized as follows. The next section includes an introduction of technologies relating to NDEA with undesirable outputs. The new technology set will be introduced in Section 2. Illustrative examples will be provided in Section 3. Section 4 is devoted to provide an applied example with real data of Spanish airports taken from Lozano *et al.* [14]. Section 5 concludes the discussion.

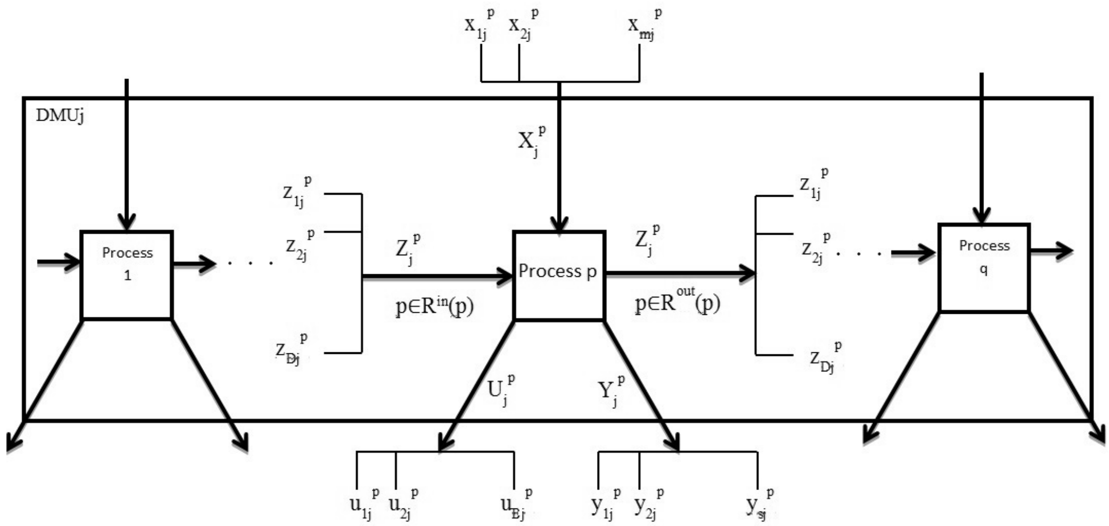

FIGURE 1. A general network DEA system.

2. Network technology set in presence of undesirable outputs

In this section we introduce production possibility set for general network structure. We consider a general structure which includes n homogeneous DMUs all of which have q processes and also have similar internal structure and internal relations, such as depicted in Figure 1.

Following the Lozano *et al.* [14], let: $I(p)$ be the set of inputs consumed by process p, $O(p)$ be the set of desirable outputs produced by process p, $U(p)$ be the set of undesirable outputs produced by process p, $R^{\text{in}}(p)$ and $R^{\text{out}}(p)$ be the sets of intermediate products consumed by and produced by process p, respectively. $P_I(i)$ be the set of processes that consume the input i, $P_O(r)$ be the set of processes that produce the desirable output r, $P_U(b)$ be the set of processes that produce the undesirable output b, $P^{\text{in}}(d)$ and $P^{\text{out}}(d)$ be the sets of processes that consume and produce the intermediate product d, respectively. Let x_{ij}^p denote the amount of input i consumed by process p of DMU$_j$ and $x_{ij} = \sum_{P\in P_I(i)} x_{ij}^p$ denote the amount of input i consumed by DMU$_j$. Let y_{ij}^p denote the amount of desirable output r produced by process p of DMU$_j$ and $y_{rj} = \sum_{P\in P_O(r)} y_{rj}^p$ for the amount of desirable output r produced by DMU_j. Let u_{bj}^p denote the amount of undesirable output b produced by process p of DMU_j and $u_{bj} = \sum_{P\in P_U(b)} u_{bj}^P$ denote the amount of undesirable output b produced by DMU$_j$. For $p \in R^{\text{in}}(p)$ and for $p \in R^{\text{out}}(p)$ let z_{dj}^p for the amount of intermediate product d consumed by and produced by process p of DMU$_j$, respectively. Also assume for each DMU that the all intermediate products produced by one DMU are completely consumed as the input of the same DMU; *i.e.* $\sum_{p\in p^{\text{out}}(d)} z_{dj}^p = \sum_{p\in p^{\text{in}}(d)} z_{dj}^p \ \ \forall d \ \ \forall j$.

In the presence of undesirable outputs, weak disposability axiom for desirable and undesirable outputs could be taken into consideration. Shepherd [20] defines this principle mathematically as follows:

Definition 2.1. Outputs are weakly disposable if $(\mathbf{x}, \mathbf{y}, \mathbf{u}) \in$T and $0 \le \theta \le 1$ implies $(\mathbf{x}, \theta\mathbf{y}, \theta\mathbf{u}) \in$T.

In order to determine production possibility set for general network DEA, at first a distinct technology must be defined for each process p. Then, with the composition of these q sub-technologies a comprehensive technology called the network technology can be defined for the general structure. Using the mentioned process, Lozano *et al.* [14] defined the technology for general network DEA with undesirable outputs as follows:

$$T_L = \left\{ (\mathbf{x},\mathbf{y},\mathbf{u}) \;\middle|\; \begin{array}{ll} \sum\limits_{p\in P_I(i)} \sum\limits_{j=1}^{n} \lambda_j^p x_{ij}^p \le x_i & \forall i \\ \sum\limits_{p\in P_O(r)} \theta^p \sum\limits_{j=1}^{n} \lambda_j^p y_{rj}^p \ge y_r & \forall r \\ \sum\limits_{p\in P_U(b)} \theta^p \sum\limits_{j=1}^{n} \lambda_j^p u_{bj}^p = u_b & \forall b \\ \sum\limits_{p\in P^{out}(d)} \theta^p \sum\limits_{j=1}^{n} \lambda_j^p z_{dj}^p - \sum\limits_{p\in P^{in}(d)} \theta^p \sum\limits_{j=1}^{n} \lambda_j^p z_{dj}^p \ge 0 & \forall d \\ \sum\limits_{j=1}^{n} \lambda_j^p = 1 & \forall p \\ \lambda_j^p \ge 0 & \forall p \; \forall j \\ 0 \le \theta^p \le 1 & \forall p \in \bigcup\limits_b P_U(b) \\ \theta^p = 1 & \forall p \notin \bigcup\limits_b P_U(b) \end{array} \right\}. \tag{2.1}$$

It should be noted that when assuming the technology to be constant return to scale (CRS), variables θ^p obtain their highest values, which is $\theta^p = 1$. In other words, in the case of CRS, there is no need to insert θ^p in defining the technology set (see Färe and Grosskopf [2]). In this paper, we define technologies just for VRS modes. Note that in order to change RTS assumption to non-decreasing return to scale (NDRS) and non-increasing return to scale (NIRS), it is just enough to modify the constraint on intensity weights to greater than or equal to ($\ge$) and smaller than or equal to ($\le$).

In set T_L uniform abatement factor θ^p is used for all pth process of all DMUs, just like what Färe and Grosskopf [2] used in "black bok" mode. Using uniform abatement factor for all DMUs may create the non-convex PPS, as Kuosmanen and Podinovski [4] illustrated in details. In the structure of NDEA, this leads to the possibility that sub-technologies to be non-convex and also creates some problems in recognizing inefficiency; which will be discussed later. So, we use distinct abatement factors θ_j^p for pth process of DMUs in technology set T_L just as Kuosmanen [3] used. Then, the technology set takes the following form:

$$T_{KK} = \left\{ (\mathbf{x},\mathbf{y},\mathbf{u}) \;\middle|\; \begin{array}{ll} \sum\limits_{p\in P_I(i)} \sum\limits_{j=1}^{n} \lambda_j^p x_{ij}^p \le x_i & \forall i \\ \sum\limits_{p\in P_O(r)} \sum\limits_{j=1}^{n} \theta_j^p \lambda_j^p y_{rj}^p \ge y_r & \forall r \\ \sum\limits_{p\in P_U(b)} \sum\limits_{j=1}^{n} \theta_j^p \lambda_j^p u_{bj}^p = u_b & \forall b \\ \sum\limits_{p\in P^{out}(d)} \sum\limits_{j=1}^{n} \theta_j^p \lambda_j^p z_{dj}^p - \sum\limits_{p\in P^{in}(d)} \sum\limits_{j=1}^{n} \lambda_j^p z_{dj}^p \ge 0 & \forall d \\ \sum\limits_{j=1}^{p} \lambda_j^p = 1 & \forall p \\ \lambda_j^p \ge 0 & \forall j \; \forall p \\ 0 \le \theta_j^p \le 1 & \forall j \; \forall p \\ \theta_j^p = 1 & \forall j \; \forall p \notin \bigcup\limits_b P_U(b) \end{array} \right\}. \tag{2.2}$$

It is clear that by imposing $\theta_1^p = \theta_2^p = \ldots = \theta_n^p$ $(p = 1,\ldots,q)$ T_L and T_{KK} are equal and also shows T_L is a special form of T_{KK}. So, we have $T_L \subseteq T_{KK}$.

In general, models that were applied to technology T_L were non-linear programing (NLP), which can change to linear form only in special cases. However, an advantage of using distinct abatement factor is that it can be presented as a linear technology set, as Kuosmanen [3] discussed. So, by setting $\lambda_j^p = \alpha_j^p + \beta_j^p$ $(p = 1,\ldots,q)$ which β_j^p

represents the part of output of process p of DMUj that is abated by scaling down, whereas α_j^p represents the part of output of process p of DMUj that remain active. *i.e.*, $\beta_j^p = (1-\theta_j^p)\lambda_j^p$ and $\alpha_j^p = \theta_j^p\lambda_j^p$, we introduce the following linear representation of the technology set for general network DEA in the presence of undesirable outputs as follows:

$$\overline{T}_{KK} = \left\{ (\mathbf{x},\mathbf{y},\mathbf{u}) \left| \begin{array}{ll} \sum\limits_{p\in P_I(i)} \sum\limits_{j=1}^{n} (\alpha_j^p + \beta_j^p) x_{ij}^p \le x_i & \forall i \\ \sum\limits_{p\in P_O(r)} \sum\limits_{j=1}^{n} \alpha_j^p y_{rj}^p \ge y_r & \forall r \\ \sum\limits_{p\in P_U(b)} \sum\limits_{j=1}^{n} \alpha_j^p u_{bj}^p = u_b & \forall b \\ \sum\limits_{p\in P^{out}(d)} \sum\limits_{j=1}^{n} \alpha_j^p z_{dj}^p - \sum\limits_{p\in P^{in}(d)} \sum\limits_{j=1}^{n} (\alpha_j^p + \beta_j^p) z_{dj}^p \ge 0 & \forall d \\ \sum\limits_{j=1}^{n} (\alpha_j^p + \beta_j^p) = 1 & \forall p \\ \alpha_j^p \ge 0, \beta_j^p \ge 0 & \forall j\ \forall p \\ \beta_j^p = 0 & \forall j\ \forall p \notin \bigcup\limits_b P_U(b) \end{array} \right. \right\}. \tag{2.3}$$

Technology set $\overline{T}_{KK}$ has more accurate result in comparison with T_L; which is illustrated by an example in Section 4. Furthermore, in the next section, we show that in the presence of undesirable outputs, Technology set $\overline{T}_{KK}$ cannot cover all network structures and needs some modifications; such as, in evaluating production systems with more than two series process with identical intermediate products which is produced or consumed by same processes and undesirable outputs are coexist.

3. A new technology set for general network DEA with undesirable output

In this section, we will provide a technology set for general network system with undesirable outputs, inspired by the model put forward in Kazemi Matin and Azizi [21].

Let us assume that z_{dj}^{pc} denote the amount of dth intermediate product produced by process p of DMUj and all or part of it is consumed by process c $(c = 1,\dots,q)$. Assume that y_{rj}^p denote the amount of rth desirable output produced by process p of DMUj and part of which (y_{rj}^{op}) exits the system as final output and the rest (y_{rj}^{Ipc}) remains in DMUj and all or part of it is consumed as input in process c $(c = 1,\dots,q)$. This means $y_{rj}^p = y_{rj}^{op} + y_{rj}^{Ipc}$. Also y_{rj}^{op} must be positive at least for one of DMUs. Because putting it equal to zero, indicates that all produced outputs will enter the system again and so, y_{rj}^p must be considered as intermediate product. Also, assume that u_{bj}^p denote the amount of bth undesirable output produced by process p of DMUj and part of which (u_{bj}^{op}) exits the system as a final undesirable output and the rest (u_{bj}^{Ipc}) remains in DMUj and all or part of it is consumed as input in process c $(c = 1,\dots,q)$. This means $u_{bj}^p = u_{bj}^{op} + u_{bj}^{Ipc}$. Also, similar to final desirable outputs, u_{bj}^{op} must be positive at least for one of DMUs, too. Kazemi Matin and Azizi [21] proposed a radial input-oriented model to evaluate DMU_0 in general network system as follows:

$$\begin{array}{lll} \theta_0 = \text{Min } h & & \\ \text{s.t.} & \sum\limits_{p=1}^{q} \sum\limits_{j=1}^{n} \lambda_j^p x_{ij}^p \le h \sum\limits_{p=1}^{q} x_{i0}^p; & i = 1,\dots,m \\ & \sum\limits_{p=1}^{q} \sum\limits_{j=1}^{n} \lambda_j^p y_{rj}^p - \sum\limits_{p=1}^{q} \sum\limits_{j=1}^{n} \sum\limits_{c=1}^{q} \lambda_j^c y_{rj}^{Ipc} \ge \sum\limits_{p=1}^{q} y_{r0}^{op}; & r = 1,\dots,s \\ & \sum\limits_{c=1}^{q} \sum\limits_{j=1}^{n} \lambda_j^p z_{dj}^{pc} \ge \sum\limits_{c=1}^{q} \sum\limits_{j=1}^{n} \lambda_j^c z_{dj}^{pc}; & d = 1,\dots,D, \quad p = 1,\dots,q \\ & \lambda_j^p \ge 0; & j = 1,\dots,n, \quad p = 1,\dots,q. \end{array} \tag{3.1}$$

The technology applied in the above model can be used in all network DEA structures. Thus, with undesirable outputs, using distinct abatement factors introduced by Kuosmanen [3] and ultimately, making the technology linear, similar to what we did in technology set $\overline{T}_{KK}$, we introduce the correct technology for general network DEA with undesirable outputs as follows:

$$\hat{T}_{KK} = \left\{ (\mathbf{x}, \mathbf{y}, \mathbf{u}) \left| \begin{array}{ll} \sum_{p=1}^{q} \sum_{j=1}^{n} (\alpha_j^p + \beta_j^p) x_{ij}^p \leq \sum_{p=1}^{q} x_i^p \ (= x_i) & \forall i \\ \sum_{p=1}^{q} \sum_{j=1}^{n} \alpha_j^p y_{rj}^p - \sum_{p=1}^{q} \sum_{j=1}^{n} \sum_{c=1}^{q} \alpha_j^c y_{rj}^{Ipc} \geq \sum_{p=1}^{q} y_r^{op} (= y_r^o) & \forall r \\ \sum_{p=1}^{q} \sum_{j=1}^{n} \alpha_j^p u_{bj}^p - \sum_{p=1}^{q} \sum_{j=1}^{n} \sum_{c=1}^{q} \alpha_j^c u_{bj}^{Ipc} = \sum_{p=1}^{q} u_b^{op} (= u_b^o) & \forall b \\ \sum_{c=1}^{q} \sum_{j=1}^{n} \alpha_j^p z_{dj}^{pc} - \sum_{c=1}^{q} \sum_{j=1}^{n} (\alpha_j^c + \beta_j^c) z_{dj}^{pc} \geq 0 & \forall d \ \forall p \\ \sum_{p=1}^{q} (\alpha_j^p + \beta_j^p) = 1 & \forall p \\ \alpha_j^p \geq 0, \beta_j^p \geq 0 & \forall j \ \forall p \\ \beta_j^p = 0 & \forall j \ \forall p \notin \bigcup_b P_U(b) \end{array} \right. \right\}. \tag{3.2}$$

We see that technology set $\widehat{T}_{KK}$ covers more general modes of the NDEA. The notable difference between technology $\overline{T}_{KK}$ and technology $\widehat{T}_{KK}$ is in the fourth constraint that relates to global balance. Balance constraint in $\overline{T}_{KK}$ reads that the amount of each intermediate product produced in the system is sufficient to satisfy the amount of that intermediate product that is consumed whereas balance constraint in $\widehat{T}_{KK}$ means that the sum of produced intermediate products of each process is at least as high as the sum of consumed corresponding intermediate products by other sub-processes. This difference can cause to obtain the completely dissimilar efficiency scores by using $\overline{T}_{KK}$ or $\widehat{T}_{KK}$ in some special cases; which is illustrated by Example 4.2 in the next section.

3.1. Dual formulations of the general network DEA technologies

By applying the duality theory of linear programming, we obtain the dual linear programs for the input-oriented radial measure building upon technology set $\hat{T}_{KK}$. Then, we seek for economic interpretation for the case distinct abatement factors used in modeling weak disposability assumption. We consider a primal input-oriented radial model building upon technology $\widehat{T}_{KK}$ as follows:

$$\begin{aligned} \theta_0 = \text{Min} \ \ & h \\ \text{s.t.} \ \ & (h\mathbf{x}_o, \mathbf{y}_o, \mathbf{u}_o) \in \hat{T}_{KK} \end{aligned} \tag{3.3}$$

or equivalently:

$$
\begin{aligned}
\theta_0 = \text{Min} & & & (primal)\\
\text{s.t.} \quad & \sum_{p=1}^{q}\sum_{j=1}^{n}\left(\alpha_j^p+\beta_j^p\right)x_{ij}^p \le h\sum_{p=1}^{q}x_{i0}^p; & & (v_i)\\
& \sum_{p=1}^{q}\sum_{j=1}^{n}\alpha_j^p y_{rj}^p - \sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{c=1}^{q}\alpha_j^c y_{rj}^{Ipc} \ge \sum_{p=1}^{q} y_{r0}^{op};\ r=1,\dots,s & & (g_r)\\
& \sum_{p=1}^{q}\sum_{j=1}^{n}\alpha_j^p u_{bj}^p - \sum_{p=1}^{q}\sum_{j=1}^{n}\sum_{c=1}^{q}\alpha_j^c u_{bj}^{Ipc} = \sum_{p=1}^{q} u_{b0}^{op};\ b=1,\dots,B & & (t_b)\\
& \sum_{c=1}^{q}\sum_{j=1}^{n}\alpha_j^p z_{dj}^{pc} \ge \sum_{c=1}^{q}\sum_{j=1}^{n}\left(\alpha_j^c+\beta_j^c\right)z_{dj}^{pc}; & d=1,\dots,D \quad p=1,\dots,q & (w_d^p)\\
& \sum_{j=1}^{n}\left(\alpha_j^p+\beta_j^p\right)=1\ ; & p=1,\dots,q & (\varphi^p)\\
& \alpha_j^p,\beta_j^p \ge 0\ ; & p=1,\dots,q \quad j=1,\dots,n &\\
& \beta_j^p = 0\ \ ; & \forall p \notin \bigcup_b P_U(b) \quad j=1,\dots,n. &
\end{aligned}
$$

This model in the technology set $\widehat{T}_{KK}$, detects the proportional reduction of inputs while keeping the desirable and undesirable outputs unchanged. The dual formulation of the above linear programming model takes the following form:

$$
\begin{aligned}
\theta_0 = \text{Max} \quad & \sum_{p=1}^{q}\sum_{r=1}^{s} g_r y_{r0}^{op} - \sum_{p=1}^{q}\sum_{b=1}^{B} t_b u_{b0}^{op} - \sum_{p=1}^{q}\varphi^p & (dual)\\
\text{s.t.} \quad & \sum_{p=1}^{q}\sum_{i=1}^{m} v_i x_{i0}^p = 1 & (\theta_0)\\
& \sum_{i=1}^{m} v_i x_{ij}^p - \sum_{r=1}^{s} g_r y_{rj}^p + \sum_{c=1}^{q}\sum_{r=1}^{s} g_r y_{rj}^{Icp} + \sum_{b=1}^{B} t_b u_{bj}^p - \sum_{c=1}^{q}\sum_{b=1}^{B} t_b u_{bj}^{Icp} &\\
& - \sum_{c=1}^{q}\sum_{d=1}^{D} w_d^p z_{dj}^{pc} + \sum_{c=1}^{q}\sum_{d=1}^{D} w_d^c z_{dj}^{cp} + \varphi^p \ge 0 \quad j=1,\dots,n \quad p=1,\dots,q & (\alpha_j^p)\\
& \sum_{i=1}^{m} v_i x_{ij}^p + \sum_{c=1}^{q}\sum_{d=1}^{D} w_d^c z_{dj}^{cp} + \varphi^p \ge 0 \quad \forall p \in \bigcup_b P_U(b) \quad j=1,\dots,n, & (\beta_j^p)\\
& v_i \ge 0 \qquad i=1,\dots,m &\\
& g_r \ge 0 \qquad r=1,\dots,s &\\
& w_d^p \ge 0 \qquad d=1,\dots,B \quad p=1,\dots,q &\\
& t_b\ free \qquad b=1,\dots,B &\\
& \varphi^p\ free \qquad p=1,\dots,p. &
\end{aligned}
\tag{3.4}
$$

To make clear, we have point out the dual variables corresponding to the constraints of the primal problem in parentheses, in front of each constraint. In the dual formulation, variables v_i, g_r, t_b and w_d^p denote the shadow prices of inputs, desirable and undesirable outputs, and intermediate products, respectively. The variable φ^p associates with the convexity constraint of the primal problem.

From the equality $\sum_{p=1}^{q}\sum_{i=1}^{m} v_i x_{i0}^p = 1$, we can add $1-\sum_{p=1}^{q}\sum_{i=1}^{m} v_i x_{i0}^p$ to objective function of dual problem. So, new objective function will be:

$$
\text{Max}\ \ 1-\left[\sum_{p=1}^{q}\varphi^p - \left(\sum_{p=1}^{q}\sum_{r=1}^{s} g_r y_{r0}^{op} - \sum_{p=1}^{q}\sum_{b=1}^{B} t_b u_{b0}^{op} - \sum_{p=1}^{q}\sum_{i=1}^{m} v_i x_{i0}^p\right)\right].
$$

The mathematical phrase inside the brackets means the normalized profit inefficiency.

TABLE 1. Data for 3 DMUs in Example 4.1.

	x	z	u	y
A	4	2	2	4
B	1	1	4	2
C	4	2	1	2

Furthermore, in order to have a good interpretation of dual models constraints, we can change the dual program formulation to follow fractional model:

$$\begin{aligned}
\theta_0 = \text{Max}\ & \frac{\sum\limits_{p=1}^{q}\sum\limits_{r=1}^{s} g_r y_{r0}^{op} - \sum\limits_{p=1}^{q}\sum\limits_{b=1}^{B} t_b u_{b0}^{op} - \sum\limits_{p=1}^{q} \varphi^p}{\sum\limits_{p=1}^{q}\sum\limits_{i=1}^{m} v_i x_{i0}^{p}} \\
\text{s.t.}\ & \frac{\sum\limits_{r=1}^{s} g_r y_{rj}^{p} - \sum\limits_{b=1}^{B} t_b u_{bj}^{p} + \sum\limits_{c=1}^{q}\sum\limits_{d=1}^{D} w_d^p z_{dj}^{pc} - \varphi^p}{\sum\limits_{i=1}^{m} v_i x_{ij}^{p} + \sum\limits_{c=1}^{q}\sum\limits_{r=1}^{s} g_r y_{rj}^{Icp} + \sum\limits_{c=1}^{q}\sum\limits_{b=1}^{B} t_b u_{bj}^{Icp} + \sum\limits_{c=1}^{q}\sum\limits_{d=1}^{D} w_d^c z_{dj}^{cp}} \le 1 \qquad j = 1,\dots,n \qquad p = 1,\dots,q \\
& \sum_{i=1}^{m} v_i x_{ij}^{p} + \sum_{c=1}^{q}\sum_{d=1}^{D} w_d^c z_{dj}^{cp} + \varphi^p \ge 0 \quad \forall p \in \bigcup_b P_U(b) \qquad j = 1,\dots,n \\
& v_i \ge 0 \qquad i = 1,\dots,m \\
& g_r \ge 0 \qquad r = 1,\dots,s \\
& w_d^p \ge 0 \qquad d = 1,\dots,B \qquad p = 1,\dots,q \\
& t_b\ free \qquad b = 1,\dots,B \\
& \varphi^p\ free \qquad p = 1,\dots,p.
\end{aligned} \tag{3.5}$$

This model finds the maximum of ratio total weighted outputs to total weighted inputs of DMU_0. The weights v_i, g_r, t_b and w_d^p are corresponding to inputs, desirable and undesirable outputs, and intermediate products of DMU_0, such that for each DMU and each process,these weight hold on the constraints. First constraints express that for any arbitrary process, aggregation of all outputs, including desirable outputs and produced intermediate products detracting the undesirable outputs, should not to be greater than aggregation of all inputs, including inputs, consumed intermediate product and part of desirable and undesirable outputs which are produced in other processes and consumed as inputs in desired process. Second constraints, which are obtained from weak disposability intensity variables β_j^p, only hold for processes that produce undesirable outputs.

4. Illustrative examples

In this section we use two examples to illustrate the differences in evaluating the efficiency of DMUs in the models defined on technologies $T_L, \overline{T}_{KK}$ and $\widehat{T}_{KK}$.

Example 4.1. This example shows the advantage of applying $\widehat{T}_{KK}$ in comparison with applying T_L. It will be seen that an incorrect solution may be obtain by using T_L.

Assume that there are three hypothetical DMUs labeled A, B and C, each of which uses one input to produce one desirable output and one undesirable output. Also, each DMU has two stages, as depicted in Figure 2. In the first stage input x is consumed and intermediate product z is produced. In second stage, intermediate product z is consumed as input and a desirable output y along with an undesirable output u are produced. Data relating to inputs, intermediate products and outputs of these DMUs are presented in Table 1.

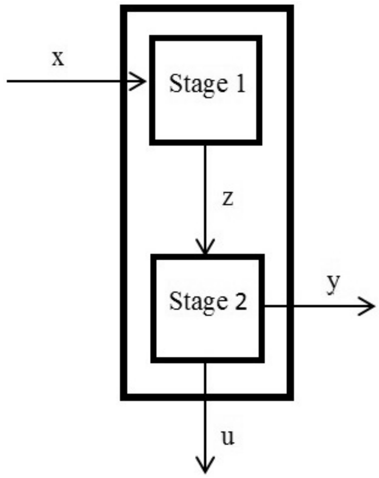

Figure 2. Structure of DMUs in example 1.

In order to evaluate the performance of the units, we consider a radial input-oriented measure, considering the internal structure of DMUs on technology set T_L as follows:

$$\begin{aligned} &\text{Min} \;\; h \\ &\text{s.t.} \quad (hx_o, y_o, u_o) \in T_L \end{aligned} \tag{4.1}$$

or equivalently:

$$\begin{aligned} &\text{Min} \;\; h \\ &\text{s.t.} \quad 4\lambda_1^1 + \lambda_2^1 + 4\lambda_3^1 \le hx_0 \\ &\qquad\; 4\lambda_1^2 + 2\lambda_2^2 + 2\lambda_3^2 \ge \theta y_0 \\ &\qquad\; 2\lambda_1^2 + 4\lambda_2^2 + \lambda_3^2 = \theta u_0 \\ &\qquad\; 2\left(\lambda_1^1 - \lambda_1^2\right) + \left(\lambda_2^1 - \lambda_2^2\right) + 2\left(\lambda_3^1 - \lambda_3^2\right) \ge 0 \\ &\qquad\; \lambda_1^1 + \lambda_2^1 + \lambda_3^1 = 1 \\ &\qquad\; \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 \\ &\qquad\; \theta \ge 1 \\ &\qquad\; \lambda_j^p \ge 0 \quad p = 1, 2 \quad j = 1, 2, 3. \end{aligned}$$

And, in this example, for technologies $\overline{T}_{KK}$ and $\widehat{T}_{KK}$ the models created are same as follows:

$$\begin{aligned} &\text{Min} \;\; h \\ &\text{s.t.} \quad (hx_o, y_o, u_o) \in \hat{T}_{KK} \end{aligned} \tag{4.2}$$

Table 2. Input-oriented radial measures using models (9) and (10).

	Model (9)				Model (10)			
	h^*	θ^*	$\lambda^{1^*}=(\lambda_1^1,\lambda_2^1,\lambda_3^1)$	$\lambda^{2^*}=(\lambda_1^2,\lambda_2^2,\lambda_3^2)$	h^*	$\mu^{1^*}=(\alpha_1^1,\alpha_2^1,\alpha_3^1)$	$\mu^{2^*}=(\alpha_1^2,\alpha_2^2,\alpha_3^2)$	$\nu^{2^*}=(\beta_1^2,\beta_2^2,\beta_3^2)$
A	1	1	(1,0,0)	(1,0,0)	1	(1,0,0)	(1,0,0)	(0,0,0)
B	1	1	(0,1,0)	(0,1,0)	1	(0,1,0)	(0,1,0)	(0,0,0)
C	1	1	(1,0,0)	(0,0,1)	0.625	(0.5,0.5,0)	(0.5,0,0)	(0,0.5,0)

or equivalently:

$$\begin{aligned}
&\text{Min} \;\; h\\
&\text{s.t.} \;\; 4\alpha_1^1+\alpha_2^1+4\alpha_3^1 \le hx_0\\
&\qquad 4\alpha_1^2+2\alpha_2^2+2\alpha_3^2 \ge y_0\\
&\qquad 2\alpha_1^2+4\alpha_2^2+\alpha_3^2 = u_0\\
&\qquad 2\left(\alpha_1^1-\alpha_1^2-\beta_1^2\right)+\left(\alpha_2^1-\alpha_2^2-\beta_2^2\right)+2\left(\alpha_3^1-\alpha_3^2-\beta_3^2\right)\ge 0\\
&\qquad \alpha_1^1+\alpha_2^1+\alpha_3^1=1\\
&\qquad \alpha_1^2+\beta_1^2+\alpha_2^2+\beta_2^2+\alpha_3^2+\beta_3^2=1\\
&\qquad \alpha_j^2\ge 0\;,\;\beta_j^p\ge 0 \quad p=1,2 \quad j=1,2,3.
\end{aligned}$$

In Table 2, results of solving models (9) and (10) are presented. As it can be seen, DMU_C is recognized as efficient in model (9), while model (10) shows it is inefficient.

Model (10) determines the target for DMU_C as follows:

$$C^*=(x_c^*,z_c^*,u_c^*,y_c^*)=\left(\sum_j \alpha_j^{1^*}x_j,\sum_j \alpha_j^{1^*}z_j,\sum_j \alpha_j^{2^*}u_j,\sum_j \alpha_j^{2^*}y_j\right)=(2.5,1.5,1,2).$$

Now, we demonstrate axiomatically that C^* is a feasible production plan. Regarding weak disposability axiom for outputs, desirable and undesirable outputs from second stage of DMU_B can be proportionally abated toward zero. As a result, point $B'=(1,1,0,0)$ belongs to PPS. Also using convexity axiom and having $\lambda=0.5$, convex composition of DMU_A and virtual unit B', which equals $\lambda A+(1-\lambda)B'=(2.5,1.5,1,2)=C^*$ belongs to PPS.

Now, using model (9) for virtual unit C^*, we have $\beta_{C^*}^*=1.6$. This means that the feasible virtual unit C^* does not belong to technology T_L.

Like what Kuosmanen and podinovski [4] did, in the given example it is possible to show that in case of using uniform abatement factors for all DMUs, the sub-technology made for stage 2 will be non-convex. So, in general, technology T_L might exclude some feasible DMUs that are feasible axiomatically.

Example 4.2. In this example we intend to illustrate the difference between technology $\overline{T}_{KK}$ and technology $\widehat{T}_{KK}$. It will be seen that an incorrect solution may be obtain by applying $\overline{T}_{KK}$. We consider four hypothetical DMUs, each unit include four stages, as depicted in Figure 3. Data relating to inputs, intermediate products and desirable and undesirable outputs are presented in Table 3.

To evaluate DMU_0, radial input-oriented models under CRS assumption, building upon technologies $\overline{T}_{KK}$ and $\widehat{T}_{KK}$ are presented as model (11) and model (12), respectively,

$$\begin{aligned}
&\text{Min} \; h\\
&\text{s.t.} \;\; (h\mathbf{x}_o,\mathbf{y}_o,\mathbf{u}_o)\in \overline{T}_{KK}
\end{aligned} \tag{4.3}$$

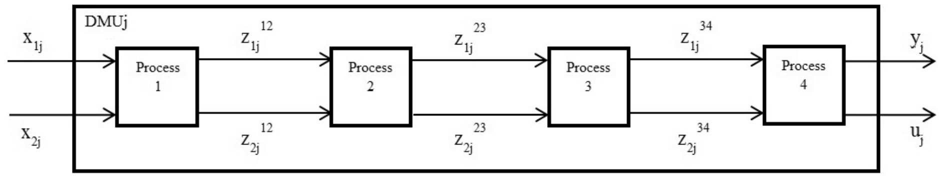

Figure 3. etwork structure of DMUs in Example 4.2.

Table 3. Inputs, intermediate products and output data for DMUs in Example 4.2.

	x_1	x_2	z_1^{12}	z_1^{23}	z_1^{34}	z_2^{12}	z_2^{23}	z_2^{34}	y	u
A	1	4	0	5	3	5	4	2	4	3
B	2	3	2	0	2	4	5	0	3	2
C	3	2	3	2	0	3	0	1	2	1
D	4	1	4	4	2	0	3	3	1	0

and equivalently:

$$\begin{aligned}
&\text{Min } h\\
&\text{s.t.}\quad \alpha_1^1+2\alpha_2^1+3\alpha_3^1+4\alpha_4^1 \le h x_{1\,0}\\
&\qquad 4\alpha_1^1+3\alpha_2^1+2\alpha_3^1+\alpha_4^1 \le h x_{2\,0}\\
&\qquad 4\alpha_1^4+3\alpha_2^4+2\alpha_3^4+\alpha_4^4 \ge y_0\\
&\qquad 3\alpha_1^4+2\alpha_2^4+\alpha_3^4 = u_0\\
&\qquad \left[2\left(\alpha_2^1-\alpha_2^2\right)+3\left(\alpha_3^1-\alpha_3^2\right)+4\left(\alpha_4^1-\alpha_4^2\right)\right]\\
&\qquad\quad +\left[5\left(\alpha_1^2-\alpha_1^3\right)+\ 2\left(\alpha_3^2-\alpha_3^3\right)+4\left(\alpha_4^2-\alpha_4^3\right)\right]\\
&\qquad\quad +\left[3\left(\alpha_1^3-\alpha_1^4-\beta_1^4\right)+2\left(\alpha_2^3-\alpha_2^4-\beta_2^4\right)+2\left(\alpha_4^3-\alpha_4^4-\beta_4^4\right)\right]\ge 0\\
&\qquad \left[5\left(\alpha_1^1-\alpha_1^2\right)+4\left(\alpha_2^1-\alpha_2^2\right)+3\left(\alpha_3^1-\alpha_3^2\right)\right]\\
&\qquad\quad +\left[4\left(\alpha_1^2-\alpha_1^3\right)+5\left(\alpha_2^2-\alpha_2^3\right)+3\left(\alpha_4^2-\alpha_4^3\right)\right]\\
&\qquad\quad +\left[2\left(\alpha_1^3-\alpha_1^4-\beta_1^4\right)+\left(\alpha_3^3-\alpha_3^4-\beta_3^4\right)+3\left(\alpha_4^3-\alpha_4^4-\beta_4^4\right)\right]\ge 0\\
&\qquad \alpha_j^1,\alpha_j^2,\alpha_j^3,\alpha_j^4,\beta_j^4\ge 0\quad j=1,\ldots,4
\end{aligned}$$

$$\begin{aligned}
&\text{Min } h\\
&\text{s.t.}\quad (h\mathbf{x}_o,\mathbf{y}_o,\mathbf{u}_o)\in \hat{T}_{KK}
\end{aligned}\tag{4.4}$$

and equivalently:

$$\begin{aligned}
&\text{Min } h\\
&\text{s.t.}\quad \alpha_1^1+2\alpha_2^1+3\alpha_3^1+4\alpha_4^1 \le h x_{1\,0}\\
&\qquad 4\alpha_1^1+3\alpha_2^1+2\alpha_3^1+\alpha_4^1 \le h x_{2\,0}\\
&\qquad 4\alpha_1^4+3\alpha_2^4+2\alpha_3^4+\alpha_4^4 \ge y_0\\
&\qquad 3\alpha_1^4+2\alpha_2^4+\alpha_3^4 = u_0\\
&\qquad 2\left(\alpha_2^1-\alpha_2^2\right)+3\left(\alpha_3^1-\alpha_3^2\right)+4\left(\alpha_4^1-\alpha_4^2\right)\ge 0\\
&\qquad 5\left(\alpha_1^2-\alpha_1^3\right)+\ 2\left(\alpha_3^2-\alpha_3^3\right)+4\left(\alpha_4^2-\alpha_4^3\right)\ge 0\\
&\qquad 3\left(\alpha_1^3-\alpha_1^4-\beta_1^4\right)+2\left(\alpha_2^3-\alpha_2^4-\beta_2^4\right)+2\left(\alpha_4^3-\alpha_4^4-\beta_4^4\right)\ge 0\\
&\qquad 5\left(\alpha_1^1-\alpha_1^2\right)+4\left(\alpha_2^1-\alpha_2^2\right)+3\left(\alpha_3^1-\alpha_3^2\right)\ge 0\\
&\qquad 4\left(\alpha_1^2-\alpha_1^3\right)+5\left(\alpha_2^2-\alpha_2^3\right)+3\left(\alpha_4^2-\alpha_4^3\right)\ge 0\\
&\qquad 2\left(\alpha_1^3-\alpha_1^4-\beta_1^4\right)+\left(\alpha_3^3-\alpha_3^4-\beta_3^4\right)+3\left(\alpha_4^3-\alpha_4^4-\beta_4^4\right)\ge 0\\
&\qquad \alpha_j^1,\alpha_j^2,\alpha_j^3,\alpha_j^4,\beta_j^4\ge 0\quad j=1,\ldots,4.
\end{aligned}$$

Table 4. Efficiency scores of DMUs in Example 4.2.

	Model(11)	Model(12)
A	0	0.47
B	0	0.26
C	0	0.22
D	0	1

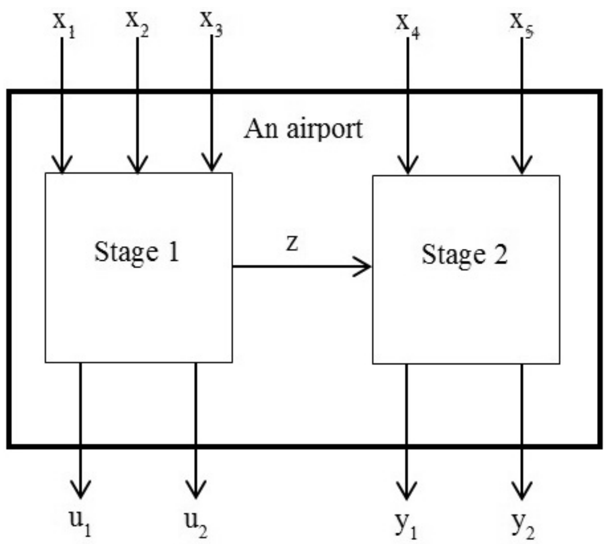

Figure 4. Network structure of airports.

In Table 4, the computed efficiency scores relating to these DMUs in evaluation with models (11) and (12) are presented.

As it can be seen, model (11) has come up with zero values for all four units in determining their efficiency score. This means that in the case of this special example in which all units include series processes with identical intermediate products, this model fails to estimate the efficiency score for DMUs.

5. An empirical application

Here, we apply the above introduced models on data relating to Spanish airports in 2008, taken from Lozano *et al.* [14]. In this evaluation, like Lozano *et al.* [14], we use directional distance function in evaluating performance of the production units building upon technologies T_L and $\widehat{T}_{KK}$.

In this example, as it can be seen in Figure 4, production system of each airport along with its internal operations has been considered as a two-stage production system. Each airport has five inputs of which three inputs x_1, x_2, and x_3 are consumed in first stage and two inputs x_4 and x_5 are consumed in second stage. Two undesirable outputs u_1 and u_2 are produced in first stage and two desirable outputs y_1 and y_2 are produced in second stage. Also, an intermediate product z is produced as desirable output in first stage which will be consumed completely as input in second stage. More explanations and data on 39 airports can be seen in Lozano *et al.* [14].

Directional distance function (DDF) which was introduced by Chambers *et al.* [22,23], is a non-radial measure that shrinks inputs and expands outputs simultaneously to weak efficiency frontier. This model measure the distance between the evaluated DMU and efficiency frontier in direction $g = (g_x, g_y)$. DDF is one of the most effective approaches to distinguish efficient units from inefficient ones and to illustrate efficient targets for inefficient units. It is usual to use directed distance function when there are undesirable outputs [24–27].

In this example, like Lozano *et al.* [14], we consider direction vector $g = (g_x, g_y, g_u) = (0, y_r, u_b)$ for DDF to be able to linearize the introduced model building upon technology T_L. Another reason for doing this is that inputs are uncontrollable or non-discretionary. So, for inputs, we have to choose $g_x = 0$. Therefore, to evaluate DMU_0, directional distance function building upon technologies T_L and will be introduced as models (13) and (14), respectively.

$$\begin{aligned} \beta_L = \;& \text{Max } \beta \\ & \text{s.t.} \quad (\mathbf{x}, (1+\beta)\mathbf{y}, (1-\beta)\mathbf{u}) \in T_L \end{aligned} \tag{5.1}$$

or equivalently:

$$\begin{aligned} \beta_L = \;& \text{Max } \beta \\ & \text{s.t.} \quad \textstyle\sum_{j=1}^{39} \lambda_j^{s1} x_{ij} \le \theta^{s1} x_{i0} && i = 1,2,3 \\ & \qquad \textstyle\sum_{j=1}^{39} \lambda_j^{s2} x_{ij} \le x_{i0} && i = 4,5 \\ & \qquad \textstyle\sum_{j=1}^{39} \lambda_j^{s2} y_{rj} \ge (1+\beta)\, y_{r0} && r = 1,2 \\ & \qquad \textstyle\sum_{j=1}^{39} \lambda_j^{s1} u_{bj} = (1-\beta)\, u_{b0} && b = 1,2 \\ & \qquad \textstyle\sum_{j=1}^{39} (\lambda_j^{s1} - \lambda_j^{s2}) z_j \ge 0 \\ & \qquad \textstyle\sum_{j=1}^{39} \lambda_j^{s1} = \theta^{s1} \\ & \qquad \textstyle\sum_{j=1}^{39} \lambda_j^{s2} = 1 \\ & \qquad \lambda_j^{s1} \ge 0 \,, \lambda_j^{s2} \ge 0 && j = 1,\ldots,39 \\ & \qquad 0 \le \theta^{s1} \le 1 \end{aligned}$$

$$\begin{aligned} \beta_{KK} = \;& \text{Max } \beta \\ & \text{s.t.} \quad (\mathbf{x}, (1+\beta)\mathbf{y}, (1-\beta)\mathbf{u}) \in \hat{T}_{KK} \end{aligned} \tag{5.2}$$

or equivalently:

$$\begin{aligned} \beta_{KK} = \;& \text{Max } \beta \\ & \text{s.t.} \quad \textstyle\sum_{j=1}^{39} \left(\alpha_j^{s1} + \beta_j^{s1}\right) x_{ij} \le x_{i0} && i = 1,2,3 \\ & \qquad \textstyle\sum_{j=1}^{39} \alpha_j^{s2} x_{ij} \le x_{i0} && i = 4,5 \\ & \qquad \textstyle\sum_{j=1}^{39} \alpha_j^{s2} y_{rj} \ge (1+\beta)\, y_{r0} && r = 1,2 \\ & \qquad \textstyle\sum_{j=1}^{39} \alpha_j^{s1} u_{bj} = (1-\beta)\, u_{b0} && b = 1,2 \\ & \qquad \textstyle\sum_{j=1}^{39} \left(\alpha_j^{s1} - \alpha_j^{s2}\right) z_j \ge 0 \\ & \qquad \textstyle\sum_{j=1}^{39} \left(\alpha_j^{s1} + \beta_j^{s1}\right) = 1 \\ & \qquad \textstyle\sum_{j=1}^{39} \alpha_j^{s2} = 1 \\ & \qquad \alpha_j^{s1} \ge 0 \,, \beta_j^{s1} \ge 0 \,,\ \alpha_j^{s2} \ge 0 && j = 1,\ldots,39. \end{aligned}$$

Note that the relationship between original DDF and model (13) is explained in detail in Lozano *et al.* [14]. Let us assume that optimal solution of model (13) is $(\beta^*, \theta^{s1^*}, \lambda_j^{s1^*}, \lambda_j^{s2^*})$ and optimal solution of model (14) is $(\beta^*, \alpha_j^{s1^*}, \beta_j^{s1^*}, \alpha_j^{s2^*})$. In this case, targets of inputs, desirable outputs, undesirable outputs and intermediate products that are introduced by models (13) and (14) will be computed as (15) and (16), respectively,

$$\begin{cases} x_i^* = \frac{1}{\theta^{s1}} \sum\limits_{j=1}^{39} \lambda_j^{s1*} x_{ij} & i = 1,2,3 \\ x_i^* = \sum\limits_{j=1}^{39} \lambda_j^{s2*} x_{ij} & i = 4,5 \\ y_r^* = \sum\limits_{j=1}^{39} \lambda_j^{s2*} y_{rj} & r = 1,2 \\ u_b^* = \sum\limits_{j=1}^{39} \lambda_j^{s1*} u_{bj} = (1-\beta^*)\, u_{b0} & b = 1,2 \\ z^* = \sum\limits_{j=1}^{39} \lambda_j^{s1*} z_j \end{cases} \tag{5.3}$$

$$\begin{cases} x_i^* = \sum\limits_{j=1}^{39} \left(\alpha_j^{s1*} + \beta_j^{s1*}\right) x_{ij} & i = 1,2,3 \\ x_i^* = \sum\limits_{j=1}^{39} \alpha_j^{s2*} x_{ij} & i = 4,5 \\ y_r^* = \sum\limits_{j=1}^{39} \alpha_j^{s2*} y_{rj} & r = 1,2 \\ u_b^* = \sum\limits_{j=1}^{39} \alpha_j^{s1*} u_{bj} = (1-\beta^*)\, u_{b0} & b = 1,2 \\ z^* = \sum\limits_{j=1}^{39} \alpha_j^{s1*} z_j. \end{cases} \tag{5.4}$$

For performance evaluation of Spanish airports, optimal solution values taken from models (13) and (14) and targets of outputs and intermediate products calculated by (15) and (16) are presented in Table 5.

As it can be seen, except for *Albacete* airport and *Salamanca* airport, obtained results of models (13) and (14) are identical in all other cases. *Albacete* airport is recognized as efficient DMU by model (13) whereas model (14) introduced it as inefficient DMU. Likewise, model (14) introduced *Salamanca* airport more inefficient than model (13).

If we take each airport as a DMU, for DMU_2 (*Albacete* airport) the target which proposed by model (14) will be as follows:

$$DMU_2^* = (x_1^*, x_2^*, x_3^*, x_4^*,\ x_5^*, z^*, u_1^*, u_2^*, y_1^*, y_2^*) = (106061.8, 2, 2, 4, 1, 2.3, 13.1, 312, 34.1, 15.8).$$

We demonstrate that the target of DMU_2 is a feasible point in the PPS.

If we consider first stage of airports, we can use weak disposability axiom to abate undesirable outputs and intermediate product (first stage outputs). So, from the first stage of the sixth airport ($S1(DMU_6)$), we will have virtual first stage of the sixth airport as $\overline{S1}(DMU_6) = (x_1, x_2, x_3, z, u_1, u_2) = (171000, 1, 2, 0, 0, 0)$ inside the PPS of first stages. Now, considering convexity axiom and choosing $\Lambda_1 = (\lambda_6 = 0.5133, \lambda_9 = 0.0013, \lambda_{10} = 0.4852, \lambda_{23} = 0.0001, \lambda_j = 0$ for other j), we will have:

$$\begin{aligned} \hat{S}1 &= (x_1, x_2, x_3, z, u_1, u_2) \\ &= \lambda_6 \overline{S1}(DMU_6) + \lambda_9 S1(DMU_9) + \lambda_{10} S1(DMU_{10}) + \lambda_{23} S1(DMU_{23}) \\ &= (106061.8, 2, 2, 2.3, 13.1, 312). \end{aligned}$$

So, virtual unit $\widehat{S}1$ belongs to PPS relating to first stages. Now, we consider second stage of the airports. From convexity axiom and with the assumption of $\Lambda_2 = (\lambda_2 = 0.9741,\ \lambda_9 = 0.0050,\ \lambda_{12} = 0.0023,\ \lambda_{20} = 0.0183,$

TABLE 5. Optimal solution of models (13) and (14) and targets of outputs and intermediate products.

	Airport	Model	$\text{Eff} = 1 - \beta^*$	z^*	u_1^*	u_2^*	y_1^*	y_2^*
1	A Coruna	β_L	0.352	18.5	429	8382.1	1935.8	533.3
		β_{KK}	0.352	18.5	429	8382.1	1935.8	533.3
2	Albacete	β_L	1	2.1	58	1376	19.3	8.9
		β_{KK}	0.227	2.3	13.1	312.1	34.1	15.8
3	Alicante	β_L	0.986	85.1	7537	140497.8	9709.3	8677.9
		β_{KK}	0.986	85.1	7537	140497.8	9709.3	8677.9
4	Almeria	β_L	0.214	16	238	4310.9	1829.5	1534.2
		β_{KK}	0.214	16.5	238	4310.9	1829.5	1534.2
5	Asturias	β_L	0.374	23.4	490	8940	2487.9	381.5
		β_{KK}	0.374	23.4	490	8940	2487.9	381.5
6	Badajoz	β_L	0.803	3.2	110	1898.7	97	11.6
		β_{KK}	0.803	3.2	110	1898.7	97	11.6
7	Barcelona	β_L	1	321.7	33036	645924.6	30272.1	103996.5
		β_{KK}	1	321.7	33036	645924.6	30272.1	103996.5
8	Bilbao	β_L	0.324	59.5	1490	26231.5	6991.9	5326.2
		β_{KK}	0.324	59.5	1490	26231.5	6991.9	5326.2
9	Cordoba	β_L	1	9.6	14	254.4	22.2	0
		β_{KK}	1	9.6	14	254.4	22.2	0
10	El Hierro	β_L	0.919	4.4	25	589.7	211.2	185.6
		β_{KK}	0.919	4.4	25	589.7	211.2	185.6
11	Fuerteventura	β_L	0.435	59.6	1706	31409.9	7029.3	4534.3
		β_{KK}	0.435	59.6	1706	31409.9	7029.3	4534.3
12	Girona-Costa Brava	β_L	1	49.9	4992	100305.6	5511	184.1
		β_{KK}	1	49.9	4992	100305.6	5511	184.1
13	Gran Canaria	β_L	0.829	104.7	6186	113043.8	11959.6	39461.1
		β_{KK}	0.829	104.7	6186	113043.8	11959.6	39461.1
14	Granada-Jaen	β_L	0.471	20.1	448	8424.6	2173.6	1120.6
		β_{KK}	0.471	20.1	448	8424.6	2173.6	1120.6
15	Ibiza	β_L	0.777	45.2	4813	118783.7	5682.9	4803.7
		β_{KK}	0.777	45.2	4813	118783.7	5682.9	4803.7
16	Jerez	β_L	0.327	19.9	384	6310.9	2181.1	1387.4
		β_{KK}	0.327	19.9	384	6310.9	2181.1	1387.4
17	La Gomera	β_L	0.736	2.5	13	309.7	52.9	9.9
		β_{KK}	0.736	2.5	13	309.7	52.9	9.9
18	La Palma	β_L	0.671	14.9	284	5563.9	1529.6	1696.9
		β_{KK}	0.671	14.9	284	5563.9	1529.6	1696.9
19	Lanzarote	β_L	0.591	64.3	3014	60047.7	7665	7652.9
		β_{KK}	0.591	64.3	3014	60047.7	7665	7652.9
20	Leon	β_L	0.201	7.3	89	1446.3	221.6	28.7
		β_{KK}	0.201	7.3	89	1446.3	221.6	28.7

$\lambda_{32} = 0.0003$, $\lambda_j = 0$ for other j), we will have:

$$
\begin{aligned}
\hat{S}2 &= (x_4, x_5, z, y_1, y_2) \\
&= \left(\sum_{j=1}^{39} \lambda_j x_{4j}, \sum_{j=1}^{39} \lambda_j x_{5j}, \sum_{j=1}^{39} \lambda_j z_j, \sum_{j=1}^{39} \lambda_j y_{1j}, \sum_{j=1}^{39} \lambda_j y_{2j}\right) \\
&= (4, 1, 2.3, 34.1, 15.8).
\end{aligned}
$$

TABLE 5. Continued.

	Airport	Model	$\text{Eff} = 1 - \beta^*$	z^*	u_1^*	u_2^*	y_1^*	y_2^*
21	Madrid Barajas	β_L	1	469.7	52526	908360	50846.5	329186.6
		β_{KK}	1	469.7	52526	908360	50846.5	329186.6
22	Malaga	β_L	0.94	117.5	14621	261116.1	13577.1	13491.8
		β_{KK}	0.94	117.5	14621	261116.1	13577.1	13491.8
23	Melilla	β_L	0.719	7.9	157	2141.2	403.2	495
		β_{KK}	0.719	7.9	157	2141.2	403.2	495
24	Murcia	β_L	0.478	25	643	11529.9	2855	1966.1
		β_{KK}	0.478	25	643	11529.9	2855	1966.1
25	Palma de Mallorca	β_L	1	193.4	26038	501486	22832.9	21395.8
		β_{KK}	1	193.4	26038	501486	22832.9	21395.8
26	Pamplona	β_L	0.55	11	366	6430.7	630	76.8
		β_{KK}	0.55	11	366	6430.7	630	76.8
27	Reus	β_L	0.485	20.8	458	8851.9	1935.9	181.5
		β_{KK}	0.485	20.8	458	8851.9	1935.9	181.5
28	Salamanca	β_L	0.09	3.5	38	596.6	114.8	12.7
		β_{KK}	0.06	3.5	25.4	394.6	116.6	12.8
29	San Sebastian	β_L	0.16	8.4	114	1784.1	742.1	117.4
		β_{KK}	0.16	8.4	114	1784.1	742.1	117.4
30	Santander	β_L	0.256	0.256	257	4564.7	1494.1	155.9
		β_{KK}	0.256	0.256	257	4564.7	1494.1	155.9
31	Santiago	β_L	0.293	29.2	588	10059.7	3272.9	4128.7
		β_{KK}	0.293	29.2	588	10059.7	3272.9	4128.7
32	Saragossa	β_L	1	14.6	1095	19547.6	595	21438.9
		β_{KK}	1	14.6	1095	19547.6	595	21438.9
33	Seville	β_L	0.642	51.5	1647	32781.1	5965.9	8288.7
		β_{KK}	0.642	51.5	1647	32781.1	5965.9	8288.7
34	Tenerife North	β_L	0.738	52.2	1315	24076.6	5347.8	26232.6
		β_{KK}	0.738	52.2	1315	24076.6	5347.8	26232.6
35	Tenerife South	β_L	0.766	80.8	4025	84890.9	10182.7	10571.5
		β_{KK}	0.766	80.8	4025	84890.9	10182.7	10571.5
36	Valencia	β_L	0.596	72.4	2979	61218.3	8114.3	18709.7
		β_{KK}	0.596	72.4	2979	61218.3	8114.3	18709.7
37	Valladolid	β_L	0.138	9.2	117	2042.1	893	590.7
		β_{KK}	0.138	9.2	117	2042.1	893	590.7
38	Vigo	β_L	0.269	20.9	412	6876.5	2213.9	2565.7
		β_{KK}	0.269	20.9	412	6876.5	2213.9	2565.7
39	Vitoria	β_L	1	12.2	669	11585.8	67.8	34989.7
		β_{KK}	1	12.2	669	11585.8	67.8	34989.7

So, virtual unit $\widehat{S}2$ belongs to PPS relating to second stages. As a result, $(\widehat{S}1, \widehat{S}2) = (106061.8, 2, 2, 4, 1, 2.3, 13.1, 312, 34.1, 15.8) = DMU_2^*$ belongs to the general production possibility set.

Furthermore, we change improvement direction in models (15) and (16) to $g = (0, 0, u_b)$. In this case, the number of different cases in optimal solutions of the models increases to five, which are reported in Table 6.

As it can be seen, in both cases model (13) introduces *Albacete* airport as efficient, while model (14) introduces the same unit as being inefficient. Also, in all cases the optimal solutions gained from model (14) are greater than or equal to the optimal solutions of model (13), because $T_L \subseteq \widehat{T}_{KK}$. This means that model (14) is more successful in recognizing sources of inefficiency than model (13) does. Therefore, it has more discriminatory power in performance evaluation of production units with general network structure, in the presence of undesirable outputs.

TABLE 6. Different optimal solution cases of models (13) and (14) with $g = (0, 0, u_b)$.

	Airport	Model	Eff $= 1 - \beta^*$	z^*	u_1^*	u_2^*	y_1^*	y_2^*
2	Albacete	β_L	1	2.1	58	1376.5	19.2	8.9
		β_{KK}	0.201	2.1	11.7	276.7	19.2	8.9
20	Leon	β_L	0.136	5.7	59.9	975.2	123.2	16.0
		β_{KK}	0.134	5.7	59.2	963.7	123.2	16.0
28	Salamanca	β_L	0.07	1.2	29.9	463.8	60.1	10.5
		β_{KK}	0.03	2.7	12.8	198.8	60.1	10.5
29	San Sebastian	β_L	0.078	4.5	55.6	872.3	403.2	281.0
		β_{KK}	0.072	5.2	51.3	805.2	403.2	281.0
37	Valladolid	β_L	0.047	5.5	39.6	693.7	479.7	444.0
		β_{KK}	0.046	5.5	38.8	679.0	479.7	444.0

6. Conclusion

When there are undesirable outputs, using weak disposability axiom for all outputs is a useful way for performance measurement of the production units. In this method, regarding the type of outputs abatement, the shape of production possibility sets will be different. Using distinct abatement factors for each decision making unit can have two main advantages. First, defined models will be planned linear programing problems. Second, the introd uced performance measurement models have more discriminatory power than those models that use technologies with uniform abatement factors. However, in some special cases of network DEA (NDEA), type of selection in balance constraint may obtain unacceptable solution for oriented models. In the present study, we choose an appropriate balance constraint and introduced a correct production technology set for general network DEA in presence of undesirable outputs. Finally, some illustrative examples are used to show advantages of the new introduced network technology set, considering undesirable outputs in production.

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