# Dejean's conjecture holds for $n \geq 30$ 

James Currie*and Narad Rampersad ${ }^{\dagger}$<br>Department of Mathematics and Statistics<br>University of Winnipeg<br>515 Portage Avenue<br>Winnipeg, Manitoba R3B 2E9 (Canada)<br>j.currie@uwinnipeg.ca<br>n.rampersad@uwinnipeg.ca

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#### Abstract

We extend Carpi's results by showing that Dejean's conjecture holds for $n \geq 30$.


## 1 Introduction

Repetitions in words have been studied starting with Thue [12, 13] at the beginning of the previous century. Much study has also been given to repetitions with fractional exponent [1, 3, 4, 5, 6, 6, 8]. If $n>1$ is an integer, then an $n$-power is a non-empty word $x^{n}$, i.e., word $x$ repeated $n$ times in a row. For rational $r>1$, a fractional $r$-power is a non-empty word $w=x^{\lfloor r\rfloor} x^{\prime}$ such that $x^{\prime}$ is the prefix of $x$ of length $(r-\lfloor r\rfloor)|x|$. For example, 01010 is a $5 / 2$-power. A basic problem is that of identifying the repetitive threshold for each alphabet size $n>1$ :

What is the infimum of $r$ such that an infinite sequence on $n$ letters exists, not containing any $r$-powers?

[^0]We call this infimum the repetitive threshold of an $n$-letter alphabet, denoted by $R T(n)$. Dejean's conjecture [4] is that

$$
R T(n)= \begin{cases}7 / 4, & n=3 \\ 7 / 5, & n=4 \\ n /(n-1) & n \neq 3,4\end{cases}
$$

The values $R T(2), R T(3), R T(4)$ were established by Thue, Dejean and Pansiot, respectively [12, 4, 11]. Moulin-Ollagnier [10] verified Dejean's conjecture for $5 \leq n \leq 11$, while Mohammad-Noori and Currie [9] proved the conjecture for $12 \leq n \leq 14$.

An exciting new development has recently occurred with the work of Carpi [3], who showed that Dejean's conjecture holds for $n \geq 33$. Verification of the conjecture is now only lacking for a finite number of values. In the present paper, we sharpen Carpi's methods to show that Dejean's conjecture holds for $n \geq 30$.

## 2 Preliminaries

The following definitions are from sections 8 and 9 of [3]: Fix $n \geq 30$. Let $m=\lfloor(n-3) / 6\rfloor$. Let $A_{m}=\{1,2, \ldots, m\}$. Let ker $\psi=\left\{v \in A_{m}^{*} \mid \forall a \in A_{m}, 4\right.$ divides $\left.|v|_{a}\right\}$. (In fact, this is not Carpi's definition of ker $\psi$, but rather the assertion of his Lemma 9.1.) A word $v \in A_{m}^{+}$is a $\psi$-kernel repetition if it has period $q$ and a prefix $v^{\prime}$ of length $q$ such that $v^{\prime} \in \operatorname{ker} \psi,(n-1)(|v|+1) \geq$ $n q-3$.

It will be convenient to have the following new definition: If $v$ has period $q$ and its prefix $v^{\prime}$ of length $q$ is in ker $\psi$, we say that $q$ is a kernel period of $v$.

As Carpi states at the beginning of section 9 of [3]:
By the results of the previous sections, at least in the case $n \geq 30$, in order to construct an infinite word on $n$ letters avoiding factors of any exponent larger than $n /(n-1)$, it is sufficient to find an infinite word on the alphabet $A_{m}$ avoiding $\psi$-kernel repetitions.

For $m=5$, Carpi produces such an infinite word, based on a paper-folding construction. He thus establishes Dejean's conjecture for $n \geq 33$. In the present paper, we give an infinite word on the alphabet $A_{4}$ avoiding $\psi$-kernel repetitions. We thus establish Dejean's conjecture for $n \geq 30$.

Definition 1. Let $f: A_{4}^{*} \rightarrow A_{4}^{*}$ be defined by $f(1)=121, f(2)=123$, $f(3)=141, f(4)=142$. Let $w$ be the fixed point of $f$.

It is useful to note that the frequency matrix of $f$, i.e.,

$$
\left[|f(i)|_{j}\right]_{4 \times 4}=\left[\begin{array}{cccc}
2 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
2 & 0 & 0 & 1 \\
1 & 1 & 0 & 1
\end{array}\right]
$$

has an inverse modulo 4 .
Remark 1. Let $q$ be a non-negative integer, $q \leq 1966$. Fix $n=32$.
R1: Word $w$ contains no $\psi$-kernel repetition $v$ with kernel period $q$.
R2: Word $w$ contains no factor $v$ with kernel period $q$ such that $|v| / q \geq$ $35 / 34$.

Note that $\frac{32}{31}-\frac{34}{31 q}=\frac{35}{34}$ when $q=\frac{34^{2}}{3}=385 \frac{1}{3}$, so neither piece of the remark implies the other. Note also that the conditions of the remark become less stringent for $n=30,31$. One also verifies that

$$
\frac{35}{34}+\frac{9}{2(1967)} \leq \frac{32}{31}-\frac{34}{31 q}
$$

for $q \geq 1967$. To show that $w$ contains no $\psi$-kernel repetitions for $n=30,31$, 32 , it thus suffices to verify R1 and to show that word $w$ contains no factor $v$ with kernel period $q \geq 1967$ such that

$$
\begin{equation*}
|v| / q \geq 35 / 34+9 / 2(1967) \tag{1}
\end{equation*}
$$

The remarks R1 and R2 are verified by computer search, so we will consider the second part of this attack. Fix $q \geq 1967$, and suppose that $v$ is a factor of $w$ with kernel period $q$, and $|v| / q \geq 35 / 34$. Without loss of generality, suppose that no extension of $v$ has period $q$. Write $v=s f(u) p$ where $s$ (resp. $p$ ) is a suffix (resp. prefix) of the image of a letter, and $|s|$ (resp. $|p|$ ) $\leq 2$.

If $|v| \leq q+2$, then $35 / 34 \leq(q+2) / q$ and $1 / 34 \leq 2 / q$, forcing $q \leq 68$. This contradicts R2. We will therefore assume that $|v| \geq q+3$.

Suppose $|s|=2$. Since $|v| \geq q+3$, write $v=s 0 z s 0 v^{\prime}$, where $|s 0 z|=q$. Examining $f$, we see that the letter $a_{s}$ preceding any occurrence of $s 0$ in $w$ is
uniquely determined if $|s|=2$. It follows that $a_{s} v$ is a factor of $w$ with kernel period $q$, contradicting the maximality of $v$. We conclude that $|s| \leq 1$.

Again considering $f$, we see that if $t$ is any factor of $w$ of length 3 , and $u_{1} t$, $u_{2} t$ are prefixes of $w$, then $\left|u_{1}\right| \equiv\left|u_{2}\right|(\bmod 3)$. Since $|v| \geq q+3$, we conclude that 3 divides $q$. Write $q=3 q_{0}$. Since $|s| \leq 1,|p| \leq 2$ and $|v| \geq q+3$, we see that $|f(u)| \geq q$. Thus $f(u)$ has a prefix of length $q=3 q_{0}$ which is in ker $\psi$. As the frequency matrix of $f$ is invertible modulo 4 , the prefix of $u$ of length $q_{0}$ is in ker $\psi$. We see that

$$
\frac{|v|}{q} \leq \frac{3|u|+3}{3 q_{0}}=\frac{|u|}{q_{0}}+\frac{1}{q_{0}} .
$$

Lemma 2. Let s be a non-negative integer. If factor $v$ of $w$ has kernel period $q$, where $q \leq 1966\left(3^{s}\right)$, then

$$
\frac{|v|}{q}<\frac{35}{34}+\frac{3}{1966} \sum_{j=0}^{s-1} 3^{-j} .
$$

Proof: If $s=0$, this is implied by R2. Suppose $t>0$ and the result holds for $s<t$. Suppose that $1966\left(3^{t-1}\right)<q \leq 1966\left(3^{t}\right)$ and there is a factor $v$ of $w$ such that $v$ has kernel period $q$. Suppose that $|v| / q \geq 35 / 34$. Without loss of generality, suppose that no extension of $v$ has period $q$. We have seen that there is a factor $u$ of $w$ with kernel period $q_{0}=q / 3$, $1966\left(3^{t-2}\right)<q_{0} \leq 1966\left(3^{t-1}\right)$ such that

$$
\begin{aligned}
|v| / q & \leq|u| / q_{0}+1 / q_{0} \\
& <\left(\frac{35}{34}+\frac{3}{1966} \sum_{j=0}^{t-2} 3^{-j}\right)+\frac{1}{q_{0}} \text { (by the induction hypothesis) } \\
& <\frac{35}{34}+\frac{3}{1966} \sum_{j=0}^{t-2} 3^{-j}+\frac{1}{1966\left(3^{t-2}\right)} \\
& =\frac{35}{34}+\frac{3}{1966} \sum_{j=0}^{t-2} 3^{-j}+\frac{3}{1966\left(3^{t-1}\right)} \\
& =\frac{35}{34}+\frac{3}{1966} \sum_{j=0}^{t-1} 3^{-j} . \square
\end{aligned}
$$

Theorem 3. Word $w$ contains no factor $v$ with kernel period $q$ such that

$$
|v| / q \geq 35 / 34+9 / 2(1966)
$$

Proof: Suppose that factor $v$ of $w$ has kernel period $q$ such that (1) holds. By Remark [1, we have $q \geq 1966$. By the previous lemma, for some non-negative $s$,

$$
|v| / q<\frac{35}{34}+\frac{3}{1966} \sum_{j=0}^{s-1} 3^{-j}<\frac{35}{34}+\frac{3}{1966} \sum_{j=0}^{\infty} 3^{-j}=\frac{35}{34}+\frac{9}{2(1966)}
$$

Corollary 4. Dejean's conjecture holds for $n=30,31,32$.

## Appendix: Computer search

Suppose that some factor $v$ of $w$ has kernel period $q \leq 1966$ and either $31(|v|+1) \geq 32 q-3$ or $|v| / q \geq 35 / 34+9 / 2(1967)$. Without loss of generality, taking such a $v$ as short as possible, we may assume that

$$
|v| \leq\left\lceil\frac{32(1966)-3}{31}-1\right\rceil=2029
$$

(We also have $\left\lceil(1966)\left(\frac{35}{34}+\frac{9}{2(1967)}\right)\right\rceil=2029$.)
If $|v|>3, v$ is a factor of $f(u)$ for some factor $u$ of $w$ with $|u| \leq(|v|+4) / 3$. For a non-negative integer $r$, let $g(r)=\lfloor(r+4) / 3\rfloor$. Since $g^{7}(2029)=2<3$, (here the exponent denotes iterated function composition) word $v$ must be a factor of $f^{7}(u)$ for some factor $u$ of $w,|u|=2$.

The word $u_{0}=23141121142$ contains all 8 factors of $w$ which have length 2. To establish R1 and R2, one thus checks that they hold for the single word $f^{7}\left(u_{0}\right)$ (which is of length 24,057 ). In fact, we performed this computer check on the word $f^{7}\left(u_{1}\right)$, where $u_{1}=11421231211231411$ contains all 13 factors of $w$ which have length 3 .

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