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OPTIMAL DESIGN OF SALES AND MAINTENANCE UNDER THE RENEWABLE WARRANTY *

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Abstract. This paper presents an integrated model to determine the optimal sales price, preventive maintenance (PM) interval and warranty period with the objective of maximizing the total profit. It is assumed that the sales growth can be featured by NHPP-Bass model over the time. Production cost, R&D cost and warranty cost involving product reliability are considered in this integrated profit model. Then, we consider a periodic PM policy, minimal repair and replacement policy in this paper and the product is deteriorated with the time goes. We also consider effects of the repair time of the repairable product. During the warranty period, manufacturer conducts the PM periodically, and if the repair time is beyond the limited repair time, the failure is replaced with a new product attached renewed warranty period. If not, the failed product is conducted with only minimal repair. Moreover, we give the numerical example and the sensitive analysis to provide insights into the influence of sales price, warranty period and PM interval.

Mathematics Subject Classification. 90B25, 90B50, 90C30.

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1. Introduction

This paper investigates the optimal pricing and warranty strategy for the automobile market like Lemon Market which can be find on the internet (Washington General Lemon Law). The features of sales growth process and warranty policy are integrated simultaneously in our proposed total profit model. In order to maximize the total profit and satisfy the customers expectations, the manufacturer should carefully determine the product sales price, warranty and PM planning. One the one hand, the purpose is to save some costs and achieve the maximal total profit from the aspect of the manufacturer; On the other hand, the product should be attached to the suitable price (market factor) and post services (warranty services) from the customers standpoint. Therefore, it is highly desirable to both manufacturers and customers to conduct suitable warranty policies and pricing strategy in the product sales process.

For the product pricing aspect which is one of the most significant market variable influenced the sales growth and undoubtedly total profit. In general, product sales quantity is directly affected by the demand which is closely related with the sales price. Indeed, a satisfied sales price relative to the functionalities of product for customers

Keywords. Sales price, preventive maintenance, warranty period, total profit, optimization.

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Notations.

P, W, μ:	sales price, warranty period and PM interval
φ_1, φ_2:	innovation factor and imitator factor
k_1, k_2:	constant of amplitude factor and time displacement factor
a, b:	price elasticity and warranty elasticity
θ:	reliability parameter
$\theta_{\max}$, $\theta_{\min}$:	maximal value and minimal value of reliability parameter
$c_{R\&D1}$, $c_{R\&D2}$:	foundational cost and additional cost of R&D cost model
c_{p1}, c_{p2}:	foundational cost and additional cost of production cost model
$R_{PM}(t)$:	failure rate function of product with preventive maintenance
$f_R(t)$, $F_R(t)$:	pdf and cdf of failure time
r, r_0:	repaired time and limited repaired time
N_{NR}:	replacement numbers
l:	resetting level
IR_i:	inter–arrival time between the $i-1$ and the i replacement
ρ:	probability of reporting failures
c_{wr}, c_{wm}, c_{wp}:	unit replacement, minimal repair and PM cost.

can often drive customers to purchase the product. Therefore, for the manufacturer aspect, it is important to formulate an attractive sales price to promote the product sales. In this paper, we jointly make the decision of the optimal sales price through our optimization profit model.

In most sales growth processes, manufacturer provides certain types of warranty policies for a repairable product to attract the customers and through the warranty cost analysis to reduce the warranty cost to maximize it total profit. In this paper, we consider the periodic PM policy which improves product reliability through slowing the product degradation process. We also adopt renewable minimal repair and replacement warranty policy for the repairable product. Since customers may claim the warranty services due to misuse by various human factors (Customers may simply buy a new product), which makes customers not to repair their failure product. Wu [27] developed the warranty cost model considering the Failed–But–Not–Reported phenomena. Therefore, in our paper, we consider the Failed–But–Not–Reported phenomena for the minimal repair policy. Moreover, we consider the repaired time for the failed product which is designed to avoid risk of too many repair claims during the warranty period. Therefore, in this paper, the PM interval and warranty period are both the decision variables for manufacturer to determine.

Then, the rest of this paper is organized as follows. Section 2 will give a literature review related with our proposed optimization profit model with the decision variables of sales price, warranty period and PM interval. Section 3 will give the product sales growth model. Section 4 will give R&D cost and production cost model. In Section 5, warranty cost model will be developed considering periodic PM policy, minimal repair policy and replacement policy. In Section 6, optimization profit model will be developed, then, numerical experiences and some insights will be given in this section. Section 7 will give our paper's conclusions and future topics.

2. Literature review

Product sales often drive various operational decision of a manufacture, such as production designing, pricing strategies and warranty services (Gickman and Berger [6]). Inversely, production and market strategies also produce a directly effect on the product sales growth. Therefore, modelling the sales growth process is important, particularly for manufacturer who is depend to sell products. Various product sales growth models have been studied in the past. Bass [1] proposed a durable product diffusion model which called Bass model considering the potential market, innovator factor and imitator factor. Niu [19] extended Bass model and gave the stochastic form of Bass model. Xie *et al.* [28] used Bass model to construct their sales growth model. In this paper, we also use the Bass model to construct our sales growth model to express the product diffusion process.

Pricing strategy is also significant for manufacturer, undoubtedly, the pricing strategy can directly affect the total profit. A suitable sales price related with the functionalities and reliability of the product can drive force for the customers to purchase the product. DeCroix [5] used the game theory to find the optimal sales price for the durable goods in the oligopoly market, and this paper also considered the customer's risk aversion. Wu *et al.* [26] developed the Hamiltonian function for the total profit and gave the dynamic price path considering two cases including zero discount rate and positive discount rate. Zhu and Cetinkaya [31] used the price strategy to customers to purchase the product.

Product designing becomes an increasingly significant factor to affect the total profit. To express the product reliability, manufacturer should carefully design its product warranty planning. A service plan is a type of optional warranty which is the manufacturer provided to customers (Jiang and Zhang [9]). Undoubtedly, a well–developed warranty planning can attract more customers and promote the product sales, then further increasing the total profit. Blischke [2] modeled the warranty cost model considering free–replacement, pro–rata and reliability–improvement policy. Murthy and Djamaludin [16] gave a review of literature of different types of product warranty policies. Free replacement, pro–rata and minimal repair are widely used in the industrial production process (Mitra and Patankar, Huang *et al.*, Chou *et al.*, Ye *et al.*, Liu *et al.*, Park *et al.*) [4, 7, 12, 14, 20, 30]. Wu *et al.* [25] developed the warranty cost model to determine the optimal burn–time considering the combination of the free replacement and pro–rata warranty policy. Park and Pham [21] developed the warranty cost model considering age replacement policy and block replacement policy. Su and Wang [23] used the two–dimensional extended warranty policy for warranty cost model. Because the warranty cost is mainly depended on the product reliability, Mettas [13] established a reliability cost model to estimate the research and development costs of different types of systems with different lifetime distributions.

Preventive maintenance is also a kind of important warranty policy which can improve the product reliability by slowing the product deterioration process. More PM actions can reduce the product failure rate, meanwhile, can increase the warranty cost. Canfield [3] developed the optimization cost model considering the periodic PM. Nakagawa [17] developed two imperfect PM policies model including PM can reduce the failure rate and PM can reduce the age with fixed time intervals. Monga and Zuo [15] developed the optimization cost model considering PM policy with deteriorating components. Nakagama and Mizutani [18] converted the usual maintenance models including replacement policies and imperfect PM policies with the infinite time span. Huang *et al.* [8] considered time and usage of product and took into account periodic preventive maintenance policy in their total profit model. Lee and Cha [11] developed the cost model considering the periodic preventive maintenance policy. Yang *et al.* [29] developed the maintenance cost model considering the preventive maintenance policy.

The purpose of this paper is to analytically study the expected total profit. Xie *et al.* [28] developed integrated profit model to determine the optimal sales price, warranty period and post sales price. Huang *et al.* [7] used the same idea but different model to develop the optimization profit model to determine the optimal sales price, warranty period and product reliability. Ladany and shore [10] developed the total profit model to determine the optimal warranty period. In this paper, we develop the integrated total profit model to make decisions on the sales price, warranty period and PM interval with the objective of maximizing the total profit.

3. Sales growth model

Product sales process is important for manufacturer. Section 3.1 will give the Bass diffusion model; Section 3.2 will give the potential market model. Then, in the Section 3.3, the sales growth model can be developed.

3.1. product sales model

Bass model (Bass [1]) is probably the most influential new product diffusion model, which is applied to a wide range of durable good consumptions. Bass model assumes that the trajectory of cumulative sales quantity of a new product follows a deterministic function of time with two parameter including the innovator factor φ_1 and the imitator factor φ_2. Bass model means that innovators are not affected in the timing of their initial purchases by the quantity of customers who have already bought the product, while imitators are affected by the quantity of previous buyers. Imitators buying decisions, in some sense, from those who have already bought. Let $n > 0$ be the size of the products potential market capacity, which is relied on the sales price and warranty period. We assume that the $Q(t)$ follows Bass model with $Q(0) = 0$. Therefore, the related model of new product diffusion of NHPP–Bass model is given by,

$$q(t) = \frac{\mathrm{d}Q(t)}{\mathrm{d}t} = [n - Q(t)]\left[\varphi_1 + \varphi_2\frac{Q(t)}{n}\right] \tag{3.1}$$

where $q(t)$ is sales rate.

In equation (3.1), n means the potential demand for the whole market as we told before, $Q(t)$ is the sales quantity as time t, therefore, $[n - Q(t)]$ is the customers which is not purchase the products. $\varphi_1 + \varphi_2[Q(t/n]$ is a probability which is initial buyers will be developed at time t given that no purchase has yet been made is a linear form of previous buyers.

$$Q(t) = n\frac{1 - \exp[-(\varphi_1 + \varphi_2 t)]}{1 + \varphi_2/\varphi_1\exp[-(\varphi_1 + \varphi_2]} \tag{3.2}$$

In particular, when the time $t \to +\infty$, the sales quantity will become the potential market quantity and the sales rate will become to zero.

$$\lim_{t\to+\infty} Q(t) = \lim_{t\to+\infty} n\frac{1 - \exp[-(\varphi_1 + \varphi_2 t)]}{1 + \varphi_2/\varphi_1\exp[-(\varphi_1 + \varphi_2]} = n \tag{3.3}$$

$$\lim_{t\to+\infty} \frac{\mathrm{d}Q(t)}{\mathrm{d}t} = n\frac{(\varphi_1 + \varphi_2)(1 + \varphi_2/\varphi_1)\exp[-(\varphi_1 + \varphi_2)t]}{(1 + \varphi_2/\varphi_1\exp[-(\varphi_1 + \varphi_2)t])^2} = 0 \tag{3.4}$$

3.2. Potential market model

Manufacturers often use the pricing approach to attract the customers. Customers buy the product will consider the product reliability and the warranty. Inversely, potential market is closely related with price and warranty period.

Glickman and Berger [6] proposed a potential market model, and the potential market will be decreased with the sales price increased and increases with the warranty period increases. Let P and W be the sales price and the warranty period respectively. The potential market is replaced log–linear function with an exponential form which developed by Glickman and Berger [6] as follows,

$$n(P, W) = k_1 P^{-a}(W + k_2) \tag{3.5}$$

where k_1, $k_2 \geq 0$, $a > 0$, and $0 < b < 1$. And k_1 is constant of amplitude factor of this model, k_2 is time displacement when the warranty period equals to zero for non–zero demand and both k_1 and k_2 are the positive constants. It is significant to indicate that Glickman and Berger [6] assumed that $a > 0$. And in this paper, we assume that the $a > 0$ which is relaxed condition in equation (3.5).

3.3. New product sales model and peak time of sales rate

For the selling period $(0, T]$, sales price is P and the warranty period is W. We combine equations (3.2) and (3.5), the cumulated sales quantity in the period of $t \leq T$ can be given by,

$$Q(t, P, W) = k_1 P^{-a}(W + k_2)\frac{1 - \exp[-(\varphi_1 + \varphi_2)t]}{1 + \varphi_2/\varphi_1\exp[-(\varphi_1 + \varphi_2]} \tag{3.6}$$

It is significant to point out that the sales process goes through five stages: development, growth, maturity, saturation, and decline, therefore, according to equation (3.6), the sales rate during the selling period can be given by,

$$q(t,P,W) = k_1 P^{-a}(W+k_2)\frac{(\varphi_1+\varphi_2)(1+\varphi_2/\varphi_1)\exp[-(\varphi_1+\varphi_2)t]}{(1+\varphi_2/\varphi_1\exp[-(\varphi_1+\varphi_2)t])^2} = 0 \tag{3.7}$$

And it is easily to prove that the sales rate is concave and that the maximal sales rate is at the peak time,

$$t^* = ln(\varphi_2/\varphi_1)/\varphi_2/\varphi_1 + \varphi_2 \tag{3.8}$$

4. *R&D* cost and production cost model

In the cost aspect, we consider research and development cost (*R&D* cost) and production cost. Sections 4.1–4.2 will discuss the *R&D* cost and production cost considering product reliability, respectively.

4.1. *R&D* cost considering product reliability

R&D cost is presented before selling the product. It usually relies on the technological science level and working effect. In the general, smaller the value of θ (reliability parameter), the higher research and development cost will be accepted by manufacturers, and the higher θ will cause the lower product reliability. Therefore, we assume that the unit *R&D* cost has exponential form and is a monotonically increased function of product reliability (Mettas [13]), and we give this model as follows,

$$C_{R\&D} = c_{R\&D1}\exp\left(c_{R\&D2}\frac{\theta_{\max}-\theta}{\theta_{\min}-\theta}\right) \tag{4.1}$$

where $c_{R\&D1}$ is the foundational cost when θ get the maximal value and $c_{R\&D2}$ measures the difficulty to enhance the product reliability which is determined by the designing and technological capacity. We deem that the increases of product reliability is difficult from a high value than from a low value. Therefore, equation (4.1) conveys that the *R&D* cost increases more abruptly from when the reliability parameter is relatedly smaller.

4.2. Production cost considering product reliability

To determine the optimal product sales price, warranty period and productivity, manufacturers should consider the related cost, such as production cost and warranty cost. Therefore, let c_P be a unit production cost. We consider the reliability effect in the production cost model, and the unit production cost increases with the product reliability increases because the production of a more reliable product needs more inputs or higher standards of resources, such as time, labor, and quality of materials. The unit production cost is obtained by,

$$C_P = c_{P1} + c_{P2}\exp\left(\gamma\frac{\theta_{\max}-\theta}{\theta_{\min}-\theta}\right) \tag{4.2}$$

The production cost has a similar format to *R&D* cost in equation (4.1), the initial unit production cost exponentially increases with the product reliability and increases sharply as the product reliability parameter θ approaches its smallest value. The constant c_{P1} represents the initial unit cost for manufacturing the allowable least reliable product, the constant c_{P2} represents the additional cost for manufacturing the most reliable product possible, and the constant γ represents the exponential increase of the initial unit cost with improvement of the product reliability.

5. Warranty cost

In this section, we introduce the warranty cost model for the background of automobile market. Manufacturer will first make a preventive examination and to decide which warranty approaches should be attached to the automobile. Therefore, we consider a minimal repair warranty and renewable replacement warranty policy. Under the policy of minimal repair warranty and renewable replacement warranty policy, the failures are only replaced with the new product when the product repaired time beyond the limited repaired time r_0, the warranty period will be renewed with the new product to customers. Moreover, if the repair time is below the limited repaired time r_0, the manufacturer will conduct the minimal repaired warranty service for the customer. We assume that the warranty period is a constant value W and W is renewed only when the product is replaced by the new product.

So, Section 5 introduces the warranty cost model related with preventive warranty cost, minimal repair and replacement cost model.

5.1. Preventive maintenance warranty cost model

In the Section 5.1, we develop a preventive maintenance cost model. To develop this cost model, we first define the failure rate function, and this function is used in Canfield [3], and in this paper, the failure rate model is developed as follows,

$$R_{PM}(t) = \sum_{i=1}^{s}[R((i-1)(\mu-l)+\mu) - R(i(\mu-l))] + R(t-sl) \tag{5.1}$$

where $s\mu \leq t \leq (s+1)\mu$, $s = 1, 2\ldots$ Let $R_{PM}(t)$ be the failure rate function of product with preventive maintenance process, and $R(\cdot)$ is initial failure rate function of product without preventive warranty process. μ is PM interval which is equal to period of time between two PM services. Then, l is resetting level and the range of l lies in $0 \leq l \leq \mu$. And l can reflect the preventive maintenance effect. Park *et al.* [20] also use renewal scale l to measure the PM effect. When $l = 0$, there is no PM effect and when $l = \mu$, there is total successful PM effect.

According to the failure rate function of product, the probability density function and cumulative density function of failure time can be given as follows,

$$\begin{aligned} f_R(t) &= R_{PM}(t)\exp\left(-\int_0^t R_{PM}(u)\,\mathrm{d}u\right) \\ F_R(t) &= 1-\exp\left(-\int_0^t R_{PM}(u)\,\mathrm{d}u\right) \end{aligned} \tag{5.2}$$

We assume that the inter–arrival time between the $i-1$ and i replacement is IR_i, and IR_i is a sequence of independent and identically distribution random variables. And the mean to inter–replacement time can be expressed by,

$$E(IR_i(\mu, W)) = \frac{\int_0^W \delta f_R(\delta)\,\mathrm{d}\delta}{F_R(\mu, W)[1-RT(r_0)]} \tag{5.3}$$

where r_0 is limited repaired time. If the repair time exceeds r_0, the decision of manufacturer is to provide a replacement service. And $RT(r)$ is the cumulated density function of repaired time r. Therefore, $E(IR_i)$ can be expressed as the product life cycle before replacing it during the warranty period of W. The replacement is appeared only when the repaired time of the failed product and it is greater than limited repaired time.

The probability of product replacements during the warranty period is following the geometric distribution, and Park and Pham [21] also give the function,

$$Pr\{N_r = N_{WR}\} = [1-F_R(\mu, W)(1-RT(r_0))]\{F_R(\mu, W)[1-RT(r_0)]\}^{N_{WR}-1}(N_{WR} = 1, 2\ldots) \tag{5.4}$$

According to equation (5.2), the expected number of replacement product during the warranty length is given by,

$$E[N_{WR}(\mu, W)] = \frac{F_R(\mu, W)[1-RT(r_0)]}{1-F_R(\mu, W)[1-RT(r_0)]} \tag{5.5}$$

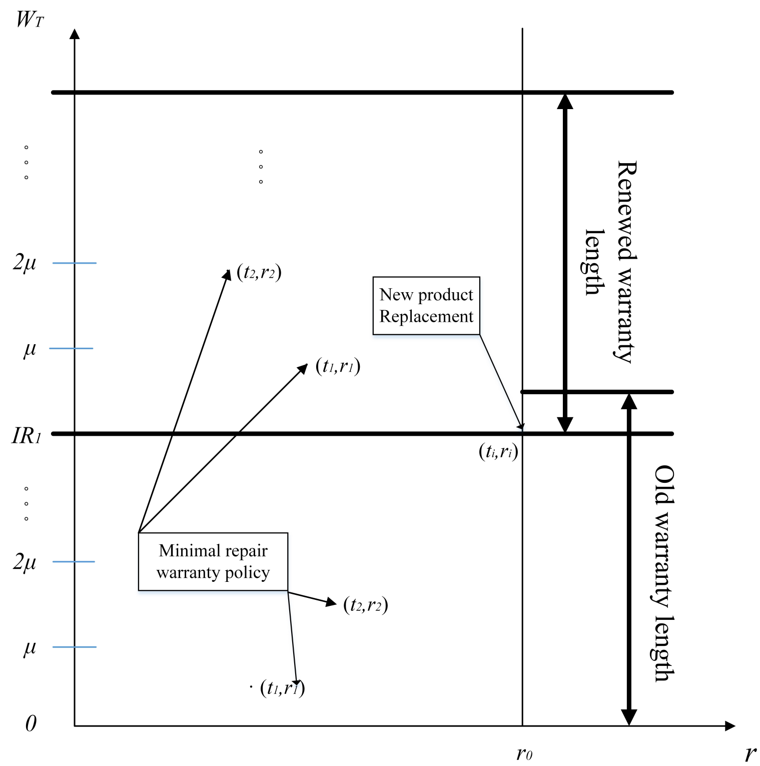

Figure 1. Preventive maintenance policy under the minimal repair warranty policy and replacement warranty policy.

Figure 1 indicates that the product is replaced for the ith preventive maintenance policy and the repaired time is beyond the limited repaired time r_0. Therefore, the renewed warranty length is appeared because the new product replacement is occurred.

Then, the expected total warranty period can be expressed as follows,

$$E[W_T(\mu, W)] = \sum_{i=1}^{n} E(IR_i(\mu, W)) + W \tag{5.6}$$

where W is the fixed period of old warranty length. $E[W_T(\mu, W)]$ is random variable which is total warranty period because the renewed warranty for n replacement plus old fixed warranty period. And the expected preventive maintenance cost model is given by,

$$\begin{aligned} E[C_{WP}(\mu, W)] &= c_{wp} \left\lfloor \left(\sum_{i=1}^{n} E(IR_i(\mu, W)) + W \right) / \mu \right\rfloor \\ &= c_{wp} \left\lfloor \left(E[N_W R(\mu, W)]) \frac{\int_0^W \delta f_R(\delta)\, d\delta}{F_R(W)[1 - RT(r_0)]} + W \right) / \mu \right\rfloor \\ &= c_{wp} \left\lfloor \left(\frac{F_R(\mu, W)[1 - RT(r_0)]}{1 - F_R(\mu, W)[1 - RT(r_0)]}) \frac{\int_0^W \delta f_R(\delta)\, d\delta}{F_R(W)[1 - RT(r_0)]} + W \right) / \mu \right\rfloor \end{aligned} \tag{5.7}$$

where c_{wp} is unit preventive maintenance cost. And the times of preventive maintenance policy is an integral value, therefore, we set the times of preventive maintenance policy is an integer.

5.2. Minimal repair warranty cost model

There are many warranty policies, such as renewable free replacement, non–renewable free replacement, pro rata, and non–renewable free minimal repair warranty policy (Xie *et al.* [28]). The minimal repair warranty policy is widely used for the products, such as automobile, computers, televisions and pumps. In this paper, we consider the policy of minimal repair warranty policy.

$$\begin{aligned} E\left(C_{WM}(\mu, W)\right) &= c_{wm}E[N(W)] \\ &= c_{wm}\left(E[N_W R(\mu, W)]\int_0^E (IR_i(\mu, W))R_{PM}(t)\mathrm{d}t + \int_0^W R_{PM}(t)\mathrm{d}t\right) \\ &= c_{wm}\left(\frac{F_R(\mu, W)[1-RT(r_0)]}{1-F_R(\mu, W)[1-RT(r_0)]}\int_0^E (IR_i(\mu, W))R_{PM}(t)\mathrm{d}t + \int_0^W R_{PM}(t)\mathrm{d}t\right) \end{aligned} \tag{5.8}$$

where c_{wm} is unit minimal repair warranty cost.

However, Failed–But–Not–Reported phenomena will be existed because of various reasons, which makes $N(W) \leq E[C_{WM}(\mu, W)]\int_0^E (IR_i(\mu, W))R_{PM}(t), \mathrm{d}t + \int_0^W R_{PM}(t), \mathrm{d}t$. We let the ρ be the probability of reporting the failures during the warranty period. According to Xie *et al.* [28], we can give the expected minimal repair warranty period considering Failed-But-Not-Reported phenomena,

$$\begin{aligned} E(C_{WM}(\mu, W)) = c_{wm}&\frac{F_R(\mu, W)[1-RT(r_0)]}{1-F_R(\mu, W)[1-RT(r_0)]} \\ &\times \frac{\rho(1-e^{-(1-\rho)\theta W})}{1-\rho}\int_0^E (IR_i(\mu, W))R_{PM}(t)\mathrm{d}t + c_{wm}\int_0^W R_{PM}(t)\mathrm{d}t \end{aligned} \tag{5.9}$$

5.3. Replacement warranty cost policy

Free replacement warranty policy is conducted when the repaired time exceed the limited repaired time and a renewable warranty period is conducted for replacing a new product. And it is easy to obtain the replacement cost by the unit replacement cost multiples the number of replacement. Therefore, the expected replacement warranty cost during the warranty period is given by,

$$E[C_{WR}(\mu, W)] = c_{wr}E[N_{WR}(\mu, W)] = c_{wr}\frac{F_R(\mu, W)[1-RT(r_0)]}{1-F_R(\mu, W)[1-RT(r_0)]} \tag{5.10}$$

where c_{wr} is unit replacement warranty cost.

5.4. Total warranty cost model

Let $E[C_W(\mu, W)]$ be the total expected warranty cost with the variables of reliability and inter-replacement time interval, and the total expected warranty cost includes renewable free replacement warranty cost, minimal repair warranty cost and preventive maintenance cost. Then, the function of total expected warranty cost is given by,

$$\begin{aligned}
E[C_W(\mu,W)] &= E(C_{WP}(\mu,W) + E(C_{WM}(\mu,W)) + E(C_{WR}(\mu,W)) \\
&= c_{wp}\left\lfloor\left(\frac{F_R(\mu,W)[1-RT(r_0)]}{1-F_R(\mu,W)[1-RT(r_0)]}\frac{\int_0^W \delta f_R(\delta)\,d\delta}{F_R(W)[1-RT(r_0)]}+W\right)/\mu\right\rfloor \\
&\quad + c_{wm}\frac{F_R(\mu,W)[1-RT(r_0)]}{1-F_R(\mu,W)[1-RT(r_0)]}\frac{\rho(1-e^{-(1-\rho)\theta W})}{1-\rho}\int_0^E (IR_i(\mu,W))R_{PM}(t)\mathrm{d}t \\
&\quad + c_{wm}\int_0^W R_{PM}(t)\mathrm{d}t + c_{wr}\frac{F_R(\mu,W)[1-RT(r_0)]}{1-F_R(\mu,W)[1-RT(r_0)]} \qquad (5.11)
\end{aligned}$$

6. Optimization profit model and optimal solutions

6.1. Optimization profit model

Let $\pi(\mu,P,W)$ be the total profit for the manufacture, which is the continuous function of decision variables of μ,θ and P. We assume that the expected failures during the time interval t is expressed as the form of $RR_{PM}(t) = ct^d$, therefore, the failure rate can be obtain as the form of $R_{PM}(t) = cdt^{d-1}$. In this study, we consider the product has a gradually increased failure rate, so we assume that $d > 1$. As d increases, the failure rate of product is increased sharply. Therefore, we assume that $d = 2$ which is suitable for the real situation . Park *et al.* [20] also considered power law process in the model, and they set $d = 3$.

We assume that repaired time pdf $RT(r)$ follows the Weibull distribution, $RT(r) = k/\lambda * (r/\lambda)^{k-1} * \exp(-(r/\lambda)^k)$, where $\lambda \geq 0$, $r \geq 0$. Therefore, according to equations (3.6), (4.1), (4.2) and (5.11), the model of total expected profit for the manufacture is given by,

$$\begin{aligned}
\pi(\mu,W,P) &= Q(P,W)[P - C_{R\&D} - C_P - E[C_W(\mu,W)]] \\
&= k_1P^{-a}(W+k_2)\frac{1-\exp[-(\varphi_1+\varphi_2 t)]}{1+\varphi_2/\varphi_1\exp[-(\varphi_1+\varphi_2]} \\
&\quad \times\left\{P - c_{R\&D1}\exp\left(c_{R\&D1}\frac{\theta_{\max}-\theta}{\theta_{\min}-\theta}\right) - \left(c_{P1}+c_{P2}\exp\left(\gamma\frac{\theta_{\max}-\theta}{\theta_{\min}-\theta}\right)\right)\right. \\
&\quad - c_{wp}\left\lfloor\left(\frac{F_R(\mu,W)[1-RT(r_0)]}{1-F_R(\mu,W)[1-RT(r_0)]})\frac{\int_0^W \delta f_R(\delta)\,d\delta}{F_R(W)[1-RT(r_0)]}+W\right)/\mu\right\rfloor \\
&\quad - c_{wm}\frac{F_R(\mu,W)[1-RT(r_0)]}{1-F_R(\mu,W)[1-RT(r_0)]}\frac{\rho(1-e^{-(1-\rho)\theta W})}{1-\rho}\int_0^E (IR_i(\mu,W))R_{PM}(t)\mathrm{d}t \\
&\quad \left. - c_{wm}\int_0^W R_{PM}(t)\mathrm{d}t - c_{wr}\frac{F_R(\mu,W)[1-RT(r_0)]}{1-F_R(\mu,W)[1-RT(r_0)]}\right\} \qquad (6.1)
\end{aligned}$$

The function of $\pi(\mu,\theta,P)$ is modeled to obtain the optimal decisions of PM interval, warranty period and product sales price with the objective of maximizing the expected total profit. Therefore, to maximize the expected total profit, the following non-linear function is given by,

$$\begin{aligned}
\max\quad & \pi(\mu,P,W) \\
\text{s.t.}\quad & l \leq \mu \\
& W_{\min} \leq W \leq W_{\max}
\end{aligned}$$

TABLE 1. Optimal solutions for different sales time.

t	μ^*	W^*	P^*	Cumulated sales quantity	Total $R\&D$ cost ($\times 10^8$)	Total production cost ($\times 10^8$)	Total Warranty cost ($\times 10^8$)	Total profit ($\times 10^8$)
1	0.5158	2.4	12 382	209 720	1.8654	2.4642	2.5131	19.124
2	0.6686	2.6	12 387	506 097	4.5015	5.9466	4.2185	48.025
3	0.5719	2.6	12 346	892 346	7.9370	10.485	4.2190	87.517
4	0.8433	2.8	12 368	1 377 710	12.254	16.188	6.5627	135.39
5	0.6444	3.0	12 393	1 931 365	17.179	22.694	10.159	189.32

where $W_{\min}$ and $W_{\max}$ are lower bound and upper bound for the warranty period W, respectively. Then, we let μ^*, W^* and P^* be the optimal PM interval, optimal product warranty period and optimal sales price, respectively. And non-linear function is the established total profit optimization model with two constraints.

6.2. Parameter value setting

The sales period of the automobile is typically long and automobile is the durable product, therefore, we assume that it is three years for selling time, $t = 3$.

For the sales growth model in equation (3.2), we assume that innovator factor $\varphi_1 = 0.04$ and imitator factor $\varphi_2 = 0.4$. In equation (3.5), we set $k_1 = 3 \times 10^{11}$ and $a = 1.2$; this means that the sales quantity will increase by 1.412 times if the sales price is discounted by 25%. And we set $k_2 = 0$ because the warranty period of automobiles is normally several years and the effect of k_2 can be ignored. According to equation (3.5), the ratio of the potential market with two-year warranty period to that with one–year warranty period are 1.12, 1.089, 1.05 when $b = 0.1, 0.15, 0.2$, respectively. In order to reflect reality of automobile sales, we choose $b = 0.2$ for our warranty policy.

In equation (4.1), we set $c_{R\&D1} = 1000$, $c_{R\&D2} = 1$. $R\&D$ cost for designing an automobile with maximal achievable reliability can be infinite and the basic R&D cost for designing is $c_{R\&D1}$ when the product reliability is allowable low..

In the production cost model (4.2). We assume that $c_{P1} = 1000$ for the fixed cost of producing a product, $c_{P2} = 10$ can be deem as the variable cost of producing a product. and $\gamma = 1$. We set $\theta_{\min} = 0.01$ of achieving a highest reliability and $\theta_{\max} = 0.4$ of achieving a lowest reliability. And the product reliability we set $\theta = 0.05$.

In the minimal repair cost model, we assume that the probability of reporting the failures equal to 0.8.

Finally, we assume that the unit replacement cost $c_{wr} = 1000$, unit minimal repair cost $c_{wm} = 100$ and uni0 PM cost $c_{wp} = 20$. We use year as unit for the warranty period and repair time. Therefore, the value of $r_0 = 1/24$ is chosen due real situation which is Lemon law suggests 15 days as a repair time. According to Park *et al.* [20], we set $\lambda = 1$ and $k = 3$.

6.3. Optimal solutions

Table 2 presents the optimal results for the instances associated with the different sales years. As shown in the Table 2, μ^*, W^* and P^* are the optimal solutions denoted by expected total profit model, and we also given the optimal cumulated sales quantity, total $R\&D$ cost, total production cost and total warranty cost. And we use year as the unit for each period. According to our parameter setting in Section 6.2, if the failed product requires more than half month of repair time $r_0 = 1/24$, it is replaced by a new product instead of being repaired. And we present the optimal solutions for different sales time, so we choose $t = \{1, 2, 3, 4, 5\}$ year.

Table 1 shows that, for 3 years selling time, under the preventive maintenance warranty policy, minimal repair warranty policy and replacement warranty policy, the optimal time interval for preventive maintenance

TABLE 2. Optimal solutions for different parameters of price elasticity.

a	μ^*	W^*	P^*	Cumulated sales quantity	Total $R\&D$ cost ($\times 10^8$)	Total production cost ($\times 10^8$)	Total Warranty cost ($\times 10^8$)	Total profit ($\times 10^8$)
1.1	0.5046	3.0	22 709	1 204 828	10.716	14.157	10.160	238.57
1.2	0.5719	2.6	12 346	892 346	7.9370	10.485	4.2190	87.517
1.3	0.9716	2.6	8963	527 351	4.6906	6.1964	4.2176	32.164

is 0.5719 years, the optimal warranty period is 2.6 years and the optimal sales price is \$12 346, and the total profit is \$8751.7 million. The total sales quantity is 0.89 million for three years selling time. For the total $R\&D$ cost and total production cost. The total $R\&D$ (\$793.7 million) cost and production cost (\$1048.5 million) is increased as the sales quantity increases and decreased as the sales quantity decreases.

6.4. Sensitive analysis

In this section, we conduct a sensitive analysis to investigate the effects of several parameters including price elasticity a, limited repair time r_0, unit replacement cost, unit minimal repair cost and unit PM cost on the optimal decisions in order to derive some strategic insights for the manufacturer. We also use the parameter settings in the Section 6.2.

6.4.1. Effect of price elasticity a

To study the effects of price elasticity a on the optimal solutions, all kinds of costs and total profit, we choose different values of price elasticity a. Table 2 shows that the optimal solutions under the parameter setting specified in Section 6.2. In Section 6.2, we set $a = 1.2$. Therefore, we test the different values of $a = \{1.1, 1.2, 1.3\}$ for the sensitive analysis.

Table 2 shows that the effects of price elasticity a reflected the importance of product sales price on the cumulated sales quantity, which inversely influences the expected total profit. When the product sales price is decreased, the revenue will be adversely influenced by the lower price but positively influenced by the increases in sales quantity. Therefore, as the price elasticity a increases, the effect of sales price on the cumulated sales quantity becomes more important. For the larger value of a, it is worth having relatively lower sales price because the important increases of the cumulated sales quantity can make up for the lower price adverse effect on the total revenue.

As the optimal sales price decreases with the price elasticity a increases, the warranty period is shorten in order to cover the total $R\&D$ cost, total production cost and total warranty cost to maximize the total profit. Then, as the price elasticity a decreases, we can analogously obtain explanations using the same idea when the price elasticity a increases.

6.4.2. Effect of limited repair time r_0

To study the effects of limited repair time r_0 on the optimal solutions, all kinds of costs and total profit, we choose different values of limited repair time r_0. Table 3 shows that the optimal solutions under the parameter setting specified in Section 6.2. In Section 6.2, we set $r_0 = 1/24$. Therefore, we test the different values of $a = \{1/24, 2/24, 3/24, 4/24, 5/24, 6/24\}$ for sensitive analysis.

Table 3 shows that the limited repair time reflects significant of the PM interval time μ on the cumulated sales quantity, which adversely reflects on the total profit. When the limited repair time is increased, the total profit is negatively affected by PM interval increases and positively affected by the cumulated sales. First of all, total profit is decreases with the repair time increases, and then the total profit is increased, because the cumulated sales quantity decreases at first can not cover the warranty cost of limited repair time increases, therefore,

TABLE 3. Optimal solutions for different limited repaired time.

r_0	μ^*	W^*	P^*	Cumulated sales quantity	Total $R\&D$ cost ($\times 10^8$)	Total production cost ($\times 10^8$)	Total Warranty cost ($\times 10^8$)	Total profit ($\times 10^8$)
1/24	0.5719	2.6	12 346	892 346	7.9370	10.485	4.2190	87.517
2/24	0.5754	4.4	12 354	990 458	8.8097	10.938	27.534	74.380
3/24	0.5697	4.4	12 358	990 045	8.8060	10.933	12.555	89.359
4/24	0.5542	4.4	12 370	988 919	8.7960	10.920	7.3636	94.551
5/24	0.5337	4.4	12 385	987 461	8.7831	10.903	5.0083	96.906
5/24	0.5322	4.4	12 386	987 437	8.7828	10.902	3.7225	98.192

TABLE 4. Optimal solutions for different unit replacement cost.

c_{wr}	μ^*	W^*	P^*	Cumulated sales quantity	Total $R\&D$ cost ($\times 10^8$)	Total production cost ($\times 10^8$)	Total Warranty cost ($\times 10^8$)	Total profit ($\times 10^8$)
1000	0.5719	2.6	12 346	892 346	7.9370	10.485	4.2190	87.517
1500	0.5719	2.6	12 347	892 143	7.9360	10.484	4.2209	87.517
2000	0.5682	2.6	12 349	891 995	7.9330	10.481	4.2229	87.513

TABLE 5. Optimal solutions for different unit minimal repaired cost.

c_{wm}	μ^*	W^*	P^*	Cumulated sales quantity	Total $R\&D$ cost ($\times 10^8$)	Total production cost ($\times 10^8$)	Total Warranty cost ($\times 10^8$)	Total profit ($\times 10^8$)
100	0.5719	2.6	12 346	892 346	7.9370	10.485	4.2190	87.517
200	0.5683	2.6	12 348	892 004	7.9343	10.482	6.3249	85.411
300	0.5683	2.6	12 349	891 909	7.9341	10.481	8.4308	83.305

the total profit will increase when the cumulated sales quantity decreases, which can cover the warranty cost of limited repair time increases.

As the cumulated sales quantity decreases with the limited repair time increase, the sales price is increased to cover the $R\&D$ cost, production cost and warranty cost to maximize the total profit.

6.4.3. Effect of unit replacement, minimal repair and PM cost

To study the effects of unit replacement, minimal repair and PM cost on the optimal solutions, all kinds of costs and total profit, we choose different values of both for each of unit replacement, minimal repair and PM cost. Tables 4–6 shows that the optimal solutions under the parameter setting specified in Section 6.2. In Section 6.2, we set $c_{wr} = 1000$, $c_{wm} = 100$ and $c_{wm} = 20$. Therefore, we test the different values of $c_{wr} = \{1000, 1500, 2000\}$, $c_{wm} = \{100, 200, 300\}$ and $c_{wm} = \{20, 30, 40\}$ for the sensitive analysis.

TABLE 6. Optimal solutions for different unit PM cost.

c_{wp}	μ^*	W^*	P^*	Cumulated sales quantity	Total $R\&D$ cost ($\times 10^8$)	Total production cost ($\times 10^8$)	Total Warranty cost ($\times 10^8$)	Total profit ($\times 10^8$)
20	0.5719	2.6	12 346	892 346	7.9370	10.485	4.2190	87.517
30	0.5775	2.6	12 339	892 835	7.9416	10.491	4.2207	87.515
40	0.5775	2.6	12 338	892 913	7.9418	10.492	4.2225	87.514

7. Conclusions and future research

As products gradually become more multi–functional and complex, the repair need higher cost and more time, which cause the manufacturer to make strategic decision on PM interval, sales price, warranty period and so on. A suitable PM action and pricing strategy are becoming the important aspects of reducing the warranty cost and increasing the expected total profit. Therefore, in this paper, we establish an integrated optimization model including product sales price, warranty period, PM interval and cumulated sales quantity. We propose the sales growth model, a new format for a production cost model and the warranty cost model for the Lemon Law market. Then, the sensitive analysis of parameter of price elasticity a, limited repair time r_0, unit PM cost, unit minimal repair cost and unit replacement cost.

The main purpose of this paper is to determine the optimal PM interval, sales price and warranty period with the objective of maximizing the total profit. In this paper, the product is deteriorated with the time goes and we introduce PM policy, minimal repair policy and replacement policy to formulate the expected total profit function. Between each PM action, the product is deteriorated by the function of failure rate and the product failure rate is decreased combined with the level of resetting. Then, the $R\&D$ cost and production cost is exponential form but different construction, these cost will be changed in the wake of changing of product reliability. Moreover, the policy of PM, minimal repair and replacement are both actually exerted in the industrial manufacturing. And these policies are also used in the background of market of Lemon automobile and Lemon Law has also published in some states of USA.

The innovations of our paper can be classified as two aspects. First of all, we proposed an exponential form of unit production cost model considering product reliability. Secondly, we consider the Failed–But–Not–Reported phenomenon of customers in our combination warranty cost model.

Finally, future topics may be given as follows. Firstly, repair time is an important factor, which affect the manufacturers and customers; therefore, repair time can be as a variable to be optimized. Secondly, for the real world, competitive mechanism is existed in the whole market, but in our paper, we only consider one manufacturer in the market; therefore, adding the competitive mechanism as a future research topic to complicate the model, which is worth for manufacturers to achieve the profit.

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