# Dejean's conjecture holds for $n \ge 30$

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November 13, 2018

#### Abstract

We extend Carpi's results by showing that Dejean's conjecture holds for  $n \ge 30$ .

#### 1 Introduction

Repetitions in words have been studied starting with Thue [12, 13] at the beginning of the previous century. Much study has also been given to repetitions with fractional exponent [1, 3, 4, 5, 6, 8]. If n > 1 is an integer, then an *n*-power is a non-empty word  $x^n$ , i.e., word x repeated n times in a row. For rational r > 1, a **fractional** r-power is a non-empty word  $w = x^{\lfloor r \rfloor} x'$  such that x' is the prefix of x of length  $(r - \lfloor r \rfloor)|x|$ . For example, 01010 is a 5/2-power. A basic problem is that of identifying the *repetitive threshold* for each alphabet size n > 1:

What is the infimum of r such that an infinite sequence on n letters exists, not containing any r-powers?

<sup>\*</sup>The author is supported by an NSERC Discovery Grant.

<sup>&</sup>lt;sup>†</sup>The author is supported by an NSERC Postdoctoral Fellowship.

We call this infimum the *repetitive threshold* of an *n*-letter alphabet, denoted by RT(n). Dejean's conjecture [4] is that

$$RT(n) = \begin{cases} 7/4, & n = 3\\ 7/5, & n = 4\\ n/(n-1) & n \neq 3, 4 \end{cases}$$

The values RT(2), RT(3), RT(4) were established by Thue, Dejean and Pansiot, respectively [12, 4, 11]. Moulin-Ollagnier [10] verified Dejean's conjecture for  $5 \le n \le 11$ , while Mohammad-Noori and Currie [9] proved the conjecture for  $12 \le n \le 14$ .

An exciting new development has recently occurred with the work of Carpi [3], who showed that Dejean's conjecture holds for  $n \ge 33$ . Verification of the conjecture is now only lacking for a finite number of values. In the present paper, we sharpen Carpi's methods to show that Dejean's conjecture holds for  $n \ge 30$ .

#### 2 Preliminaries

The following definitions are from sections 8 and 9 of [3]: Fix  $n \ge 30$ . Let  $m = \lfloor (n-3)/6 \rfloor$ . Let  $A_m = \{1, 2, \ldots, m\}$ . Let ker  $\psi = \{v \in A_m^* | \forall a \in A_m, 4 \text{ divides } |v|_a\}$ . (In fact, this is not Carpi's definition of ker  $\psi$ , but rather the assertion of his Lemma 9.1.) A word  $v \in A_m^+$  is a  $\psi$ -kernel repetition if it has period q and a prefix v' of length q such that  $v' \in \ker \psi$ ,  $(n-1)(|v|+1) \ge nq-3$ .

It will be convenient to have the following new definition: If v has period q and its prefix v' of length q is in ker  $\psi$ , we say that q is a **kernel period** of v.

As Carpi states at the beginning of section 9 of [3]:

By the results of the previous sections, at least in the case  $n \geq 30$ , in order to construct an infinite word on n letters avoiding factors of any exponent larger than n/(n-1), it is sufficient to find an infinite word on the alphabet  $A_m$  avoiding  $\psi$ -kernel repetitions.

For m = 5, Carpi produces such an infinite word, based on a paper-folding construction. He thus establishes Dejean's conjecture for  $n \ge 33$ . In the present paper, we give an infinite word on the alphabet  $A_4$  avoiding  $\psi$ -kernel repetitions. We thus establish Dejean's conjecture for  $n \ge 30$ .

**Definition 1.** Let  $f : A_4^* \to A_4^*$  be defined by f(1) = 121, f(2) = 123, f(3) = 141, f(4) = 142. Let w be the fixed point of f.

It is useful to note that the frequency matrix of f, i.e.,

$$[|f(i)|_j]_{4\times 4} = \begin{bmatrix} 2 & 1 & 0 & 0\\ 1 & 1 & 1 & 0\\ 2 & 0 & 0 & 1\\ 1 & 1 & 0 & 1 \end{bmatrix}$$

has an inverse modulo 4.

**Remark 1.** Let q be a non-negative integer,  $q \leq 1966$ . Fix n = 32.

- R1: Word w contains no  $\psi$ -kernel repetition v with kernel period q.
- R2: Word w contains no factor v with kernel period q such that  $|v|/q \ge 35/34$ .

Note that  $\frac{32}{31} - \frac{34}{31q} = \frac{35}{34}$  when  $q = \frac{34^2}{3} = 385\frac{1}{3}$ , so neither piece of the remark implies the other. Note also that the conditions of the remark become **less** stringent for n = 30, 31. One also verifies that

$$\frac{35}{34} + \frac{9}{2(1967)} \le \frac{32}{31} - \frac{34}{31q}$$

for  $q \ge 1967$ . To show that w contains no  $\psi$ -kernel repetitions for n = 30, 31, 32, it thus suffices to verify R1 and to show that word w contains no factor v with kernel period  $q \ge 1967$  such that

$$|v|/q \ge 35/34 + 9/2(1967). \tag{1}$$

The remarks R1 and R2 are verified by computer search, so we will consider the second part of this attack. Fix  $q \ge 1967$ , and suppose that v is a factor of w with kernel period q, and  $|v|/q \ge 35/34$ . Without loss of generality, suppose that no extension of v has period q. Write v = sf(u)p where s (resp. p) is a suffix (resp. prefix) of the image of a letter, and |s| (resp.  $|p|) \le 2$ .

If  $|v| \le q+2$ , then  $35/34 \le (q+2)/q$  and  $1/34 \le 2/q$ , forcing  $q \le 68$ . This contradicts R2. We will therefore assume that  $|v| \ge q+3$ .

Suppose |s| = 2. Since  $|v| \ge q + 3$ , write v = s0zs0v', where |s0z| = q. Examining f, we see that the letter  $a_s$  preceding any occurrence of s0 in w is uniquely determined if |s| = 2. It follows that  $a_s v$  is a factor of w with kernel period q, contradicting the maximality of v. We conclude that  $|s| \leq 1$ .

Again considering f, we see that if t is any factor of w of length 3, and  $u_1t$ ,  $u_2t$  are prefixes of w, then  $|u_1| \equiv |u_2| \pmod{3}$ . Since  $|v| \geq q+3$ , we conclude that 3 divides q. Write  $q = 3q_0$ . Since  $|s| \leq 1$ ,  $|p| \leq 2$  and  $|v| \geq q+3$ , we see that  $|f(u)| \geq q$ . Thus f(u) has a prefix of length  $q = 3q_0$  which is in ker  $\psi$ . As the frequency matrix of f is invertible modulo 4, the prefix of u of length  $q_0$  is in ker  $\psi$ . We see that

$$\frac{|v|}{q} \le \frac{3|u|+3}{3q_0} = \frac{|u|}{q_0} + \frac{1}{q_0}.$$

**Lemma 2.** Let s be a non-negative integer. If factor v of w has kernel period q, where  $q \leq 1966(3^s)$ , then

$$\frac{|v|}{q} < \frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{s-1} 3^{-j}.$$

**Proof:** If s = 0, this is implied by R2. Suppose t > 0 and the result holds for s < t. Suppose that  $1966(3^{t-1}) < q \leq 1966(3^t)$  and there is a factor v of w such that v has kernel period q. Suppose that  $|v|/q \geq 35/34$ . Without loss of generality, suppose that no extension of v has period q. We have seen that there is a factor u of w with kernel period  $q_0 = q/3$ ,  $1966(3^{t-2}) < q_0 \leq 1966(3^{t-1})$  such that

$$\begin{aligned} |v|/q &\leq |u|/q_0 + 1/q_0 \\ &< \left(\frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{t-2} 3^{-j}\right) + \frac{1}{q_0} \text{ (by the induction hypothesis)} \\ &< \frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{t-2} 3^{-j} + \frac{1}{1966(3^{t-2})} \\ &= \frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{t-2} 3^{-j} + \frac{3}{1966(3^{t-1})} \\ &= \frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{t-1} 3^{-j} . \Box \end{aligned}$$

**Theorem 3.** Word w contains no factor v with kernel period q such that

$$|v|/q \ge 35/34 + 9/2(1966)$$

**Proof:** Suppose that factor v of w has kernel period q such that (1) holds. By Remark 1, we have  $q \ge 1966$ . By the previous lemma, for some non-negative s,

$$|v|/q < \frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{s-1} 3^{-j} < \frac{35}{34} + \frac{3}{1966} \sum_{j=0}^{\infty} 3^{-j} = \frac{35}{34} + \frac{9}{2(1966)}.\Box$$

**Corollary 4.** Dejean's conjecture holds for n = 30, 31, 32.

### Appendix: Computer search

Suppose that some factor v of w has kernel period  $q \leq 1966$  and either  $31(|v|+1) \geq 32q-3$  or  $|v|/q \geq 35/34+9/2(1967)$ . Without loss of generality, taking such a v as short as possible, we may assume that

$$|v| \le \left\lceil \frac{32(1966) - 3}{31} - 1 \right\rceil = 2029.$$
  
(We also have  $\left\lceil (1966) \left( \frac{35}{34} + \frac{9}{2(1967)} \right) \right\rceil = 2029.$ )

If |v| > 3, v is a factor of f(u) for some factor u of w with  $|u| \le (|v|+4)/3$ . For a non-negative integer r, let  $g(r) = \lfloor (r+4)/3 \rfloor$ . Since  $g^7(2029) = 2 < 3$ , (here the exponent denotes iterated function composition) word v must be a factor of  $f^7(u)$  for some factor u of w, |u| = 2.

The word  $u_0 = 23141121142$  contains all 8 factors of w which have length 2. To establish R1 and R2, one thus checks that they hold for the single word  $f^7(u_0)$  (which is of length 24,057). In fact, we performed this computer check on the word  $f^7(u_1)$ , where  $u_1 = 11421231211231411$  contains all 13 factors of w which have length 3.

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