Combinatorial approximation of maximum k-vertex cover in bipartite graphs within ratio 0.7

Vangelis Th. Paschos*

PSL* Research University, Université Paris-Dauphine, LAMSADE CNRS UMR 7243 paschos@lamsade.dauphine.fr

June 28, 2021

Abstract

We propose a purely combinatorial algorithm for MAX k-VERTEX COVER in bipartite graphs, achieving approximation ratio 0.7. The only combinatorial algorithms currently known until now for this problem are the natural greedy algorithm, that achieves ratio $(e^{-1})/e = 0.632$, and an easy 2/3-approximation algorithm presented in [4].

1 Introduction

In the MAX k-VERTEX COVER problem, a graph G = (V, E) with |V| = n and |E| = m is given together with an integer $k \leq n$. The goal is to find a subset $K \subseteq V$ with k elements such that the total number of edges covered by K is maximized. This problem is **NP**-hard even in bipartite graphs [2, 5].

The approximation of MAX k-VERTEX COVER has been originally studied in [6], where an approximation $1 - \frac{1}{e}$ was proved, achieved by the natural greedy algorithm. This ratio is tight even in bipartite graphs [3]. In [1], using a sophisticated linear programming method, the approximation ratio for MAX k-VERTEX COVER is improved up to $\frac{3}{4}$. Finally, by an easy reduction from MIN VERTEX COVER, it can be shown that MAX k-VERTEX COVER in general graphs does not admit a polynomial time approximation schema (PTAS), unless $\mathbf{P} = \mathbf{NP}$ [7].

Obviously, the result of [1] immediately applies to the case of bipartite graphs. Very recently, [5] has improved this ratio in bipartite graphs up to $^{8/9}$, always using involved linear programming techniques, but the existence of a PTAS for such graphs always remains open.

Finally, let us note that MAX k-VERTEX COVER is polynomial in regular bipartite graphs or in semi-regular ones, where the vertices of each color class have the same degree. Indeed, in both cases it suffices to chose k vertices in the color class of maximum degree.

Our principal motivation for this paper is to study in what extent combinatorial methods for MAX k-VERTEX COVER compete with linear programming ones. In other words, what is the ratios' level, a purely combinatorial algorithm can guarantee? In this purpose,

^{*}Also, Institut Universitaire de France

we devise an algorithm that builds five distinct solutions and returns the best among them; for this algorithm, we prove a worst case 0.7-approximation ratio. Let us note that a similar issue is presented in [8] for MAX CUT where a 0.531-ratio combinatorial algorithm is given. Comparison of classes of methods with respect to their abilities to solve problems seems to be a very interesting research issue. This may bring new insights to both the problems handled and the methods themselves. Furthermore, such studies may exhibit interesting and funny mathematical problems.

Note finally that in [4], an easy 2/3-approximation algorithm is presented, together with a very complex one where a computer assisted analysis was giving a ratio of 0.792. But this ratio is impossible to be proved analytically.

2 Preliminaries

Consider a bipartite graph $B(V_1, V_2, E)$, fix an optimal solution O for MAX k-VERTEX COVER (i.e., a vertex-set on k vertices covering a maximum number of edges in E) and denote by k_1 and k_2 the cardinalities of the subsets O_1 and O_2 of O lying in the colorclasses V_1 and V_2 , respectively. W.l.o.g., we assume $k_1 \leq k_2$ and we set:

$$k_1 = \mu \cdot k_2, \quad \mu \leqslant 1 \tag{1}$$

$$k = k_1 + k_2 = (1 + \mu) \cdot k_2 \tag{2}$$

Denote by $\delta(V')$, $V' \subseteq V = V_1 \cup V_2$, the number of edges covered by V' and by opt(B) the value of an optimal solution (i.e., the number of edges covered by O).

Let S_i , i = 1, 2, be the k_i vertices of V_i that cover the most of edges. Obviously, S_i is the set of the k_i largest degree vertices in V_i (breaking ties arbitrarily) and the following hold:

$$\begin{aligned} \delta(S_1) &\geq \delta(O_1) \\ \delta(S_2) &\geq \delta(O_2) \end{aligned}$$
(3)

In what follows, we call "best" vertices, a set of vertices that cover the most of uncovered edges¹ in B. Furthermore, we will also use the following additional notations and conventions (we assume that vertices in both V_1 and V_2 are ordered in decreasing degree order), where all the greek letters used imply parameters that are all smaller than, or equal to, 1:

- $\delta(O_1)$: the number of edges covered by O_1 ; for conciseness we set $\delta(O_1) = \alpha \cdot \operatorname{opt}(B)$;
- $\beta_1 \cdot \delta(O_1) = \beta_1 \cdot \alpha \cdot \operatorname{opt}(B)$: the number of edges covered by $S_1 \cap O_1$;
- $\delta'(O_2)$: the number of *private* edges covered by O_2 , i.e., the edges already covered by O_1 are not counted up to $\delta'(O_2)$; obviously, $\delta'(O_2) = (1 \alpha) \cdot \operatorname{opt}(B)$;
- $\theta \cdot \delta(O_1)$: the number of edges (if any) from O_1 that go "below" O_2 (recall V_1 and V_2 are ordered in decreasing degree order);
- $\gamma \cdot \delta'(O_2)$: symmetrically, it denotes the number of edges of O_2 that go below the vertices of O_1 ;
- $\zeta \cdot \delta(O_1)$: suppose that after taking the k best vertices of V_1 , there still remain, say, k'_1 vertices of O_1 that have not been encountered yet; then, $\zeta \cdot \delta(O_1)$ is the number of edges covered by those vertices;

¹For instance, saying "we take S_1 plus the k_2 best vertices in V_2 , this means that we take S_1 and then k_2 vertices of highest degree in $B[(V_1 \setminus S_1), V_2]$.

• $\lambda \cdot \delta'(O_2)$: this is the symmetric of the quantity $\zeta \cdot \delta(O_1)$ for the pair (V_2, O_2) (supposing that the number of vertices in O_2 that have not been encountered is k'_2).

In Figure 1, the edge-sets defined by the parameters above are illustrated. Heavy lines within rectangles V_1 and V_2 represent the borders of S_1 and S_2 (the upper ones) and those of the k best vertices (the lower ones). Edges from O_1 ($\arg(\delta(O_1))$) are not shown in the the figure. They can go everywhere in V_2 . Private edges of O_2 ($\arg(\delta'(O_2))$) are shown as heavy lack lines (the set of edges δ'_2). They can go everywhere in $V_1 \setminus O_1$.

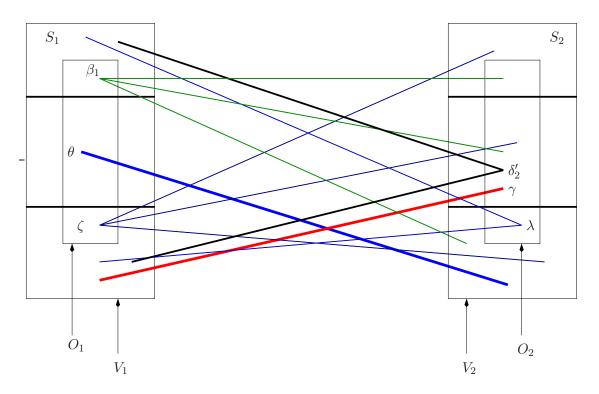


Figure 1: The edge-sets induced by the several parameters.

The basic idea of the algorithm is quite simple. It computes the best among five solutions built. It is presented in Section 3 while, in Section 4, we show that its approximation ratio is bounded below by 0.7.

3 The algorithm ...

The algorithm guesses the cardinalities k_1 , k_2 of O_1 and O_2 , respectively, builds the five solutions specified just below and returns the best among them:

SOL₁: take S_1 plus the k_2 remaining best vertices from V_2 ;

SOL₂: take S_2 plus the k_1 remaining best vertices from V_1 ;

SOL₃: take the k best vertices of V_2 ;

SOL₄: take the best between the following two solutions:

1. the k best vertices of V_1 ;

2. the best $2 \cdot k_1$ vertices of V_1 plus the remaining $k - 2 \cdot k_1$ best vertices of V_2 .

Let us note that the algorithm above, since it runs for any value of k_1 and k_2 , it will run for $k_1 = k$ and $k_2 = k$. So, it is optimal for the instances of [3], where the greedy algorithm attains the ratio $(e^{-1})/e$.

In what follows in this section, in Lemmata 1 to 4, we analyze the solutions built by the algorithm and provide several expressions for the ratios achieved by each of them. All these ratios are expressed as functions of the parameters specified in Section 2. In order to simplify notations from now on we shall write opt instead of opt(B).

Lemma 1. The approximation ratio achieved by solution SOL_1 is the maximum of the following quantities:

$$1 - \alpha + \beta_1 \cdot \alpha \tag{4}$$

$$\alpha + \gamma \cdot (1 - \alpha) \tag{5}$$

Furthermore, if S_1 and O_1 coincide (i.e., $S_1 \cap O_1 = S_1$), SOL₁ is optimal.

Proof. For (4), S_1 covers, by (3), more than $\delta(O_1) = \alpha \cdot \text{opt}$ edges. Decompose this edgeset into a set X of edges covered by $S_1 \setminus (S_1 \cap O_1)$ and the set of edges of size $\beta_1 \cdot \alpha \cdot \text{opt}$ of edges covered by $S_1 \cap O_1$. On the other hand, the k_2 remaining best vertices in V_2 will cover more edges than the k_2 remaining best vertices in O_2 , that cover more than $(1 - \alpha) \cdot \text{opt} - |X|$ edges, qed.

For (5), whenever S_1 does not coincide with O_1 , there are vertices of O_1 that ly below S_1 . Since $\gamma \cdot \delta'(O_2)$ is the number of edges from O_2 that go belong O_1 , these edges will be not counted up in the set of edges covered by S_1 .

Finally, if S_1 and O_1 coincide, SOL₁ will cover $\alpha \cdot \text{opt} + (1 - \alpha) \cdot \text{opt} = \text{opt edges}$.

Lemma 2. The approximation ratio achieved by solution SOL_2 is bounded below by:

$$1 - \alpha + \alpha \cdot \theta \tag{6}$$

Proof. The proofs is similar with the one of Lemma 1 for (5).

Lemma 3. The approximation ratio achieved by solution SOL_3 is the maximum of the following quantities:

$$1 - \lambda \cdot (1 - \alpha) - \alpha \cdot \theta \tag{7}$$

$$(1 - \alpha) \cdot (1 + \lambda \cdot \mu) \tag{8}$$

Proof. If after taking the k best vertices of V_2 the whole of O_2 has been encountered, all but $\theta \cdot \delta(O_1)$ edges of the optimum have been covered. In this case, an appoximation ratio $1 - \alpha \cdot \theta$ is achieved.

Otherwise, by the definition of $\lambda \cdot \delta'(O_2)$:

$$opt - \lambda \cdot \delta'(O_2) - \theta \cdot \delta(O_1) = opt \cdot (1 - \lambda \cdot (1 - \alpha) - \theta \cdot \alpha)$$

edges of the optimum are covered.

On the other hand, taking the k best vertices of V_2 , consists of first taking S_2 (covering $(1-\alpha) \cdot \text{opt}$ edges) and then the k_1 best vertices below it. Furthermore, below the k best vertices, the group of the k'_2 "worst" vertices of O_2 has average degree at least $\lambda \cdot \delta'(O_2)/k'_2$. Since the algorithm takes k_1 "better" vertices, they will cover at least:

$$\frac{k_1}{k'_2} \cdot \lambda \cdot \delta'(O_2) \ge \frac{k_1}{k_2} \cdot \lambda \cdot (1-\alpha) \cdot \operatorname{opt} \stackrel{(1)}{=} \mu \cdot \lambda \cdot (1-\alpha) \cdot \operatorname{opt}$$

which proves (8).

Lemma 4. The approximation ratio achieved by solution SOL_4 is the maximum of the following quantities:

$$1 - \zeta \cdot \alpha - \gamma \cdot (1 - \alpha) \tag{9}$$

$$(2 - \beta_1) \cdot \alpha + \frac{(1 - \mu) \cdot \zeta \cdot \alpha}{\mu} \tag{10}$$

$$\frac{(1-\mu) + \alpha \cdot (2+\mu) - \alpha \cdot \beta_1}{2} + \frac{(1-2\cdot\mu) \cdot \zeta \cdot \alpha}{2\cdot\mu} \tag{11}$$

Proof. Let us first note that, if after taking the k best vertices in V_1 all the vertices of O_1 are captured, the approximation ratio achieved is $1 - \gamma \cdot (1 - \alpha)$ since only $\gamma \cdot \delta'(O_2) = \gamma \cdot (1 - \alpha) \cdot \text{opt}$ edges of the optimum are not covered. Suppose now that k'_1 vertices of O_1 are not captured. In this case, the k vertices taken from V_1 cover:

opt
$$-\zeta \cdot \delta(O_1) - \gamma \cdot \delta'(O_2) = (1 - \zeta \cdot \alpha - \gamma \cdot (1 - \alpha)) \cdot opt$$

For (10) and (11) now, observe first that the k vertices taken from V_1 can be seen as the union of k/k_1 consecutive k_1 -groups (called clusters in what follows) and that, by (1) and (2), $k/k_1 = 1 + \mu/\mu$. Assume also that the $k - k'_1$ of O_1 encountered among the k best vertices of V_1 are included in the π first clusters. Denote by κ_i the number of vertices of O_1 in the *i*-th cluster, $i = 1, \ldots, \pi$, and suppose that the "optimal" κ_i vertices of cluster *i* cover $\beta_i \cdot \delta(O_1) = \beta_i \cdot \alpha \cdot \text{opt}$ edges.

Claim 1. Consider cluster *i* and denote by $\bar{O}_{1,i}$ the part of O_1 not captured by clusters $1, 2, \ldots, i - 1$ (so, $\bar{O}_{1,i} = \sum_{j=i}^{\pi} \kappa_j + k'_1$). Then, the vertices of cluster *i* will cover at least $(1 - \sum_{j=1}^{i-1} \beta_j) \cdot \alpha \cdot \text{opt}$ edges.

In order to prove Claim 1, observe that the part of $\delta(O_1)$ covered by $\overline{O}_{1,i}$ is:

$$\delta\left(\bar{O}_{1,i}\right) = \delta\left(O_{1}\right) - \sum_{j=1}^{i-1} \beta_{j} \cdot \delta\left(O_{1}\right) = \left(1 - \sum_{j=1}^{i-1} \beta_{j}\right) \cdot \alpha \cdot \text{opt}$$

and that the $\delta(\bar{O}_{1,i})$ edges are covered by $\sum_{j=i}^{\pi} \kappa_j + k'_1 = k_1 - (\sum_{j=1}^{i-1} \kappa_j) \leq k_1$ vertices, while cluster *i* contains exactly k_1 vertices with degree at least as large as those of $\bar{O}_{1,i}$. An easy average argument derives then that the vertices of cluster *i* will cover at least:

$$k_1 \cdot \frac{\left(1 - \sum_{j=1}^{i-1} \beta_j\right) \cdot \alpha \cdot \operatorname{opt}}{k_1 - \left(\sum_{j=1}^{i-1} \kappa_j\right)} \ge \left(1 - \sum_{j=1}^{i-1} \beta_j\right) \cdot \alpha \cdot \operatorname{opt}$$

edges, qed.

Consider the two first groups clusters taken from V_1 . The first of them (S_1) covers more than $\delta(O_1) = \alpha \cdot \text{opt}$ edges (by (3)) while, by Claim 1, the second one will cover more than $(k_1/k_1-\kappa_1) \cdot (1-\beta_1) \cdot \delta(O_1) \ge (1-\beta_1) \cdot \alpha \cdot \text{opt}$ edges. Observe also that, by (1) and (2), $k/k_1 = 1+\mu/\mu$. In any of the remaining $1+\mu/\mu - 2 = 1-\mu/\mu$ clusters, their vertices obviously cover more than $\zeta \cdot \delta(O_1) = \zeta \cdot \alpha \cdot \text{opt}$ edges (indeed, by the average argument of Claim 1, more than $k_1 \cdot \zeta \cdot \delta(O_1)/k'_1 \ge \zeta \cdot \delta(O_1)$). We so have:

$$|\mathrm{SOL}_4| \ge \left[(2-\beta_1) + \frac{1-\mu}{\mu} \cdot \zeta \right] \cdot \delta(O_1) = \left[(2-\beta_1) + \frac{1-\mu}{\mu} \cdot \zeta \right] \cdot \alpha \cdot \mathrm{opt}$$

that proves (10).

Let us now get some more insight in the value of SOL_4 . By extending the discussion just above, the k_1 vertices of cluster *i* will cover more than:

$$\frac{k_1}{k_1 - \sum_{j=1}^{i-1} \kappa_j} \cdot \left(\sum_{j=i}^{\pi} \beta_j + \zeta\right) \cdot \delta(O_1) \ge \left(1 - \sum_{j=1}^{i-1} \beta_j\right) \cdot \delta(O_1) \tag{12}$$

Furthermore, as seen previously, all clusters below the π first ones containing the $k_1 - k'_1$ captured vertices of O_1 , will cover more than $\zeta \cdot \delta(O_1)$ each.

Hence, summing (12) for i = 1 to π , taking into account the remark just above, and setting $\beta_0 = 0$, the following holds:

$$|\text{SOL}_{4}| \geq \left(\sum_{i=0}^{\pi-1} \left(1 - \sum_{j=0}^{i} \beta_{j}\right) + \left(\frac{1+\mu}{\mu} - \pi\right) \cdot \zeta\right) \cdot \delta(O_{1})$$

$$= \left(\pi - \sum_{i=1}^{\pi-1} (\pi - i) \cdot \beta_{i} + \left(\frac{1+\mu}{\mu} - \pi\right) \cdot \zeta\right) \cdot \delta(O_{1})$$

$$= \left(\pi - \pi \cdot \sum_{i=1}^{\pi-1} \beta_{i} + \sum_{i=1}^{\pi-1} i \cdot \beta_{i} + \left(\frac{1+\mu}{\mu} - \pi\right) \cdot \zeta\right) \cdot \delta(O_{1})$$
(13)

Observe now that:

$$\pi \cdot \sum_{i=1}^{\pi-1} \beta_i = \pi \cdot (1 - \beta_\pi - \zeta) \cdot \delta(O_1)$$
(14)

and combine (14) with (13). Then, the latter becomes:

$$|\text{SOL}_{4}| \geq \left(\pi - \pi \cdot (1 - \beta_{\pi} - \zeta) + \sum_{i=1}^{\pi-1} i \cdot \beta_{i} + \left(\frac{1+\mu}{\mu} - \pi\right) \cdot \zeta\right) \cdot \delta(O_{1})$$

$$= \left(\sum_{i=1}^{\pi} i \cdot \beta_{i} + \pi\zeta + \left(\frac{1+\mu}{\mu} - \pi\right) \cdot \zeta\right) \cdot \delta(O_{1})$$

$$= \left(\sum_{i=1}^{\pi} i \cdot \beta_{i} + \left(\frac{1+\mu}{\mu}\right) \cdot \zeta\right) \cdot \delta(O_{1})$$

$$= \left(\sum_{i=1}^{\pi} \beta_{i} + \sum_{i=2}^{\pi} \beta_{i} + \sum_{i=3}^{\pi} (i-2) \cdot \beta_{i} + \frac{1+\mu}{\mu} \cdot \zeta\right) \cdot \delta(O_{1})$$

$$= \left((1 - \zeta) + (1 - \beta_{1} - \zeta) + \sum_{i=3}^{\pi} (i-2) \cdot \beta_{i} + \frac{1+\mu}{\mu} \cdot \zeta\right) \cdot \delta(O_{1})$$

$$= \left((2 - \beta_{1}) + \sum_{i=3}^{\pi} (i-2) \cdot \beta_{i} + \frac{1-\mu}{\mu} \cdot \zeta\right) \cdot \delta(O_{1})$$

$$\geq \left((2 - \beta_{1}) + \sum_{i=3}^{\pi} \cdot \beta_{i} + \frac{1-\mu}{\mu} \cdot \zeta\right) \cdot \delta(O_{1})$$
(15)

Set $\sum_{i=3}^{\pi} \cdot \beta_i \cdot \delta(O_1) = X$. These edges are covered by both O_1 and SOL₄. Then, (15) becomes:

$$|\mathrm{SOL}_4| \ge \left((2-\beta_1) + \frac{1-\mu}{\mu} \cdot \zeta \right) \cdot \delta(O_1) + X = \left((2-\beta_1) + \frac{1-\mu}{\mu} \cdot \zeta \right) \cdot \alpha \cdot \mathrm{opt} + X$$
(16)

On the other hand, consider Item 2 in SOL₄. The $2 \cdot k_1$ best vertices from V_1 cover $(1-\zeta) \cdot \delta(O_1) - X$ edges of the optimum. Let $Y + [(1-\zeta) \cdot \delta(O_1) - X]$ be the total number of edges covered by those vertices. Then, best $k - 2 \cdot k_1$ vertices of V_2 will cover at least as many edges as the $k - 2 \cdot k_1$ best vertices of O_2 , that will cover at least $((k-2\cdot k_1)/k_2) \cdot \delta'(O_2) - Y$. Putting all this together, we get:

$$|\text{SOL}_4| \geq Y + [(1-\zeta)\delta(O_1) - X] + \frac{k-2 \cdot k_1}{k_2} \cdot \delta'(O_2) - Y$$
$$= \left((1-\zeta) \cdot \alpha + \frac{k-2 \cdot k_1}{k_2} \cdot (1-\alpha) \right) \cdot \text{opt} - X$$
$$\stackrel{(1),(2)}{=} ((1-\zeta) \cdot \alpha + (1-\mu) \cdot (1-\alpha)) \cdot \text{opt} - X$$
(17)

Expression (16) is increasing with X, while (17) is decreasing. Equality of them, leads after some easy algebra to:

$$X = \left(\frac{(1-\mu) - \alpha \cdot (2-\mu) + \beta_1 \cdot \alpha}{2} - \frac{\zeta \cdot \alpha}{2 \cdot \mu}\right) \cdot \text{opt}$$
(18)

Embedding (18) to (16) and dividing the ratio obtained by opt, derives the ratio claimed by (11). \blacksquare

4 ... and its approximation ratio

The objective of this section is to prove the following theorem.

Theorem 1. MAX *k*-VERTEX COVER is combinatorially approximable within ratio 0.7.

Proof. For the proof we propose an exhaustive parameter-elimination method (very probably non-optimal) that has the advantage to be quite simple. It consists of subsequently eliminating parameters from the ratios proved in Lemmata 1 to 4 until two ratios that are only functions of μ are got. These ratios have opposite monotonies with respect to this parameter, hence by equalizing them we determine a lower bound for the overall ratio of the algorithm.

Elimination of θ : ratios (6) and (7)

Equalizing ratios given by (6) and (7) leads to $2\alpha \cdot \theta = \alpha - \lambda \cdot (1 - \alpha) \Rightarrow \alpha \cdot \theta = \alpha - \lambda \cdot (1 - \alpha/2)$ and embedding it in (6) derives:

$$\frac{2 - \alpha \cdot (1 - \lambda) - \lambda}{2} \tag{19}$$

Elimination of λ : ratios (8) and (19)

Equalizing ratios given by (8) and (19) gives $\lambda = \alpha/(1-\alpha) \cdot (1+2\cdot\mu)$. This, together with (8), derives:

$$1 - \alpha \cdot \frac{1 + \mu}{1 + 2 \cdot \mu} \tag{20}$$

Elimination of γ : ratios (5) and (9)

It gives $\gamma = \frac{1-\alpha-\zeta \cdot \alpha}{2 \cdot (1-\alpha)}$ and the ratio obtained is:

$$\frac{1+\alpha-\zeta\cdot\alpha}{2}\tag{21}$$

Elimination of ζ : ratios (10) and (21)

We have:

$$(2 - \beta_1) \cdot \alpha + \frac{(1 - \mu) \cdot \zeta \cdot \alpha}{\mu} = \frac{1 + \alpha - \zeta \cdot \alpha}{2}$$

$$\Rightarrow \quad \frac{2 - \mu}{\mu} \cdot \zeta \cdot \alpha = 1 + \alpha - 2 \cdot (2 - \beta_1) \cdot \alpha = 1 - 3 \cdot \alpha + 2 \cdot \beta_1 \cdot \alpha$$

$$\Rightarrow \quad \zeta = \frac{\mu}{2 - \mu} \cdot \frac{1 - 3 \cdot \alpha + 2 \cdot \beta_1 \cdot \alpha}{\alpha}$$
(22)

and embedding (22) in (10), we get:

$$(2 - \beta_{1}) \cdot \alpha + \frac{1 - \mu}{2 - \mu} \cdot (1 - \alpha \cdot (3 - 2 \cdot \beta_{1}))$$

$$= \frac{(2 - \mu) \cdot (2 - \beta_{1}) \cdot \alpha + (1 - \mu) - \alpha \cdot (1 - \mu) \cdot (3 - 2 \cdot \beta_{1})}{2 - \mu}$$

$$= \frac{(1 - \mu) + \alpha \cdot (1 + \mu - \mu \cdot \beta_{1})}{2 - \mu}$$
(23)

First elimination of β_1 : ratios (4) and (23)

We have:

$$1 - \alpha + \beta_1 \cdot \alpha = \frac{(1 - \mu) + \alpha \cdot (1 + \mu - \mu \cdot \beta_1)}{2 - \mu}$$

$$\Rightarrow 2 \cdot \beta_1 \cdot \alpha = (1 - \mu) + \alpha \cdot (1 + \mu) - (2 - \mu) + \alpha \cdot (2 - \mu) = -1 + 3 \cdot \alpha$$

$$\Rightarrow \beta_1 = \frac{3 \cdot \alpha - 1}{2 \cdot \alpha}$$
(24)

Now, combination of (4) and (24) derives:

$$1 - \alpha + \beta_1 \cdot \alpha = 1 - \alpha + \frac{3 \cdot \alpha - 1}{2} = \frac{1 + \alpha}{2}$$
(25)

First ratio function of μ : combination of ratios (20) and (25)

Ratio (20) is decreasing with α , while ratio (25) is increasing. Combination of them allows elimination α in order to get a first ratio that is only a function of μ . Equalizing (20) and (25) gives:

$$1 - \alpha \cdot \frac{1 + \mu}{1 + 2 \cdot \mu} = \frac{1 + \alpha}{2} \Rightarrow \alpha \cdot (4 \cdot \mu + 3) = 2 \cdot \mu + 1$$

$$\Rightarrow \quad \alpha = \frac{2 \cdot \mu + 1}{4 \cdot \mu + 3}$$
(26)

and embedding (26) in (20) derives:

$$1 - \alpha \cdot \frac{1 + \mu}{1 + 2 \cdot \mu} = 1 - \frac{2 \cdot \mu + 1}{4 \cdot \mu + 3} \cdot \frac{1 + \mu}{1 + 2 \cdot \mu} = \frac{2 + 3 \cdot \mu}{3 + 4 \cdot \mu}$$
(27)

Second elimination of β_1 : ratios (4) and (11)

Revisit ratio (11) and observe that its last term $(1-2\cdot\mu)\cdot\zeta\cdot\alpha/2\cdot\mu$ is negative when $\mu \ge 1/2$. On the other hand, ratio (27) is increasing with μ and bounded below by 0.7 as long as $\mu \ge 1/2$. We so seek an "interesting" ratio when $\mu \le 1/2$ and, in this case $(1-2\cdot\mu)\cdot\zeta\cdot\alpha/2\cdot\mu \ge 0$ and can be omitted.

Hence, combination of ratios (4) and (11), for $\mu \leq 1/2$, leads to:

$$\frac{(1-\mu)+\alpha\cdot(2+\mu)-\alpha\cdot\beta_1}{2} = 1-\alpha+\beta_1\cdot\alpha$$

$$\Rightarrow \quad (1-\mu)+\alpha\cdot(2+\mu)-\alpha\cdot\beta_1 = 2-2\cdot\alpha+2\cdot\beta_1\cdot\alpha$$

$$\Rightarrow \quad 3\cdot\beta_1\cdot\alpha = -(1+\mu)+\alpha\cdot(4+\mu) \Rightarrow \beta_1 = \frac{\alpha\cdot(4+\mu)-(1+\mu)}{3\cdot\alpha}$$
(28)

Then, combining (4) and (28), derives this time:

$$1 - \alpha + \beta_1 \cdot \alpha = \frac{3 - 3 \cdot \alpha + \alpha \cdot (4 + \mu) - (1 + \mu)}{3} = \frac{2 - \mu + \alpha \cdot (1 + \mu)}{3}$$
(29)

Second ratio function of μ : combination of ratios (29) and (20)

Once again, ratio (20) is decreasing with α , while ratio (29) is increasing. Combination of them allows elimination α in order to get a second ratio exclusively function of μ . Equalizing (20) and (29) gives:

$$1 - \alpha \cdot \frac{1+\mu}{1+2\cdot\mu} = \frac{2-\mu+\alpha\cdot(1+\mu)}{3} \Rightarrow 2\cdot\alpha \cdot \frac{(1+\mu)\cdot(2+\mu)}{3\cdot(1+2\cdot\mu)} = \frac{1+\mu}{3}$$
$$\Rightarrow \quad \alpha = \frac{1+2\cdot\mu}{2\cdot(2+\mu)} \tag{30}$$

and embedding (30) in (20) derives:

$$1 - \alpha \cdot \frac{1 + \mu}{1 + 2 \cdot \mu} = \frac{3 + \mu}{4 + 2 \cdot \mu}$$
(31)

Final ratio

As noted above, ratio (27) increases with μ , while (31) decreases. The value of μ guaranteeing equality of these ratios also gives a lower bound for them. This value is $\mu = 1/2$ and, with this value, both ratios become 0.7.

Acknowledgement. The very fruitful and stimulating discussions I had with Edouard Bonnet and Georgios Stamoulis are gratefully acknowledged. Many thanks to Edouard Bonnet for his very pertinent comments on a first draft of the paper.

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