RAIRO-Theor. Inf. Appl. 52 (2018) 55–86
https://doi.org/10.1051/ita/2018003

RAIRO - Theoretical Informatics and Applications
www.rairo-ita.org

THE INCLUSION STRUCTURE OF PARTIALLY LOSSY QUEUE MONOIDS AND THEIR TRACE SUBMONOIDS[☆]

Chris Köcher, Dietrich Kuske[*] and Olena Prianychnykova

Abstract. We model the behavior of a lossy fifo-queue as a monoid of transformations that are induced by sequences of writing and reading. To have a common model for reliable and lossy queues, we split the alphabet of the queue into two parts: the forgettable letters and the letters that are transmitted reliably. We describe this monoid by means of a confluent and terminating semi-Thue system and then study some of the monoid's algebraic properties. In particular, we characterize completely when one such monoid can be embedded into another as well as which trace monoids occur as submonoids. Surprisingly, these are precisely those trace monoids that embed into the direct product of two free monoids – which gives a partial answer to a question raised by Diekert *et al.* at STACS 1995.

Mathematics Subject Classification. 68Q70, 20M35

Received July 14, 2017. Accepted March 20, 2018.
Published online July 11, 2018.

1. Introduction

Queues (alternatively: fifo queues or channels) form a basic storage mechanism that allows to append items at the end and to read the first item from the queue. Providing a finite state automaton with access to a queue results in a Turing complete computation model [2] such that virtually all decision problems on such devices become undecidable.

Situation changes to the better if one replaces the reliable queue by some unreliable version. The most studied version are lossy queues that can nondeterministically lose any item at any moment [1, 4, 11, 23]: in that case reachability, safety properties over traces, inevitability properties over states, and fair termination are decidable (although of prohibitive complexity, see, *e.g.*, [7]). A practically more realistic version are priority queues where items of high priority can erase any previous item of low priority. Concretely, elements of even priority $2i$ can be erased by all elements of priority *at least* $2i$ and the elements of odd priority $2i+1$ can be erased by all elements of priority *strictly larger than* $2i+1$. Then, if all items have even priority, safety and inevitability properties are decidable. But if there is at least one item of non-minimal, but odd priority, then these problems become undecidable (*cf.* [14]).

[☆]This paper provides complete and streamlined proofs of the results in C. Köcher *et al.* (2016) and C. Köcher *et al.* (2017). The authors acknowledge support from the DFG-Project "Speichermechanismen als Monoide", KU 1107/9-1.

Keywords and phrases: Lossy queue, transformation monoid.

Fachgebiet Automaten und Logik, Technische Universität Ilmenau, 98684 Ilmenau, Germany.

[*] Corresponding author: dietrich.kuske@tu-ilmenau.de

In this paper, we study partially lossy queues that can be understood as a model between lossy and priority queues. Seen as a version of lossy queues, their alphabet is divided into two sets of reliable and forgettable letters where only items from the second set can be lost. Seen as a version of priority queues, partially lossy queues use only two priorities (0 and 1).

We describe the behavior of such a partially lossy queue by a monoid as was done, *e.g.*, for pushdowns in [15] and for reliable queues in [13]:. A partially lossy queue is given by its alphabet A as well as the subset $X \subseteq A$ of letters that the queue will transmit reliably. Note that writing a symbol into a queue is always possible (resulting in a longer queue), but reading a symbol is possible only if the symbol is at the beginning of the queue (or is preceded by forgettable symbols, only). Thus, basic actions define partial functions on the possible queue contents. The generated transformation monoid is called *partially lossy queue monoid* or *plq monoid* $\mathcal{Q}(A, X)$. Then $\mathcal{Q}(A, A)$ models the behavior of a reliable queue with alphabet A [13] and $\mathcal{Q}(A, \emptyset)$ the fully lossy queue that can forget any symbol [18].

The first part of this paper presents a complete infinite semi-Thue system for the monoid $\mathcal{Q}(A, X)$. The resulting normal forms imply that two sequences of actions are equivalent if their subsequences of write and of read actions, respectively, coincide and if the induced transformations agree on the shortest queue that they are defined on.

This result is rather similar, although technically more involved, than the corresponding result on the monoid $\mathcal{Q}(A, A)$ of the reliable queue from [13]. In that paper, it is also shown that $\mathcal{Q}(A, A)$ embeds into $\mathcal{Q}(B, B)$ provided B is not a singleton. This is an algebraic formulation of the well-known fact that the reliable queue with two symbols can simulate any other reliable queue. The second part of the current paper is concerned with the embeddability relation between the monoids $\mathcal{Q}(A, X)$. Clearly, the monoid $\mathcal{Q}(A, \emptyset)$ of the fully lossy queue embeds into $\mathcal{Q}(B, \emptyset)$ whenever $|A| \leq |B|$ by looking at A as a subset of B. Joining this almost trivial idea with the (non-trivial) idea from [13], one obtains an embedding of $\mathcal{Q}(A, X)$ into $\mathcal{Q}(B, Y)$ provided the second queue has at least as many forgettable letters as the first and its number of non-forgettable letters is at least the number of non-forgettable letters of the first queue or at least two (*i.e.*, $|A \setminus X| \leq |B \setminus Y|$ and $\mathsf{min}\{|X|, 2\} \leq |Y|$). We prove that, besides these cases, an embedding exists only in case the second queue has precisely one non-forgettable letter and properly more forgettable letters than the first queue (*i.e.*, $|Y| = 1$ and $|A \setminus X| < |B \setminus Y|$). As for the reliable queue, this algebraically mirrors the intuition that a partially lossy queue can simulate another partially lossy queue in these cases, only. In particular, a reliable queue does not simulate a fully lossy queue and *vice versa* and a fully lossy queue cannot simulate another fully lossy queue with more (forgettable) letters. Hence, these results show that the class of submonoids of a plq monoid $\mathcal{Q}(A, X)$ depends heavily on the number of forgettable and non-forgettable letters.

Another important class of monoids are the so-called trace monoids which were introduced into computer science by Mazurkiewicz [24] to model the behavior of concurrent systems. From [6] we know that each trace monoid can be embedded into the direct product of free monoids. A still open question is to ask for the exact number of such factors to embed a given trace monoid. The strongest result in this respect is due to Kunc [21]: Given a C_3- and C_4-free dependence alphabet (where C_n is the cycle on n vertices) and a number k, it is decidable whether the trace monoid embeds into the direct product of k free monoids. Here, we extend this result to all dependence alphabets but only for $k = 2$. More precisely, we give a complete and decidable characterization of all independence alphabets whose generated trace monoid embeds into the direct product of two free monoids. This is the case if all letters in the independence alphabet (Γ, I) have degree at most 1 or the independence alphabet is a complete bipartite graph with some additional isolated vertices. The – at least for the authors – surprising result is that these are exactly the trace monoids embedding into the plq monoid with at least one non-forgettable and one further letter or at least three forgettable letters.

To complete the picture, we also provide a similar characterization for trace monoids embedding into $\mathcal{Q}(\{a, b\}, \emptyset)$: here, the complete bipartite component is replaced by a star graph. In any case, the direct product of $(\mathbb{N}, +)$ and $\{a, b\}^*$ embeds into $\mathcal{Q}(A, X)$, but $(\mathbb{N}, +)^3$ is not a submonoid of $\mathcal{Q}(A, X)$ for arbitrary finite sets A and $X \subseteq A$ (a conjecture formulated in [13]).

In summary, we study properties of the transformation monoid of a partially lossy queue that were studied for the reliable queue in [13]. We find expected similarities (semi-Thue system), differences (embeddability relation), and surprising similarities (trace submonoids).

2. Preliminaries

At first we need some basic definitions. So let A be an alphabet. A word $u \in A^*$ is a *prefix* of $v \in A^*$ iff $v \in uA^*$. Similarly, u is a *suffix* of v iff $v \in A^*u$. Furthermore u is a *subword* of v iff there are $k \in \mathbb{N}$, $a_1, a_2, \dots, a_k \in A$ and $w_1, w_2, \dots, w_{k+1} \in A^*$ such that $u = a_1a_2 \dots a_k$ and $v = w_1a_1w_2a_2 \dots w_ka_kw_{k+1}$, *i.e.*, we obtain u if we drop some letters from v. In this case we write $u \preceq v$. Note that $\preceq$ is a partial ordering on A^*. Let $X \subseteq A$. Then we define the *projection* $\pi_X \colon A^* \to X^*$ on X by

$$\pi_X(\varepsilon) = \varepsilon \quad \text{and} \quad \pi_{\mathrm{X}}(\mathrm{au}) = \begin{cases} a\pi_X(u) & \text{if a} \in \mathrm{X} \\ \pi_X(u) & \text{otherwise} \end{cases} \tag{2.1}$$

for each $a \in A$ and $u \in A^*$. Moreover, u is an X*-subword* of v (denoted $u \preceq_X v$) if $\pi_X(v) \preceq u \preceq v$, *i.e.*, if we obtain u from v by dropping some letters not in X. Since the projection π_X is idempotent and monotone wrt. the subword order, $u \preceq_X v$ implies $\pi_X(v) = \pi_X(\pi_X(v)) \preceq \pi_X(u) \preceq \pi_X(v)$, *i.e.*, $\pi_X(u) = \pi_X(v)$.

Note that $\preceq_\emptyset$ is the subword relation $\preceq$ since $\pi_\emptyset(v) = \varepsilon$ and $\preceq_A$ is the equality relation since $\pi_A(v) = v$.

3. Definition and basic properties

We want to model the behavior of an unreliable queue that stores entries from the alphabet A. The unreliability of the queue stems from the fact that it can forget certain letters that we collect in the alphabet $A \setminus X$. In other words, letters from $X \subseteq A$ are *non-forgettable* and those from $A \setminus X$ are *forgettable*. Note that this unreliability extends the approach from [13] where we considered reliable queues (*i.e.*, $A = X$).

Definition 3.1. A *lossiness alphabet* is a tuple $\mathcal{L} = (A, X)$ where A is an alphabet with $|A| \geq 2$ and $X \subseteq A$.

So let $\mathcal{L} = (A, X)$ be a lossiness alphabet. Then A is the set of all possible queue entries. Hence, the states of the queue are the words from A^*. Furthermore we have some basic controllable actions on these queues: writing of a symbol $a \in A$ (denoted by a) and reading of $a \in A$ (denoted by $\overline{a}$). Thereby we assume that the set $\overline{A}$ of all these reading operations $\overline{a}$ is a disjoint copy of A. So $\Sigma_{\mathcal{L}} := A \cup \overline{A}$ is the set of all operations on the partially lossy queue. For a word $u = a_1a_2 \dots a_n \in A^*$ we write $\overline{u}$ for the word $\overline{a_1}\,\overline{a_2} \dots \overline{a_n}$.

Formally, the action $a \in A$ appends the letter a to the state of the queue. The action $\overline{a} \in \overline{A}$ tries to cancel the letter a from the beginning of the current state of the queue. If this state does not start with a then the operation $\overline{a}$ is not defined. The lossiness of the queue is modeled by allowing it to forget arbitrary letters from $A \setminus X$ of its content at any moment. These ideas lead to the following definition.

Definition 3.2. Let $\mathcal{L} = (A, X)$ be a lossiness alphabet. The set of transitions $\vdash_{\mathcal{L}} \subseteq (A^* \times \Sigma_{\mathcal{L}}^*)^2$ of a partially lossy queue is given by the following rules for each $q, q' \in A^*$, $a \in A$, and $u \in \Sigma_{\mathcal{L}}^*$:

(i) $(q, au) \vdash_{\mathcal{L}} (qa, u)$
(ii) $(aq, \overline{a}u) \vdash_{\mathcal{L}} (q, u)$
(iii) if $q' \preceq_X q$ then $(q, u) \vdash_{\mathcal{L}} (q', u)$

Furthermore we define

$$\Delta_{\mathcal{L}} \colon A^* \times \Sigma_{\mathcal{L}}^* \to 2^{A^*} \colon (q, u) \mapsto \{q' \in A^* \mid (q, u) \vdash_{\mathcal{L}}^* (q', \varepsilon)\}.$$

Intuitively the set $\Delta_{\mathcal{L}}(q, u)$ is the set of all possible states that can be reached from state q by the execution of the actions from the sequence u. Using rule (iii), one obtains that this set is downward closed under $\preceq_X$.

Note that consecutive applications of the rule (iii) can be joint into a single application of this rule. In particular $(q,u) \vdash^*_{\mathcal{L}} (q',u)$ iff $(q,u) \vdash_{\mathcal{L}} (q',u)$. The application of rule (iii) followed by rule (i) can be reordered, *i.e.*,

$$(q,au) \vdash_{\mathcal{L}} (q',au) \vdash_{\mathcal{L}} (q'a,u) \Rightarrow (q,au) \vdash_{\mathcal{L}} (qa,u) \vdash_{\mathcal{L}} (q'a,u)\,,$$

but not *vice versa* since, with $a \notin X$ and $q \notin A^*a$, we have $(q,au) \vdash_{\mathcal{L}} (qa,u) \vdash_{\mathcal{L}} (q,u)$, but not $(q,au) \vdash_{\mathcal{L}} (q',au) \vdash_{\mathcal{L}} (q,u)$ for any $q' \in A^*$. Symmetrically, the application of rule (ii) followed by rule (iii) can be reordered:

$$(aq,\overline{a}u) \vdash_{\mathcal{L}} (q,u) \vdash_{\mathcal{L}} (r,u) \Rightarrow (aq,\overline{a}u) \vdash_{\mathcal{L}} (ar,\overline{a}u) \vdash_{\mathcal{L}} (r,u)\,,$$

but not *vice versa* since, with $a,b \in A$, $a \neq b$, and $q \in A^*$, we have $(baq,\overline{a}u) \vdash_{\mathcal{L}} (aq,\overline{a}u) \vdash_{\mathcal{L}} (q,u)$, but not $(baq,\overline{a}u) \vdash_{\mathcal{L}} (q',u) \vdash_{\mathcal{L}} (q,u)$ for any $q' \in A^*$. In summary, any sequence of applications of the rules (i)?(iii) can be reordered and grouped such that sequences of applications of rules (i) alternate with sequences of applications of rules (ii), always interspersed with a single application of a rule (iii).

This semantics is similar to the "standard semantics" from Appendix A in [7] where a lossy queue can lose any message at any time. The main part of that paper considers the "write-lossy semantics" where lossiness is modeled by the effect-less writing of messages into the queue. The authors show that these two semantics are equivalent ([7], Appendix A) and similar remarks can be made about priority queues [14]. A third possible semantics could be termed "read-lossy semantics" where lossiness is modeled by the loss of any messages that reside in the queue before the one that shall be read. In that case, the queue forgets letters only when necessary and this necessity occurs when one wants to read a letter that is, in the queue, preceded by some forgettable letters. Thereby, if the letter cannot be read since it does not occur in the queue or it is preceded by a non-forgettable letter, the queue will end up in an error state which we denote by $\perp$.

Next, we will define this semantics and show afterwards that both are equivalent in a precise sense.

Definition 3.3. Let $\mathcal{L} = (A,X)$ be a lossiness alphabet and $\perp \notin A$. Then the map $\circ_{\mathcal{L}} \colon (A^* \cup \{\perp\}) \times \Sigma^*_{\mathcal{L}} \to (A^* \cup \{\perp\})$ is defined for each $q \in A^*$, $a,b \in A$, and $u \in \Sigma^*_{\mathcal{L}}$ as follows:

(i) $q \circ_{\mathcal{L}} \varepsilon = q$

(ii) $q \circ_{\mathcal{L}} au = qa \circ_{\mathcal{L}} u$

(iii) $bq \circ_{\mathcal{L}} \overline{a}u = \begin{cases} q \circ_{\mathcal{L}} v & \text{if } a = b \\ q \circ_{\mathcal{L}} \overline{a}u & \text{if } b \in A \setminus (X \cup \{a\}) \\ \perp & \text{otherwise} \end{cases}$

(iv) $\varepsilon \circ_{\mathcal{L}} \overline{a}u = \perp \circ_{\mathcal{L}} u = \perp$

We will say "$q \circ_{\mathcal{L}} u$ is undefined" when $q \circ_{\mathcal{L}} u = \perp$.

Consider the definition of $q \circ_{\mathcal{L}} \overline{a}$. There, if the queue does not end up in the error state $\perp$, the word $a(q \circ_{\mathcal{L}} \overline{a})$ is the smallest suffix of q that contains all the occurrences of the letter a and its complementary prefix consists of forgettable entries, only. Hence, to apply $\overline{a}$, the queue first "forgets" the prefix and then "delivers" the letter a that is now at the first position.

Our first lemma proves that the function $\circ_{\mathcal{L}}$ is monotone in the first argument. Formally, this is true if the non-word $\perp$ is considered as the minimal element of the image $A^* \cup \{\perp\}$ of the function $\circ_{\mathcal{L}}$.

Lemma 3.4. *Let $\mathcal{L} = (A,X)$ be a lossiness alphabet, $q, q^+ \in A^*$, and $u \in \Sigma^*_{\mathcal{L}}$. If $q \preceq_X q^+$ and $q \circ_{\mathcal{L}} u \neq \perp$, then $q \circ_{\mathcal{L}} u \preceq_X q^+ \circ_{\mathcal{L}} u$ (in particular, $q^+ \circ_{\mathcal{L}} u \neq \perp$).*

Proof. The proof proceeds by induction on the length of the word u. If $u = \varepsilon$, then $q \circ_{\mathcal{L}} u = q \preceq_X q^+ = q^+ \circ_{\mathcal{L}} u$.

Next let $u = a \in A$. Then $q \circ_{\mathcal{L}} u = qa \preceq_X q^+a = q^+ \circ_{\mathcal{L}} a$.

Next let $u = \overline{a} \in \overline{A}$. Since $q \circ_{\mathcal{L}} u \neq \perp$, there is $p \in (A \setminus (X \cup \{a\}))^*$ with $q = pa(q \circ_{\mathcal{L}} u)$. Now $pa(q \circ_{\mathcal{L}} u) = q \preceq_X q^+$ implies the existence of words $p^+ \in (A \setminus \{a\})^*$ and $r \in A^*$ such that $q^+ = p^+ar$, $p \preceq_X p^+$, and $q \circ_{\mathcal{L}} u \preceq_X r$.

From $p \preceq_X p^+$, we obtain $\varepsilon = \pi_X(p) = \pi_X(p^+)$ and therefore $p^+ \in (A \setminus (X \cup \{a\}))^*$. Consequently, $r = q^+ \circ_{\mathcal{L}} u$ implying $q \circ_{\mathcal{L}} u \preceq_X q^+ \circ_{\mathcal{L}} u$.

Finally, let $|u| \geq 2$. Then there exist words $u_1, u_2 \in \Sigma_{\mathcal{L}}^*$ with $u = u_1 u_2$ and $|u_1|, |u_2| < |u|$. Since $|u_1| < |u|$, the induction hypothesis implies $q \circ_{\mathcal{L}} u_1 \preceq_X q^+ \circ_{\mathcal{L}} u_1$. Then we obtain

$$\begin{aligned} q \circ_{\mathcal{L}} u &= (q \circ_{\mathcal{L}} u_1) \circ_{\mathcal{L}} u_2 \\ &\preceq_X (q^+ \circ_{\mathcal{L}} u_1) \circ_{\mathcal{L}} u_2 && \text{(by the ind. hyp. since } q \circ_{\mathcal{L}} u_1 \preceq_X q^+ \circ_{\mathcal{L}} u_1 \text{ and } |u_2| < |u|) \\ &= q^+ \circ_{\mathcal{L}} u\,. \end{aligned}$$
□

Let $q \in A^*$ and $u \in \Sigma_{\mathcal{L}}^*$ such that $q \circ_{\mathcal{L}} u \neq \bot$. Examining the above definition, one easily sees $(q, u) \vdash_{\mathcal{L}}^* (q \circ_{\mathcal{L}} u, \varepsilon)$, *i.e.*, $q \circ_{\mathcal{L}} u \in \Delta_{\mathcal{L}}(q, u)$. We next show that $\Delta_{\mathcal{L}}(q, u)$ equals the set of X-subwords of $q \circ_{\mathcal{L}} u$.

In other words, $q \circ_{\mathcal{L}} u$ describes the set $\Delta_{\mathcal{L}}(q, u)$ completely which proves the equivalence of the "standard semantics" and "read-lossy semantics" as described above.

Theorem 3.5. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet, $u, v \in \Sigma_{\mathcal{L}}^*$, and $q \in A^*$. Then*

$$\Delta_{\mathcal{L}}(q, u) = \Delta_{\mathcal{L}}(q, v) \iff q \circ_{\mathcal{L}} u = q \circ_{\mathcal{L}} v\,.$$

Proof. We prove that

$$r \in \Delta_{\mathcal{L}}(q, u) \iff r \preceq_X q \circ_{\mathcal{L}} u$$

holds for all $q, r \in A^*$ and $u \in \Sigma_{\mathcal{L}}^*$.

We start with the proof of the implication "⇒" which proceeds by induction on the length of the word u. The case $u = \varepsilon$ is obvious since then, $\Delta_{\mathcal{L}}(q, u) = \{r \in A^* \mid r \preceq_X q\}$ and $q \circ_{\mathcal{L}} u = q$.

So let $|u| \geq 1$, *i.e.*, there are $\alpha \in \Sigma_{\mathcal{L}}$ and $u' \in \Sigma_{\mathcal{L}}^*$ such that $u = \alpha u'$. Let $r \in \Delta_{\mathcal{L}}(q, u)$. Then there exist $r', r'' \in A^*$ with

$$(q, \alpha u) \vdash_{\mathcal{L}}^* (r', \alpha u) \vdash_{\mathcal{L}} (r'', u) \vdash_{\mathcal{L}}^* (r, \varepsilon)\,.$$

From the first part of this sequence, we obtain $r' \preceq_X q$, the second implies $r'' = r' \circ_{\mathcal{L}} \alpha$, the third one (by the induction hypothesis) $r \preceq_X r'' \circ_{\mathcal{L}} u$. Using Lemma 3.4, this implies

$$r \preceq_X r'' \circ_{\mathcal{L}} u = (r' \circ_{\mathcal{L}} \alpha) \circ_{\mathcal{L}} u = r' \circ_{\mathcal{L}} \alpha u \preceq_X q \circ_{\mathcal{L}} \alpha u\,.$$

This finishes the inductive proof of the implication "⇒".

For the converse implication, note that $r \preceq_X q \circ_{\mathcal{L}} u$ implies $q \circ_{\mathcal{L}} u \neq \bot$. But then $(q, u) \vdash_{\mathcal{L}}^* (q \circ_{\mathcal{L}} u, \varepsilon) \vdash_{\mathcal{L}}^* (r, \varepsilon)$ implies $r \in \Delta_{\mathcal{L}}(q, u)$. □

Given this equivalence of the two semantics considered, it makes sense to not distinguish sequences of actions that behave the same on each and every queue. This identification leads to the central definition of this paper:

Definition 3.6. Let $\mathcal{L} = (A, X)$ be a lossiness alphabet and $u, v \in \Sigma_{\mathcal{L}}^*$. Then u and v act *equally* (denoted by $u \equiv_{\mathcal{L}} v$) if $q \circ_{\mathcal{L}} u = q \circ_{\mathcal{L}} v$ holds for each $q \in A^*$.

Since $q \circ_{\mathcal{L}} uv = (q \circ_{\mathcal{L}} u) \circ_{\mathcal{L}} v$, the resulting relation $\equiv_{\mathcal{L}}$ is a congruence on the free monoid $\Sigma_{\mathcal{L}}^*$. Hence, the quotient $\mathcal{Q}(\mathcal{L}) := \Sigma_{\mathcal{L}}^*/_{\equiv_{\mathcal{L}}}$ is a monoid which we call *partially lossy queue monoid* or *plq monoid* induced by $\mathcal{L}$.

Example 3.7. Let $A = \{a, b\}$. Then we have

$$\varepsilon \circ_{(A,\emptyset)} ba\overline{a} = ba \circ_{(A,\emptyset)} \overline{a} = \varepsilon \text{ and } \varepsilon \circ_{(A,\emptyset)} b\overline{a}a = \bot$$

implying $ba\overline{a} \not\equiv_{(A,\emptyset)} b\overline{a}a$.

On the other hand,

$$\varepsilon \circ_{(A,A)} ba\overline{a} = ba \circ_{(A,A)} \overline{a} = \bot = \varepsilon \circ_{(A,A)} b\overline{a}a\,.$$

It can be verified that, even more, $q \circ_{(A,A)} ba\overline{a} = q \circ_{(A,A)} b\overline{a}a$ holds for all $q \in A^*$ implying $ba\overline{a} \equiv_{(A,A)} b\overline{a}a$.

Remark 3.8. Let $A = \{a\}$ be a singleton and $X \subseteq A$. Then (A, X) is not a lossiness alphabet (since we required $|A| \geq 2$). Nevertheless, the above results hold also in this case. Note that $a^{n+1} \circ_{\mathcal{L}} \overline{a} = a^n$ for any $n \geq 0$ (independent of whether $X = A$ or $X = \emptyset$). Hence $\mathcal{Q}(A, A) = \mathcal{Q}(A, \emptyset)$ is the bicyclic semigroup. Therefore, we excluded this case in the definition of lossiness alphabets.

On the first sight, the equality of $\mathcal{Q}(\{a\}, \{a\})$ and $\mathcal{Q}(\{a\}, \emptyset)$ seems to be counterintuitive. But it comes along with the following simple observation: Let $\mathcal{A}$ be a partially blind one-counter automaton (*i.e.*, $\mathcal{A}$ is a PDA with a unary pushdown alphabet). Then $\mathcal{A}$ can be understood as a finite automaton with a queue over $(\{a\}, \{a\})$. Let $\mathcal{B}$ be an extension of $\mathcal{A}$ by some ε-transitions that are decreasing the counter. Alternatively, we can understand $\mathcal{B}$ as the automaton $\mathcal{A}$ where the lossiness alphabet is replaced by $(\{a\}, \emptyset)$. Then both, $\mathcal{A}$ and $\mathcal{B}$, accept the same language.

3.1. Basic properties

Next we want to give basic properties of the equivalence $\equiv_{\mathcal{L}}$. The following lemma lists some equations that hold in the plq monoid (later, we will show that these equations characterize the plq monoid completely, *cf.* Thm. 3.15).

Lemma 3.9. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet, $a, b \in A$, $x \in X$, and $w \in A^*$. Then the following hold:*

(i) $b\overline{a} \equiv_{\mathcal{L}} \overline{a}b$ *if* $a \neq b$ *(iii)* $xwa\overline{a} \equiv_{\mathcal{L}} xw\overline{a}a$
(ii) $a\overline{ab} \equiv_{\mathcal{L}} \overline{a}a\overline{b}$ *(iv)* $awa\overline{a} \equiv_{\mathcal{L}} aw\overline{a}a$

At first we take a look at equations (i)–(iii) (with $|w|_a = 0$ for simplicity). In order for a queue $q \in A^*$ to be defined after execution of the actions, the letter a must already be contained in q preceded by forgettable letters only. Since, in all cases, $\overline{a}$ is the first read operation, $\overline{a}$ reads this occurrence of a from q. Hence it does not matter whether we write b (a, resp.) before or after this reading of a. In equation (iv) we are in the same situation after execution of the leading write operation a. Therefore we can commute the read and write operations in all these situations.

Consider the case $X = A$, *i.e.*, a reliable queue. For this situation, Lemma 3.5 from [13] proves (i) and (ii). Statement (iii) is only shown for the special case $w = \varepsilon$. But, provided $X = A$, the general case follows from this special one. Thus, the above lemma generalizes Lemma 3.5 from [13].

Proof. At first, we show (i). So, let $a, b \in A$ be distinct letters and $q \in A^*$. Consider $q \circ_{\mathcal{L}} \overline{a}b \neq \bot$. Then there are words $p \in (A \setminus (X \cup \{a\}))^*$ and $r \in A^*$ with $q = par$. Hence we obtain

$$\begin{aligned} q \circ_{\mathcal{L}} \overline{a}b &= par \circ_{\mathcal{L}} \overline{a}b = r \circ_{\mathcal{L}} b = rb \\ &= parb \circ_{\mathcal{L}} \overline{a} = qb \circ_{\mathcal{L}} \overline{a} = q \circ_{\mathcal{L}} b\overline{a}\,. \end{aligned}$$

Conversely, we consider now $q \circ_{\mathcal{L}} b\overline{a} \neq \bot$. Then there is a prefix pa of qb with $p \in (A \setminus (X \cup \{a\}))^*$. Since $a \neq b$ this prefix is proper. Hence, there is $r \in A^*$ with $qb = parb$. Then, similarly we obtain $q \circ_{\mathcal{L}} b\overline{a} = q \circ_{\mathcal{L}} \overline{a}b$. So, we proved $q \circ_{\mathcal{L}} b\overline{a} = q \circ_{\mathcal{L}} \overline{a}b$ for any $q \in A^*$ and distinct $a, b \in A$ implying (i).

Now we prove (ii). Let $a, b \in A$ (not necessarily distinct) and $q \in A^*$. First we consider $q \circ_{\mathcal{L}} \overline{a} a \overline{b} \neq \bot$. Then there are words $p \in (A \setminus (X \cup \{a\}))^*$ and $r \in A^*$ such that $q = par$. Hence we obtain

$$\begin{aligned} q \circ_{\mathcal{L}} \overline{a} a \overline{b} &= par \circ_{\mathcal{L}} \overline{a} a \overline{b} = r \circ_{\mathcal{L}} a \overline{b} = ra \circ_{\mathcal{L}} \overline{b} \\ &= para \circ_{\mathcal{L}} \overline{ab} = par \circ_{\mathcal{L}} a \overline{ab} = q \circ_{\mathcal{L}} a \overline{ab} \,. \end{aligned}$$

Now, consider $q \circ_{\mathcal{L}} a \overline{ab} \neq \bot$. Again, there are words $p \in (A \setminus (X \cup \{a\}))^*$ and $r \in A^*$ with $qa = par$. From

$$\bot \neq q \circ_{\mathcal{L}} a \overline{ab} = qa \circ_{\mathcal{L}} \overline{ab} = par \circ_{\mathcal{L}} \overline{ab} = r \circ_{\mathcal{L}} \overline{b}$$

we obtain that $r \neq \varepsilon$, *i.e.*, there is $r' \in A^*$ such that $r = r'a$ and, hence, $q = par'$. Then we can infer

$$\begin{aligned} q \circ_{\mathcal{L}} a \overline{ab} &= r \circ_{\mathcal{L}} \overline{b} = r'a \circ_{\mathcal{L}} \overline{b} \\ &= r' \circ_{\mathcal{L}} a \overline{b} = par' \circ_{\mathcal{L}} \overline{a} a \overline{b} = q \circ_{\mathcal{L}} \overline{a} a \overline{b} \,. \end{aligned}$$

Hence we demonstrated (ii).

Finally, we show (iii) and (iv). Let $x \in X \cup \{a\}$ and $q \in A^*$. First consider the case $q \circ_{\mathcal{L}} xwa\overline{a} \neq \bot$. Then there is a prefix pa of $qxwa$ with $p \in (A \setminus (X \cup \{a\}))^*$. Since $x \in X \cup \{a\}$, the word pa is a prefix of qx. Hence there exists $r \in A^*$ with $qx = par$. We obtain

$$\begin{aligned} q \circ_{\mathcal{L}} xwa\overline{a} &= qxwa \circ_{\mathcal{L}} \overline{a} = parwa \circ_{\mathcal{L}} \overline{a} \\ &= rwa = rw \circ_{\mathcal{L}} a \\ &= (parw \circ_{\mathcal{L}} \overline{a}) \circ_{\mathcal{L}} a \\ &= qxw \circ_{\mathcal{L}} \overline{a} a = q \circ_{\mathcal{L}} xw\overline{a} a \,. \end{aligned}$$

Now, consider the case $q \circ_{\mathcal{L}} xw\overline{a} a \neq \bot$. Then also $q \circ_{\mathcal{L}} xw\overline{a} = qxw \circ_{\mathcal{L}} \overline{a} \neq \bot$. Hence there exist $p \in (A \setminus (X \cup \{a\}))^*$ and $r \in A^*$ with $qxw = par$. We obtain

$$\begin{aligned} q \circ_{\mathcal{L}} xw\overline{a} a &= qxw \circ_{\mathcal{L}} \overline{a} a = par \circ_{\mathcal{L}} \overline{a} a = r \circ_{\mathcal{L}} a = ra \\ &= para \circ_{\mathcal{L}} \overline{a} = qxwa \circ_{\mathcal{L}} \overline{a} = q \circ_{\mathcal{L}} xwa\overline{a} \,. \end{aligned}$$

Since this holds for arbitrary $a \in A$, $x \in X \cup \{a\}$, and $w \in A^*$, we demonstrated (iii) and (iv). ☐

Note that the equations from Lemma 3.9 preserve the relative order of write resp. read operations. We will next show that this holds for every equation $u \equiv_{\mathcal{L}} v$. To do this we need the definition of the following projections:

Definition 3.10. Let $\mathcal{L} = (A, X)$ be a lossiness alphabet. The projections $\mathsf{wrt}, \mathsf{rd} \colon \Sigma_{\mathcal{L}}^* \to A^*$ on write and read operations are defined for any $u \in \Sigma_{\mathcal{L}}^*$ by $\mathsf{wrt}(u) = \pi_A(u)$ and $\overline{\mathsf{rd}(u)} = \pi_{\overline{A}}(u)$.

In a nutshell, the projection wrt deletes all letters from $\overline{A}$ from a word. Dually, the projection rd deletes all letters from A from a word *and then suppresses the overlines*. For instance $\mathsf{wrt}(a\overline{a}b) = ab$ and $\mathsf{rd}(a\overline{a}b) = a$.

Before we prove the preservation of projections in equivalence classes we need another simple lemma that allows to separate the read and write operations provided $q \circ_{\mathcal{L}} u$ is defined.

Lemma 3.11. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet, $q \in A^*$, and $u \in \Sigma_{\mathcal{L}}^*$ such that $q \circ_{\mathcal{L}} u \neq \bot$. Then*

$$q \circ_{\mathcal{L}} u = q \circ_{\mathcal{L}} \mathsf{wrt}(u) \, \overline{\mathsf{rd}(u)} \,.$$

Proof. Let $a, b \in A$ such that $q \circ_{\mathcal{L}} \overline{a}b \neq \bot$. Then there are $p \in (A \setminus (X \cup \{a\}))^*$ and $r \in A^*$ such that $q = par$. Hence we have

$$q \circ_{\mathcal{L}} \overline{a}b = r \circ_{\mathcal{L}} b = rb = parb \circ_{\mathcal{L}} \overline{a} = q \circ_{\mathcal{L}} b\overline{a}\,,$$

i.e., we can commute $\overline{a}$ and b in this case. By induction on the (minimal) number of transpositions needed to transform u into $\mathsf{wrt}(u)\,\overline{\mathsf{rd}(u)}$ we get our lemma. □

Now we can prove the actual statement.

Proposition 3.12. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet and $u, v \in \Sigma_{\mathcal{L}}^*$ with $u \equiv_{\mathcal{L}} v$. Then we have $\mathsf{rd}(u) = \mathsf{rd}(v)$ and $\mathsf{wrt}(u) = \mathsf{wrt}(v)$.*

Proof. We first prove $\mathsf{rd}(u) = \mathsf{rd}(v)$. By symmetry, we can assume $|\mathsf{rd}(u)| \leq |\mathsf{rd}(v)|$. If $\mathsf{rd}(v) = \varepsilon$, then this implies immediately $\mathsf{rd}(u) = \mathsf{rd}(v)$. Hence it remains to consider the case $\mathsf{rd}(v) \neq \varepsilon$. Then there is a letter $a \in A$ with $\mathsf{rd}(v) \in A^*\,a$. Since $|A| \geq 2$, there is $b \in A$ with $a \neq b$. Then we have

$$\begin{aligned}
\bot \neq\ & b^{|v|}\,\mathsf{wrt}(u) \\
&= \mathsf{rd}(u)\,b^{|v|} \circ_{\mathcal{L}} u \\
&= \mathsf{rd}(u)\,b^{|v|} \circ_{\mathcal{L}} v && \text{(since } u \equiv_{\mathcal{L}} v\text{)} \\
&= \mathsf{rd}(u)\,b^{|v|} \circ_{\mathcal{L}} \mathsf{wrt}(v)\,\overline{\mathsf{rd}(v)} && \text{(by Lem. 3.11)} \\
&= \mathsf{rd}(u)\,b^{|v|}\,\mathsf{wrt}(v) \circ_{\mathcal{L}} \overline{\mathsf{rd}(v)} =: q\,.
\end{aligned}$$

Since $|\mathsf{rd}(v)| \geq |\mathsf{rd}(u)|$, the word q is a suffix of $b^{|v|}\,\mathsf{wrt}(v)$. Suppose it is a proper suffix. When reading $\mathsf{rd}(v)$ from the queue $\mathsf{rd}(u)\,b^{|v|}\,\mathsf{wrt}(v)$, the last letter read is $a \neq b$. It follows that the result q is a proper suffix of $\mathsf{wrt}(v)$. But then $|q| < |\mathsf{wrt}(v)| \leq |v| \leq |b^{|v|}\,\mathsf{wrt}(u)|$, which contradicts the above calculation leading to $b^{|v|}\,\mathsf{wrt}(u) = q$. Hence q is not a proper suffix, *i.e.*, $q = b^{|v|}\,\mathsf{wrt}(v)$. But then $\mathsf{rd}(u)\,b^{|v|}\,\mathsf{wrt}(v) \circ_{\mathcal{L}} \overline{\mathsf{rd}(v)} = q = b^{|v|}\,\mathsf{wrt}(v)$ implies $\mathsf{rd}(u) \circ_{\mathcal{L}} \overline{\mathsf{rd}(v)} = \varepsilon \neq \bot$. Since $|\mathsf{rd}(v)| \geq |\mathsf{rd}(u)|$, this is only possible with $\mathsf{rd}(u) = \mathsf{rd}(v)$.

Now the second claim follows easily:

$$\begin{aligned}
\mathsf{wrt}(u) &= \mathsf{rd}(u) \circ_{\mathcal{L}} u \\
&= \mathsf{rd}(v) \circ_{\mathcal{L}} v && \text{(since } \mathsf{rd}(u) = \mathsf{rd}(v) \text{ and } u \equiv_{\mathcal{L}} v\text{)} \\
&= \mathsf{wrt}(v)
\end{aligned}$$

□

By Example 3.7, the converse implication of Proposition 3.12 does not hold in general. But from the statements in the following subsection we can obtain a third property which ensures the reversal in combination with these two properties.

3.2. A semi-Thue system for $\mathcal{Q}(\mathcal{L})$

In this subsection, we prove that $\equiv_{\mathcal{L}}$ is the least congruence on the free monoid $\Sigma_{\mathcal{L}}^*$ that satisfies the equations from Lemma 3.9 (*cf.* Thm. 3.15 below). This is achieved by constructing a terminating and confluent semi-Thue system from those equations and by showing that every equivalence class of $\equiv_{\mathcal{L}}$ contains a unique word in normal form. Later we will often use this normal form instead of the corresponding equivalence class wrt. $\equiv_{\mathcal{L}}$.

Ordering the equations from Lemma 3.9, the semi-Thue system $\mathcal{R}_{\mathcal{L}}$ consists of the following rules for $a, b \in A$, $x \in X$, and $w \in A^*$:

(a) $b\overline{a} \to \overline{a}b$ if $a \neq b$

(b) $a\overline{ab} \to \overline{a}a\overline{b}$

(c) $xwa\overline{a} \to xw\overline{a}a$

(d) $awa\overline{a} \to aw\overline{a}a$

Hence the idea of this semi-Thue system is to pull read operations to the left as long as the equations from Lemma 3.9 permit.

A word is irreducible wrt. this semi-Thue system if, and only if, it belongs to the set

$$\mathsf{NF}_{\mathcal{L}} = \overline{A}^* \left(\bigcup_{a \in A} (A \setminus (X \cup \{a\}))^* a\overline{a} \right)^* A^*$$

since these are precisely those words that do not contain any rule's left-hand side as a factor.

Lemma 3.13. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet. The semi-Thue system $\mathcal{R}_{\mathcal{L}}$ is terminating and confluent.*

Proof. To prove termination we order the alphabet $\Sigma_{\mathcal{L}}$ such that $\overline{a} < b$ for each $a, b \in A$. Then we see that for any rule $l \to r$ from $\mathcal{R}_{\mathcal{L}}$ the word l is length-lexicographically properly smaller than r (*i.e.*, $|l| < |r|$ or $|l| = |r|$ and there are $p \in \Sigma_{\mathcal{L}}^*$ and $\alpha, \beta \in \Sigma_{\mathcal{L}}$ with $l \in p\alpha\Sigma_{\mathcal{L}}^*$, $r \in p\beta\Sigma_{\mathcal{L}}^*$, and $\alpha < \beta$). Since this ordering is well-founded the semi-Thue system is terminating.

Due to termination of $\mathcal{R}_{\mathcal{L}}$ it suffices to show that it is locally confluent. The only overlaps of left-hand sides are as follows (with $a, b \in A$ and $u, v, w \in A^*$):

- $xw(\overline{a}a\overline{b}) \leftarrow xw(a\overline{a}\overline{b}) = (xwa\overline{a})\overline{b} \to (xw\overline{a}a)\overline{b}$ for $x \in \{a\} \cup X$ with rules of type (b) and (c) or (d), resp.
- $(xuyv\overline{a}a) \leftarrow (xuyva\overline{a}) = xu(yva\overline{a}) \to xu(yv\overline{a}a)$ for $x, y \in \{a\} \cup X$ with rules of type (c) or (d), resp.

Hence $\mathcal{R}_{\mathcal{L}}$ is confluent. □

Let $u \in \Sigma_{\mathcal{L}}^*$. Since the semi-Thue system is terminating and confluent, there is a unique irreducible word $\mathsf{nf}_{\mathcal{L}}(u) \in \mathsf{NF}_{\mathcal{L}}$ with $u \to^* \mathsf{nf}_{\mathcal{L}}(u)$, the *normal form* of u. Because of the shape of the irreducible word $\mathsf{nf}_{\mathcal{L}}(u) \in \mathsf{NF}_{\mathcal{L}}$, there are $n \in \mathbb{N}$, letters $a_i \in A$, and words $x, z \in A^*$ and $y_i \in (A \setminus (X \cup \{a_i\}))^*$ (for $1 \le i \le n$) such that

$$\mathsf{nf}_{\mathcal{L}}(u) = \overline{x}\,(y_1 a_1 \overline{a_1})\,(y_2 a_2 \overline{a_2})\,\ldots\,(y_n a_n \overline{a_n})\,z$$

(note that n, x, y_i, z, and a_i are unique). We define

$$\mathsf{rd}_1(u) = x \text{ and } \mathsf{rd}_2(u) = a_1 a_2 \ldots a_n$$

such that $\mathsf{rd}(\mathsf{nf}_{\mathcal{L}}(u)) = \mathsf{rd}_1(u)\,\mathsf{rd}_2(u)$. Since, by the form of the rules, $\mathsf{rd}(u) = \mathsf{rd}(\mathsf{nf}_{\mathcal{L}}(u))$, this implies $\mathsf{rd}(u) = \mathsf{rd}_1(u)\,\mathsf{rd}_2(u)$. Finally note that $\mathsf{nf}_{\mathcal{L}}(u)$ is completely determined by the triple

$$\chi(u) := (\mathsf{wrt}(u), \mathsf{rd}(u), \mathsf{rd}_2(u))$$

that we call the *characteristic of* u.

Remark 3.14. While $\mathsf{rd}_1(u)$ is defined using the semi-Thue system $\mathcal{R}_{\mathcal{L}}$, it also has a natural meaning in terms of $\circ_{\mathcal{L}}$: from the shape of $\mathsf{nf}_{\mathcal{L}}(u)$ we can infer that $\mathsf{rd}_1(u)$ is the shortest word $q \in A^*$ such that $q \circ_{\mathcal{L}} \mathsf{nf}_{\mathcal{L}}(u)$ is defined. By Lemma 3.9 we have $u \equiv_{\mathcal{L}} \mathsf{nf}_{\mathcal{L}}(u)$. Hence, $\mathsf{rd}_1(u)$ is also the shortest word $q \in A^*$ such that $q \circ_{\mathcal{L}} u$ is defined.

Now, as promised before we show the relation between $\equiv_{\mathcal{L}}$ and $\mathcal{R}_{\mathcal{L}}$.

Theorem 3.15. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet and $u, v \in \Sigma_{\mathcal{L}}^*$. Then we have*

$$u \equiv_{\mathcal{L}} v \iff \mathsf{nf}_{\mathcal{L}}(u) = \mathsf{nf}_{\mathcal{L}}(v)\,.$$

Proof. First let $u \equiv_{\mathcal{L}} v$. By Proposition 3.12 we have $\mathsf{wrt}(u) = \mathsf{wrt}(v)$ and $\mathsf{rd}(u) = \mathsf{rd}(v)$. Since the normal forms of u and v are completely given by their characteristics and since $\mathsf{rd}_2(u)$ is determined by $\mathsf{rd}(u)$ and $\mathsf{rd}_1(u)$, it remains to prove $\mathsf{rd}_1(u) = \mathsf{rd}_1(v)$. For the following calculation let

$$\mathsf{nf}_{\mathcal{L}}(u) = \overline{\mathsf{rd}_1(u)}\,(y_1 a_1 \overline{a_1})\;\ldots\;(y_n a_n \overline{a_n})\,z\,.$$

Then we get

$$\begin{aligned}
\perp \neq z &= \varepsilon \circ_{\mathcal{L}} z \\
&= y_n a_n \circ_{\mathcal{L}} \overline{a_n}\, z && \text{(since } y_n \in (A \setminus (X \cup \{a_n\}))^*\text{)} \\
&= \varepsilon \circ_{\mathcal{L}} y_n a_n \overline{a_n}\, z \\
&\;\;\vdots \\
&= \varepsilon \circ_{\mathcal{L}} (y_1 a_1 \overline{a_1})\;\ldots\;(y_n a_n \overline{a_n})\,z \\
&= \mathsf{rd}_1(u) \circ_{\mathcal{L}} \overline{\mathsf{rd}_1(u)}\,(y_1 a_1 \overline{a_1})\;\ldots\;(y_n a_n \overline{a_n})\,z \\
&= \mathsf{rd}_1(u) \circ_{\mathcal{L}} \mathsf{nf}_{\mathcal{L}}(u) \\
&= \mathsf{rd}_1(u) \circ_{\mathcal{L}} u && \text{(since } u \equiv_{\mathcal{L}} \mathsf{nf}_{\mathcal{L}}(u) \text{ by Lem. 3.9)} \\
&= \mathsf{rd}_1(u) \circ_{\mathcal{L}} v && \text{(since } u \equiv_{\mathcal{L}} v\text{)} \\
&= \mathsf{rd}_1(u) \circ_{\mathcal{L}} \mathsf{nf}_{\mathcal{L}}(v) && \text{(since } v \equiv_{\mathcal{L}} \mathsf{nf}_{\mathcal{L}}(v) \text{ by Lem. 3.9)}
\end{aligned}$$

Since $\overline{\mathsf{rd}_1(v)}$ is a prefix of $\mathsf{nf}_{\mathcal{L}}(v)$, this implies

$$\perp \neq \mathsf{rd}_1(u) \circ_{\mathcal{L}} \overline{\mathsf{rd}_1(v)}\,.$$

Since $\overline{\mathsf{rd}_1(v)}$ consists of read-actions, only, this implies $|\mathsf{rd}_1(u)| \geq |\mathsf{rd}_1(v)|$. Now, by symmetry, these two words have the same length. But then $\perp \neq \mathsf{rd}_1(u) \circ_{\mathcal{L}} \overline{\mathsf{rd}_1(v)}$ implies $\mathsf{rd}_1(u) = \mathsf{rd}_1(v)$ and therefore $\mathsf{nf}_{\mathcal{L}}(u) = \mathsf{nf}_{\mathcal{L}}(v)$.

The converse implication follows easily by two applications of Lemma 3.9:

$$u \equiv_{\mathcal{L}} \mathsf{nf}_{\mathcal{L}}(u) = \mathsf{nf}_{\mathcal{L}}(v) \equiv_{\mathcal{L}} v\,.$$ □

Remark 3.16. As a consequence, all words from $[u]$ share the same characteristic. Hence, this theorem allows us to speak of the characteristic of the equivalence class $[u]$. With this characteristic in mind we can also apply the functions wrt, rd, rd_1, and rd_2 to equivalence classes instead of words.

Since we can compute the normal form of a word u (we can restrict $\mathcal{R}_{\mathcal{L}}$ to the rules of length at most $|u|$) we can infer the following statement:

Corollary 3.17. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet. Then the following* word problem *of $\mathcal{Q}(\mathcal{L})$ is decidable: Given $u, v \in \Sigma_{\mathcal{L}}^*$, does $u \equiv_{\mathcal{L}} v$ hold?*

In [19], the first author of this paper considers more questions in this direction. Namely, it is shown that the rational membership problem (*i.e.*, given $u \in \Sigma_{\mathcal{L}}^*$ and an NFA $\mathcal{A}$, is there $v \in L(\mathcal{A})$ with $u \equiv_{\mathcal{L}} v$?) is NL-complete. On the negative side, it is also shown that universality, inclusion, and emptiness of intersection for rational sets in $\mathcal{Q}(\mathcal{L})$ are undecidable.

Another consequence from Theorem 3.15 is the following: Recall that $a\overline{a}\overline{b} \equiv_{\mathcal{L}} \overline{a}a\overline{b}$ and $a\overline{a} \not\equiv_{\mathcal{L}} \overline{a}a$, *i.e.*, in general, we cannot cancel in the monoid $\mathcal{Q}(\mathcal{L})$. The above theorem allows to show that we can cancel read operations from the left and write operations from the right:

Corollary 3.18. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet, $u, v \in \Sigma_{\mathcal{L}}^*$, and $x, y \in A^*$. Then $\overline{x}uy \equiv_{\mathcal{L}} \overline{x}vy$ implies $u \equiv_{\mathcal{L}} v$.*

Proof. The rules of the semi-Thue system $\mathcal{R}_\mathcal{L}$ imply $\mathsf{nf}_\mathcal{L}(\overline{x}\,u\,y) = \overline{x}\,\mathsf{nf}_\mathcal{L}(u)\,y$ and $\mathsf{nf}_\mathcal{L}(\overline{x}\,v\,y) = \overline{x}\,\mathsf{nf}_\mathcal{L}(v)\,y$. Hence $\overline{x}\,u\,y \equiv_\mathcal{L} \overline{x}\,v\,y$ implies by Theorem 3.15

$$\overline{x}\,\mathsf{nf}_\mathcal{L}(u)\,y = \mathsf{nf}_\mathcal{L}(\overline{x}\,u\,y) = \mathsf{nf}_\mathcal{L}(\overline{x}\,v\,y) = \overline{x}\,\mathsf{nf}_\mathcal{L}(v)\,y$$

and therefore (by cancellation in the free monoid $\Sigma_\mathcal{L}^*$) $\mathsf{nf}_\mathcal{L}(u) = \mathsf{nf}_\mathcal{L}(v)$. Again by the above theorem, this implies $u \equiv_\mathcal{L} v$. □

For later use, we now describe the characteristic of the word $u\overline{v}$ for $u, v \in A^*$. We have $\mathsf{wrt}(u\overline{v}) = u$ and $\mathsf{rd}(u\overline{v}) = v$. It remains to describe $\mathsf{rd}_2(u\overline{v})$.

Lemma 3.19. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet. Then $\mathsf{rd}_2(u\overline{v})$ is the longest suffix v_2 of v that satisfies $\pi_X(u_1) \preceq v_2 \preceq u_1$ for some prefix u_1 of u.*

Proof. Since $\mathsf{rd}_1(u\overline{v})\mathsf{rd}_2(u\overline{v}) = \mathsf{rd}(u\overline{v}) = v$, the word $\mathsf{rd}_2(u\overline{v})$ is a suffix of v. Note that $\mathsf{nf}_\mathcal{L}(u\overline{v}) = \overline{\mathsf{rd}_1(u\overline{v})}(y_1a_1\overline{a_1})(y_2a_2\overline{a_2})\dots(y_ma_m\overline{a_m})z$ for letters $a_i \in A$ with $\mathsf{rd}_2(u\overline{v}) = a_1a_2\dots a_m$, words $y_i \in (A \setminus (X \cup \{a_i\}))^*$ and $z \in A^*$. Consequently $u_1 := y_1a_1y_2a_2\dots y_ma_m$ is a prefix of $\mathsf{wrt}(u\overline{v}) = u$ and $\pi_X(u_1) \preceq a_1a_2\dots a_m \preceq u_1$.

Now let $v = v_1v_2$ and $u = u_1u_2$ such that $\pi_X(u_1) \preceq v_2 \preceq u_1$. We can assume u_1 to be the minimal prefix of u satisfying $\pi_X(u_1) \preceq v_2 \preceq u_1$. Then there are letters $a_1, a_2, \dots, a_n \in A$ with $v_2 = a_1a_2\dots a_n$ and words $y_i \in (A \setminus (X \cup \{a_i\}))^*$ with $u_1 = y_1a_1\,y_2a_2\,\dots\,y_na_n$. Then we obtain

$$\begin{aligned}\bot \neq u_2 &= y_1a_1\,y_2a_2\,\dots\,y_na_n\,u_2 \circ_\mathcal{L} \overline{a_1a_2\dots a_n}\\ &= u_1u_2 \circ_\mathcal{L} \overline{v_2}\\ &= v_1u \circ_\mathcal{L} \overline{v}\\ &= v_1 \circ_\mathcal{L} u\overline{v} = v_1 \circ_\mathcal{L} \mathsf{nf}_\mathcal{L}(u\overline{v})\,.\end{aligned}$$

Since $\overline{\mathsf{rd}_1(u\overline{v})}$ is a prefix of $\mathsf{nf}_\mathcal{L}(u\overline{v})$, this implies $v_1 \circ_\mathcal{L} \overline{\mathsf{rd}_1(u\overline{v})} \neq \bot$ and therefore $|\mathsf{rd}_1(u\overline{v})| \leq |v_1|$. Now $\mathsf{rd}_1(u\overline{v})\mathsf{rd}_2(u\overline{v}) = v = v_1v_2$ implies $|v_2| \leq |\mathsf{rd}_2(u\overline{v})|$. □

4. Injectivity of homomorphisms into plq monoids

The main result of this section is Theorem 4.6 that provides a necessary condition on a homomorphism φ into $\mathcal{Q}(\mathcal{L})$ to be injective. This condition will prove immensely useful in our investigation of submonoids of $\mathcal{Q}(\mathcal{L})$ in the following two sections. It states that if the images of x and y under an embedding φ perform the same sequences of read and write operations, respectively, then x and y can be equated by putting them into a certain context.

A trivial example for an injective homomorphism φ into some $\mathcal{Q}(\mathcal{L})$ is the identity of $\mathcal{Q}(A, \emptyset)$. We first prove in Proposition 4.5 that this embedding satisfies the said condition and later we derive the general case. Therefore, *from now on until this proposition we consider the monoid $\mathcal{Q}(A, \emptyset)$, i.e., $X = \emptyset$*. In other words, we consider the so-called fully lossy queues where all letters are forgettable.

The following notion will be useful in the calculations performed in this monoid:

Definition 4.1. Let A be an alphabet and $u, v \in A^*$. The *subword-suffix* of u and v is the longest suffix $\mathsf{sws}(u, v)$ of v that is a subword of u.

Example 4.2. Since ab is a subword of $abba$ and cab is not, we have $\mathsf{sws}(abba, ab) = ab = \mathsf{sws}(abba, cab)$. In general, by our assumption of $X = \emptyset$, Lemma 3.19 implies $\mathsf{rd}_2(u\overline{v}) = \mathsf{sws}(u, v)$ for any words $u, v \in A^*$.

The first result of this section (Thm. 4.3) describes the normal form of the product of two elements from $\mathcal{Q}(A, \emptyset)$ in terms of their normal forms. Lemma 3.19 solves this problem in case the first factor belongs to $[A^*]$ and the second to $[\overline{A}^*]$ for arbitrary sets $X \subseteq A$.

Theorem 4.3. *Let $\mathcal{L} = (A, \emptyset)$ be a lossiness alphabet, $u, v \in \Sigma_{\mathcal{L}}^*$, and $w = \mathsf{sws}(\mathsf{wrt}(u), \mathsf{rd}_2(u)\mathsf{rd}_1(v))$. Then*

$$\begin{array}{rcrcl} \mathsf{rd}_2(uv) & = & & w & \mathsf{rd}_2(v) \text{ and} \\ \mathsf{rd}(u)\mathsf{rd}_1(v) & = & \mathsf{rd}_1(uv) & w & . \end{array}$$

It follows that the characteristics of uv can be expressed in terms of the characteristics $\chi(u)$ and $\chi(v)$ of the two factors:

$$\chi(uv) = (\mathsf{wrt}(u)\mathsf{wrt}(v), \mathsf{rd}(u)\mathsf{rd}(v), \mathsf{sws}(\mathsf{wrt}(u), \mathsf{rd}_2(u)\mathsf{rd}_1(v))\,\mathsf{rd}_2(v)) .$$

Proof. There are letters a_i and b_i in A and words x_i over A such that we have

$$\begin{aligned} \mathsf{nf}_{(A,\emptyset)}(u) &= \overline{\mathsf{rd}_1(u)}(x_1 a_1 \overline{a_1}) \dots (x_k a_k \overline{a_k}) x_{k+1} \text{ and} \\ \mathsf{nf}_{(A,\emptyset)}(v) &= \overline{a_{k+1}\, a_{k+2}\, \dots\, a_{k+\ell}}(y_1 b_1 \overline{b_1}) \dots (y_m b_m \overline{b_m}) y_{m+1} . \end{aligned}$$

Then we obtain

$$\begin{aligned} uv &\equiv_{(A,\emptyset)} \overline{\mathsf{rd}_1(u)}\,(x_1 a_1 \overline{a_1}) \dots (x_k a_k \overline{a_k})\, x_{k+1}\ v \\ &\equiv_{(A,\emptyset)} \overline{\mathsf{rd}_1(u)}\, x_1 a_1 \dots x_k a_k\, x_{k+1}\, \overline{a_1\, \dots\, a_k}\ v && \text{(by Lem. 3.19)} \\ &= \overline{\mathsf{rd}_1(u)}\ \underbrace{\mathsf{wrt}(u)}_{=:u'}\ \underbrace{\overline{\mathsf{rd}_2(u)}\ \overline{\mathsf{rd}_1(v)}}_{=:\overline{v'}}\,(y_1 b_1 \overline{b_1}) \dots (y_m b_m \overline{b_m})\, y_{m+1} . \end{aligned}$$

There are words $z_i \in A^*$ such that

$$\mathsf{nf}_{(A,\emptyset)}(u'\overline{v'}) = \overline{a_1 \dots a_n}\,(z_{n+1} a_{n+1} \overline{a_{n+1}})\ \dots\ (z_{k+\ell} a_{k+\ell} \overline{a_{k+\ell}})\, z_{k+\ell+1} .$$

Note that, by Lemma 3.19, $w = \mathsf{sws}(\mathsf{wrt}(u), \mathsf{rd}_2(u)\mathsf{rd}_1(v)) = a_{n+1} \dots a_{k+\ell}$. It follows that

$$\begin{aligned} uv &\equiv_{(A,\emptyset)} \overline{\mathsf{rd}_1(u)}\, u'\, \overline{v'}\,(y_1 b_1 \overline{b_1}) \dots (y_m b_m \overline{b_m})\, y_{m+1} \\ &\equiv_{(A,\emptyset)} \overline{\mathsf{rd}_1(u)}\, \overline{a_1 \dots a_n}\,(z_{n+1} a_{n+1} \overline{a_{n+1}})\ \dots\ (z_{k+\ell} a_{k+\ell} \overline{a_{k+\ell}})\, z_{k+\ell+1}(y_1 b_1 \overline{b_1}) \dots (y_m b_m \overline{b_m})\, y_{m+1} \\ &\equiv_{(A,\emptyset)} \overline{\mathsf{rd}_1(u)}\, \overline{a_1 \dots a_n}\,(z_{n+1} a_{n+1} \overline{a_{n+1}})\ \dots\ (z_{k+\ell} a_{k+\ell} \overline{a_{k+\ell}})\ \mathsf{nf}_{(A,\emptyset)}(z_{k+\ell+1}(y_1 b_1 \overline{b_1}) \dots (y_m b_m \overline{b_m})\, y_{m+1}) \\ &=: W . \end{aligned}$$

Note that the normal form of $z_{k+\ell+1}(y_1 b_1 \overline{b_1}) \dots (y_m b_m \overline{b_m})\, y_{m+1}$ belongs to

$$(A \setminus \{b_1\})^* b_1 \overline{b_1}\,(A \setminus \{b_2\})^* b_2 \overline{b_2}\ \dots\ (A \setminus \{b_m\})^* b_m \overline{b_m}\, A^* .$$

Consequently, the word W above is the normal form of uv. From this normal form, we obtain $\mathsf{rd}_2(uv) = a_{n+1} \dots a_{k+\ell}\, b_1 \dots b_m$ which equals $w\,\mathsf{rd}_2(v)$. Hence we proved the first equation.

For the second one, observe the following:

$$\mathsf{rd}_1(uv)\, w\, \mathsf{rd}_2(v) = \mathsf{rd}_1(uv)\, \mathsf{rd}_2(uv) = \mathsf{rd}(uv) = \mathsf{rd}(u)\, \mathsf{rd}(v) = \mathsf{rd}(u)\, \mathsf{rd}_1(v)\, \mathsf{rd}_2(v) . \qquad \square$$

The following lemma provides us with a similar statement for powers of an element of $\mathcal{Q}(A, \emptyset)$.

Lemma 4.4. *Let $\mathcal{L} = (A, \emptyset)$ be a lossiness alphabet and $u \in \Sigma_{\mathcal{L}}^*$. For $i \geq 1$ define words s_i and t_i over A inductively by*

$$\begin{aligned} s_1 &= \varepsilon\,, & & & s_{i+1} &= \mathsf{sws}\big(\mathsf{wrt}(u)s_i, \mathsf{rd}_2(u)\mathsf{rd}(u^{i-1})\mathsf{rd}_1(u)\big)\ , \\ t_1 &= \varepsilon\,, & &\text{and} & t_{i+1} &= \mathsf{sws}\big(\mathsf{wrt}(u^i), t_i\mathsf{rd}_2(u)\mathsf{rd}_1(u)\big)\ . \end{aligned}$$

Then

$$\mathsf{rd}_2(u^i) = s_i\mathsf{rd}_2(u)\,,\ \mathsf{rd}(u^{i-1})\mathsf{rd}_1(u) = \mathsf{rd}_1(u^i)\,s_i\,, \mathrm{and s_i = t_i}\,.$$

Proof. We first show the first two equations by simultaneous induction on $i \geq 1$: The case $i = 1$ is obvious since $s_1 = \varepsilon$. Now let $i \geq 2$. Then we have

$$\begin{aligned} \mathsf{rd}_2(u^i) &= \mathsf{rd}_2(uu^{i-1}) \\ &= \mathsf{sws}\big(\mathsf{wrt}(u), \mathsf{rd}_2(u)\mathsf{rd}_1(u^{i-1})\big)\ \mathsf{rd}_2(u^{i-1}) && \text{(by Thm. 4.3)} \\ &= \mathsf{sws}\big(\mathsf{wrt}(u), \mathsf{rd}_2(u)\mathsf{rd}_1(u^{i-1})\big)\ s_{i-1}\mathsf{rd}_2(u) && \text{(by ind. hyp.)} \\ &= \mathsf{sws}\big(\mathsf{wrt}(u)s_{i-1}, \mathsf{rd}_2(u)\mathsf{rd}_1(u^{i-1})s_{i-1}\big)\ \mathsf{rd}_2(u) \\ &= \mathsf{sws}\big(\mathsf{wrt}(u)s_{i-1}, \mathsf{rd}_2(u)\mathsf{rd}(u^{i-2})\mathsf{rd}_1(u)\big)\ \mathsf{rd}_2(u) && \text{(by ind. hyp.)} \\ &= s_i\mathsf{rd}_2(u) \end{aligned}$$

and

$$\begin{aligned} \mathsf{rd}_1(u^i)s_i &= \mathsf{rd}_1(u^i)\,\mathsf{sws}\big(\mathsf{wrt}(u)s_{i-1}, \mathsf{rd}_2(u)\mathsf{rd}(u^{i-2})\mathsf{rd}_1(u)\big) \\ &= \mathsf{rd}_1(u^i)\,\mathsf{sws}\big(\mathsf{wrt}(u)s_{i-1}, \mathsf{rd}_2(u)\mathsf{rd}_1(u^{i-1})s_{i-1}\big) && \text{(by ind. hyp.)} \\ &= \mathsf{rd}_1(uu^{i-1})\ \ \mathsf{sws}\big(\mathsf{wrt}(u), \mathsf{rd}_2(u)\mathsf{rd}_1(u^{i-1})\big)\ s_{i-1} \\ &= \mathsf{rd}(u)\mathsf{rd}_1(u^{i-1})s_{i-1} && \text{(by Thm. 4.3)} \\ &= \mathsf{rd}(u)\mathsf{rd}(u^{i-2})\mathsf{rd}_1(u) && \text{(by ind. hyp.)} \\ &= \mathsf{rd}(u^{i-1})\mathsf{rd}_1(u)\,. \end{aligned}$$

To demonstrate the third equation, we prove $\mathsf{rd}_2(u^i) = t_i\mathsf{rd}_2(u)$ by induction on $i \geq 1$. It is trivial for $i = 1$. For the induction step we have $i \geq 2$ and therefore:

$$\begin{aligned} \mathsf{rd}_2(u^i) &= \mathsf{rd}_2(u^{i-1}u) \\ &= \mathsf{sws}\big(\mathsf{wrt}(u^{i-1}), \mathsf{rd}_2(u^{i-1})\mathsf{rd}_1(u)\big)\ \mathsf{rd}_2(u) && \text{(by Thm. 4.3)} \\ &= \mathsf{sws}\big(\mathsf{wrt}(u^{i-1}), t_{i-1}\mathsf{rd}_2(u)\mathsf{rd}_1(u)\big)\ \mathsf{rd}_2(u) && \text{(by the ind. hyp.)} \\ &= t_i\mathsf{rd}_2(u) \end{aligned}$$

Now $s_i = t_i$ follows from $s_i\mathsf{rd}_2(u) = \mathsf{rd}_2(u^i) = t_i\mathsf{rd}_2(u)$. ☐

We next infer that if u and v agree in their subsequences of read and write operations, respectively, then they can be equated by multiplication with a large power of one of them.

Proposition 4.5. *Let $\mathcal{L} = (A, \emptyset)$ be a lossiness alphabet and $u, v \in \Sigma_{\mathcal{L}}^*$ with $\mathsf{wrt}(u) = \mathsf{wrt}(v)$ and $\mathsf{rd}(u) = \mathsf{rd}(v)$. Then there is $w \in u^* \cup v^*$ with $\mathsf{nf}_{(A,\emptyset)}(wvw) = \mathsf{nf}_{(A,\emptyset)}(wuw)$.*

Proof. W.l.o.g., we can assume that $\mathsf{rd}_1(u)$ is a prefix of $\mathsf{rd}_1(v)$. If $\mathsf{wrt}(u) = \varepsilon$ or $\mathsf{rd}(u) = \varepsilon$ then the claim holds with $w = \varepsilon$ since $u = v$. So we can assume $\mathsf{wrt}(u), \mathsf{rd}(u) \neq \varepsilon$. Let $s_i \in A^*$ for any $i \geq 1$ be defined as in Lemma 4.4.

At first we suppose there is $i \geq 1$ with $|\mathsf{wrt}(v)| \leq |\mathsf{rd}_1(u^i)|$. Then we have

$$\begin{aligned}
\mathsf{sws}\big(\mathsf{wrt}(v), \mathsf{rd}_2(v)\,\mathsf{rd}_1(u^i)\big)\ s_i &= \mathsf{sws}\big(\mathsf{wrt}(v), \mathsf{rd}_1(u^i)\big)\ s_i && (\text{by } |\mathsf{wrt}(v)| \leq |\mathsf{rd}_1(u^i)|)\\
&= \mathsf{sws}\big(\mathsf{wrt}(v), \mathsf{rd}_2(u)\,\mathsf{rd}_1(u^i)\big)\ s_i && (\text{by } |\mathsf{wrt}(v)| \leq |\mathsf{rd}_1(u^i)|)\\
&= \mathsf{sws}\big(\mathsf{wrt}(v)s_i, \mathsf{rd}_2(u)\,\mathsf{rd}_1(u^i)s_i\big)\\
&= \mathsf{sws}\big(\mathsf{wrt}(v)s_i, \mathsf{rd}_2(u)\,\mathsf{rd}(u^{i-1})\,\mathsf{rd}_1(u)\big) && (\text{by Lem. 4.4})\\
&= s_{i+1}\,.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathsf{rd}_2(vu^i) &= \mathsf{sws}\big(\mathsf{wrt}(v), \mathsf{rd}_2(v)\,\mathsf{rd}_1(u^i)\big)\ \mathsf{rd}_2(u^i) && (\text{by Thm. 4.3})\\
&= \mathsf{sws}\big(\mathsf{wrt}(v), \mathsf{rd}_2(v)\,\mathsf{rd}_1(u^i)\big)\ s_i\mathsf{rd}_2(u) && (\text{by Lem. 4.4})\\
&= s_{i+1}\,\mathsf{rd}_2(u)\\
&= \mathsf{rd}_2(u^{i+1}) && (\text{by Lem. 4.4})\,.
\end{aligned}$$

Note that $\mathsf{rd}(vu^i) = \mathsf{rd}(u^{i+1})$ and $\mathsf{wrt}(vu^i) = \mathsf{wrt}(u^{i+1})$ follow from $\mathsf{rd}(v) = \mathsf{rd}(u)$ and $\mathsf{wrt}(v) = \mathsf{wrt}(u)$. Hence $\chi(vu^i) = \chi(u^{i+1})$ and therefore $vu^i \equiv_{(A,\emptyset)} u^{i+1}$.

Now we assume that $|\mathsf{wrt}(v)| > |\mathsf{rd}_1(u^i)|$ for each $i \geq 1$. Then there is $i \geq 1$ such that $|\mathsf{rd}_1(u^i)|$ is maximal. By the definition of $\mathsf{rd}_1(u^i)$ and $\mathsf{rd}_1(u^{i+1})$, there are words $x, y \in \Sigma_{\mathcal{L}}^*$ with

$$\mathsf{nf}_{(A,\emptyset)}(u^i) = \overline{\mathsf{rd}_1(u^i)}\,x \text{ and } \mathsf{nf}_{(A,\emptyset)}(u^{i+1}) = \overline{\mathsf{rd}_1(u^{i+1})}\,y\,.$$

But then

$$\overline{\mathsf{rd}_1(u^i)}\,x\,u = \mathsf{nf}_{(A,\emptyset)}(u^i)\,u \equiv_{(A,\emptyset)} u^i u \equiv_{(A,\emptyset)} \mathsf{nf}_{(A,\emptyset)}(u^{i+1}) = \overline{\mathsf{rd}_1(u^{i+1})}\,y$$

implies

$$\overline{\mathsf{rd}_1(u^i)}\,x\,u \to^* \overline{\mathsf{rd}_1(u^{i+1})}\,y$$

where $\to$ is the derivation relation of the semi-Thue system $\mathcal{R}_{(A,\emptyset)}$. Since the rules of $\mathcal{R}_{(A,\emptyset)}$ move read operations to the left, we obtain $|\mathsf{rd}_1(u^i)| \leq |\mathsf{rd}_1(u^{i+1})|$. By the choice of i, we have $\mathsf{rd}_1(u^i) = \mathsf{rd}_1(u^{i+1})$. Let $t_i, t_{i+1} \in A^*$ be defined as in Lemma 4.4. Then we have

$$\begin{aligned}
\mathsf{rd}_1(u^i)\,t_i\,\mathsf{rd}_2(u)\,\mathsf{rd}_1(u)\,\mathsf{rd}_2(u) &= \mathsf{rd}_1(u^i)\,\mathsf{rd}_2(u^i)\,\mathsf{rd}(u) && (\text{by Lem. 4.4})\\
&= \mathsf{rd}(u^i)\,\mathsf{rd}(u)\\
&= \mathsf{rd}(u^{i+1})\\
&= \mathsf{rd}_1(u^{i+1})\,\mathsf{rd}_2(u^{i+1})\\
&= \mathsf{rd}_1(u^i)\,t_{i+1}\,\mathsf{rd}_2(u) && (\text{by Lem. 4.4})\,.
\end{aligned}$$

i.e., $t_{i+1} = t_i \mathsf{rd}_2(u)\mathsf{rd}_1(u)$. Since $\mathsf{rd}_1(v)$ is a prefix of $\mathsf{rd}_1(u)$ we get

$$t_i\,\mathsf{rd}_2(u)\,\mathsf{rd}_1(v) \preceq t_i\,\mathsf{rd}_2(u)\,\mathsf{rd}_1(u) = t_{i+1} \preceq \mathsf{wrt}(u^i)$$

and therefore $\mathsf{sws}\big(\mathsf{wrt}(u^i), t_i\,\mathsf{rd}_2(u)\,\mathsf{rd}_1(v)\big) = t_i\,\mathsf{rd}_2(u)\,\mathsf{rd}_1(v)$. Consequently,

$$\begin{aligned}
\mathsf{rd}_2(u^i v) &= \mathsf{sws}\big(\mathsf{wrt}(u^i), t_i\,\mathsf{rd}_2(u)\,\mathsf{rd}_1(v)\big)\ \mathsf{rd}_2(v) && \text{(by Thm. 4.3)}\\
&= t_i\,\mathsf{rd}_2(u)\,\mathsf{rd}_1(v)\,\mathsf{rd}_2(v)\\
&= t_i\,\mathsf{rd}_2(u)\,\mathsf{rd}_1(u)\,\mathsf{rd}_2(u) && \text{(by } \mathsf{rd}(u)=\mathsf{rd}(v))\\
&= t_{i+1}\,\mathsf{rd}_2(u)\\
&= \mathsf{rd}_2(u^{i+1}) && \text{(by Lem. 4.4)}\,.
\end{aligned}$$

Similar to the case above we get $\chi(u^i v) = \chi(u^{i+1})$ and therefore $u^i v \equiv_{(A,\emptyset)} u^{i+1}$.

Note that in both cases we have $wvw \equiv_{(A,\emptyset)} wuw$ for $w = u^i$. □

From this proposition, we can infer the announced necessary condition for a homomorphism into $\mathcal{Q}(\mathcal{L})$ to be injective (where $X \subseteq A$ is not necessarily empty).

Theorem 4.6. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet, $\mathcal{M}$ be a monoid, $\varphi\colon \mathcal{M} \hookrightarrow \mathcal{Q}(\mathcal{L})$ be an embedding, and $x, y \in \mathcal{M}$ such that $\mathsf{wrt}(\varphi(x)) = \mathsf{wrt}(\varphi(y))$ and $\mathsf{rd}(\varphi(x)) = \mathsf{rd}(\varphi(y))$.*

Then there is $z \in \mathcal{M}$ with $zxz = zyz$.

Proof. For notational simplicity, let $\varphi(x) = [u]$ and $\varphi(y) = [v]$.

By Proposition 4.5, there is $w \in u^* \cup v^*$ such that $wvw \equiv_{(A,\emptyset)} wuw$. Using Theorem 3.15, this implies $\mathsf{nf}_{(A,\emptyset)}(wvw) = \mathsf{nf}_{(A,\emptyset)}(wuw)$. As the semi-Thue system $\mathcal{R}_{\mathcal{L}}$ contains all the rules from $\mathcal{R}_{(A,\emptyset)}$ we get $\mathsf{nf}_{\mathcal{L}}(wvw) = \mathsf{nf}_{\mathcal{L}}(wuw)$ and therefore $wvw \equiv_{\mathcal{L}} wuw$. Since $w \in u^* \cup v^*$ there is $z \in x^* \cup y^* \subseteq \mathcal{M}$ such that $\varphi(z) = [w]$. Then we have $\varphi(zyz) = \varphi(zxz)$. The injectivity of φ now implies $zyz = zxz$. □

5. Embeddings between plq monoids

We now characterize when one plq monoid embeds into another plq monoid.

Theorem 5.1. *Let $\mathcal{L}_A = (A, X)$ and $\mathcal{L}_B = (B, Y)$ be two lossiness alphabets. Then $\mathcal{Q}(\mathcal{L}_A) \hookrightarrow \mathcal{Q}(\mathcal{L}_B)$ holds iff all of the following properties hold:*

(A) $|A \setminus X| \leq |B \setminus Y|$, i.e., $\mathcal{L}_B$ has at least as many forgettable letters as $\mathcal{L}_A$.
(B) If $Y = \emptyset$, then also $X = \emptyset$, i.e., if $\mathcal{L}_B$ consists of forgettable letters only, then so does $\mathcal{L}_A$.
(C) If $|Y| = 1$, then $|A \setminus X| < |B \setminus Y|$ or $|X| \leq 1$, i.e., if $\mathcal{L}_B$ has exactly one non-forgettable letter and exactly as many forgettable letters as $\mathcal{L}_A$, then $\mathcal{L}_A$ contains at most one non-forgettable letter.

In particular, $\mathcal{Q}(A, A)$ embeds into $\mathcal{Q}(B, B)$ whenever $|B| \geq 2$, *i.e.*, this theorem generalizes Corollary 5.4 from [13]. We prove it in Propositions 5.4 and 5.9.

5.1. Preorder of embeddability

The embeddability of monoids is reflexive and transitive, *i.e.*, a preorder. Before diving into the proof of Theorem 5.1, we derive from it an order-theoretic description of this preorder on the class of all plq monoids (see the reflexive and transitive closure of the graph on the right). The plq monoid $\mathcal{Q}(\mathcal{L})$ is, up to isomorphism, completely given by the numbers $m = |X|$ and $n = |A \setminus X|$ of non-forgettable and of forgettable letters, respectively. Therefore, we describe this preorder in terms of pairs of natural numbers. We write $(m, n) \to (m', n')$ if

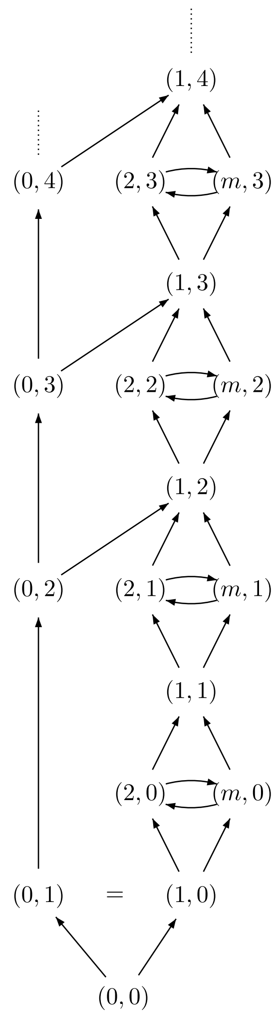
$$\mathcal{Q}([m+n], [m]) \hookrightarrow \mathcal{Q}([m'+n'], [m'])$$

where, as usual, $[n] = \{1, 2, \ldots, n\}$. Then Theorem 5.1 reads as follows: If $m, n, m', n' \in \mathbb{N}$ with $m+n, m'+n' \geq 2$, then $(m, n) \to (m', n')$ iff all of the following properties hold:

(A) $n \leq n'$
(B) If $m' = 0$, then $m = 0$
(C) If $m' = 1$, then $m \leq 1$ or $n < n'$

Then we get immediately for all appropriate natural numbers $m, n, n' \in \mathbb{N}$:

- if $m \geq 2$, then $(2, n) \to (m, n) \to (2, n)$
- $(2, n) \to (2, n')$ iff $n \leq n'$
- $(1, n) \to (2, n')$ iff $n \leq n'$
- $(0, n) \to (2, n')$ iff $n \leq n'$
- $(2, n) \to (1, n')$ iff $n < n'$
- $(1, n) \to (1, n')$ iff $n \leq n'$
- $(0, n) \to (1, n')$ iff $n \leq n'$
- $(2, n) \not\to (0, n')$
- $(1, n) \not\to (0, n')$
- $(0, n) \to (0, n')$ iff $n \leq n'$



These facts allow to derive the graph on the right (where m stands for an arbitrary number at least 3). Note that the nodes $(1, 0)$, $(0, 1)$, and $(0, 0)$ represent the monoids $\mathcal{Q}(\{a\}, \{a\})$, $\mathcal{Q}(\{a\}, \emptyset)$, and $\mathcal{Q}(\emptyset, \emptyset)$, respectively (that we do not formally consider as plq monoids).

First look at the nodes not of the form $(0, n)$. They form an alternating chain of infinite equivalence classes $\{(m, n) \mid m \geq 2\}$ and single nodes $(1, n)$. The infinite equivalence class of all nodes of the form $(m, 0)$ with $m \geq 2$ corresponds to the monoids of fully reliable queues considered in [13].

The nodes of the form $(0, n)$ also form a chain of single nodes (these nodes depict the fully lossy queue monoids from [18]). The single node number n (*i.e.*, $(0, 2+n)$) from this chain is directly below the single node number $2+n$ (*i.e.*, $(1, 2+n)$) of the alternating chain.

5.2. Sufficiency in Theorem 5.1

Let $\mathcal{L}_A = (A, X)$ and $\mathcal{L}_B = (B, Y)$ be two lossiness alphabets. Suppose Conditions (A)–(C) from Theorem 5.1 hold.

First suppose $|X| \leq |Y|$. By Condition (A), there exists an embedding $\varphi' \colon \Sigma_{\mathcal{L}_A} \to \Sigma_{\mathcal{L}_B}$ mapping X into Y, $A \setminus X$ into $B \setminus Y$ and satisfying $\varphi'(\overline{a}) = \overline{\varphi'(a)}$ for each $a \in A$. We identify φ' with the induced homomorphism from $\Sigma^*_{\mathcal{L}_A}$ into $\Sigma^*_{\mathcal{L}_B}$. For any rule $u \to v$ with $u, v \in A^*$ from the semi-Thue system $\mathcal{R}_{\mathcal{L}_A}$, we get $\varphi'(u) \equiv_{\mathcal{L}_B} \varphi'(v)$

by Lemma 3.9. Hence, by Theorem 3.15, we therefore get

$$u \equiv_{\mathcal{L}_A} v \implies \varphi'(u) \equiv_{\mathcal{L}_B} \varphi'(v)$$

for any $u, v \in \Sigma^*_{\mathcal{L}_A}$. Since the injective homomorphism φ' maps words over A in normal form to words over B in normal form, we also have the converse implication. Hence $\varphi' \colon \Sigma^*_{\mathcal{L}_A} \hookrightarrow \Sigma^*_{\mathcal{L}_B}$ induces an embedding $\mathcal{Q}(\mathcal{L}_A) \hookrightarrow \mathcal{Q}(\mathcal{L}_B)$.

In the rest of this section, we assume $|X| > |Y|$. Then, the basic idea is as follows: the submonoids of $\mathcal{Q}(\mathcal{L}_A)$ generated by X and $A \setminus X$, respectively, are isomorphic to $\mathcal{Q}(X, X)$ and $\mathcal{Q}(A \setminus X, \emptyset)$, respectively, and intersect in the unit element $[\varepsilon]$. Since $|A \setminus X| \leq |B \setminus Y|$ by (A), we find an embedding φ_1 of $\mathcal{Q}(A \setminus X, \emptyset)$ into $\mathcal{Q}(B \setminus Y, \emptyset)$. If $|Y| \geq 2$, then [13] provides an embedding φ_2 from $\mathcal{Q}(X, X)$ into $\mathcal{Q}(Y, Y)$. If $|Y| \leq 1$, then we have the problem that no such embedding φ_2 exists. But we can, by (C), "borrow" a letter b_2 from $B \setminus Y$ to construct an embedding φ_2 from $\mathcal{Q}(X, X)$ into $\mathcal{Q}(Y \cup \{b_2\}, Y)$ similar to [13]. To embed $\mathcal{Q}(\mathcal{L}_A)$ into $\mathcal{Q}(\mathcal{L}_B)$, we need a joint extension of these two embeddings of submonoids.

Since it is easier to work with words than with elements of $\mathcal{Q}(\mathcal{L}_A)$, we construct an injective homomorphism $\varphi' \colon \Sigma^*_{\mathcal{L}_A} \to \Sigma^*_{\mathcal{L}_B}$ satisfying

$$u \equiv_{\mathcal{L}_A} v \iff \varphi'(u) \equiv_{\mathcal{L}_B} \varphi'(v) \tag{5.1}$$

for any words $u, v \in \Sigma^*_{\mathcal{L}_A}$.

The construction of φ'

By Condition (A), there exists an injective mapping $\varphi_1 \colon A \setminus X \hookrightarrow B \setminus Y$. Since $|X| > |Y|$, Condition (B) implies $Y \neq \emptyset$. Let $b_1 \in Y$ be arbitrary. If $|Y| > 1$, then choose $b_2 \in Y \setminus \{b_1\}$. Otherwise, we have $1 = |Y| < |X|$. Hence, by Condition (C), the mapping φ_1 is not surjective. Thus, we can choose $b_2 \in B \setminus (Y \cup \{\varphi_1(a) \mid a \in A \setminus X\})$. With $X = \{x_1, x_2, \ldots, x_n\}$, we set

$$\varphi'(a) = \begin{cases} \varphi_1(a) & \text{if } a \in A \setminus X \\ b_1^{|A|+i} b_2 b_1^{|A|-i} b_2 & \text{if } a = x_i \end{cases} \qquad \text{and} \qquad \varphi'(\overline{a}) = \overline{\varphi'(a)} \text{ for } a \in A\,.$$

It is easy to see that φ' maps $\Sigma^*_{\mathcal{L}_A}$, A^*, and $\overline{A}^*$ injectively into $\Sigma^*_{\mathcal{L}_B}$, B^*, and $\overline{B}^*$, respectively. The following lemmas demonstrate the two implications in (5.1).

Lemma 5.2. *If $u, v \in \Sigma^*_{\mathcal{L}_A}$ with $u \equiv_{\mathcal{L}_A} v$, then $\varphi'(u) \equiv_{\mathcal{L}_B} \varphi'(v)$.*

Proof. By Theorem 3.15, it suffices to show this for any of the equations $u \equiv_{\mathcal{L}_A} v$ in Lemma 3.9.

(i) Let $a_1, a_2 \in A$ be distinct. Let $v_i = \varphi'(a_i)$ such that $\varphi'(\overline{a_i}) = \overline{v_i}$. We have $\mathsf{wrt}(\varphi'(a_1\,\overline{a_2})) = v_1 = \mathsf{wrt}(\varphi'(\overline{a_2}\,a_1))$ and $\mathsf{rd}(\varphi'(a_1\,\overline{a_2})) = v_2 = \mathsf{rd}(\varphi'(\overline{a_2}\,a_1))$. Note that $\mathsf{rd}_2(v_1\overline{v_2}) = \varepsilon$ can be shown by Lemma 3.19 distinguishing four cases depending on whether $a_i \in X$ or not. We obtain

$$\begin{aligned} \mathsf{rd}_2(\varphi'(a_1\,\overline{a_2})) &= \mathsf{rd}_2(v_1\,\overline{v_2}) \\ &= \varepsilon \\ &= \mathsf{rd}_2(\overline{v_2}\,v_1) && \text{(since } \overline{v_2}\,v_1 \in \mathsf{NF}_{\mathcal{L}_B}) \\ &= \mathsf{rd}_2(\varphi'(\overline{a_2}\,a_1))\,. \end{aligned}$$

Hence $\chi(\varphi'(a_1\,\overline{a_2})) = \chi(\varphi'(\overline{a_2}\,a_1))$ implying $\varphi'(a_1\,\overline{a_2}) \equiv_{\mathcal{L}_B} \varphi'(\overline{a_2}\,a_1)$.

(ii) Let $a_1, a_2 \in A$ be arbitrary ($a_1 = a_2$ is allowed). We have to show $\varphi'(a_1\overline{a_1}\overline{a_2}) \equiv_{\mathcal{L}_B} \varphi'(\overline{a_1}a_1\overline{a_2})$. As before, let $v_i = \varphi'(a_i)$ such that $\varphi'(\overline{a_i}) = \overline{v_i}$.

First suppose $|v_1| \leq |v_2|$. By Lemma 3.19, $\mathsf{rd}_2(v_1\overline{v_1}\overline{v_2})$ is a suffix of v_1v_2 of length at most $|v_1| \leq |v_2|$. Hence we get

$$\mathsf{rd}_2(v_1\overline{v_1}\overline{v_2}) = \mathsf{rd}_2(v_1\overline{v_2})\,.$$

Alternatively, $|v_1| > |v_2|$ implies $v_1 \in \{b_1, b_2\}^*$ and $v_2 \in B \setminus \{b_1, b_2\}$. Hence, from Lemma 3.19, we get

$$\mathsf{rd}_2(v_1\overline{v_1}\overline{v_2}) = \varepsilon = \mathsf{rd}_2(v_1\overline{v_2})\,.$$

Since all rules in the semi-Thue system $\mathcal{R}_{\mathcal{L}_B}$ try to move read actions from $\overline{B}$ to the left, we get

$$\mathsf{nf}_{\mathcal{L}_B}(\overline{v_1}\, v_1\, \overline{v_2}) = \overline{v_1}\, \mathsf{nf}_{\mathcal{L}_B}(v_1\, \overline{v_2})$$

and therefore

$$\mathsf{rd}_2(\overline{v_1}\, v_1\, \overline{v_2}) = \mathsf{rd}_2(v_1\, \overline{v_2}) = \mathsf{rd}_2(v_1\overline{v_1}\, \overline{v_2})\,.$$

Now $\mathsf{wrt}(\overline{v_1}v_1\overline{v_2}) = v_1 = \mathsf{wrt}(v_1\overline{v_1}\overline{v_2})$ and $\mathsf{rd}(\overline{v_1}v_1\overline{v_2}) = v_1v_2 = \mathsf{rd}(v_1\overline{v_1}\overline{v_2})$ finish the proof of $\overline{v_1}v_1\overline{v_2} \equiv_{\mathcal{L}_B} v_1\overline{v_1}\overline{v_2}$. By the choice of v_1 and v_2, this implies $\varphi'(a_1\overline{a_1}\overline{a_2}) \equiv_{\mathcal{L}_B} \varphi'(\overline{a_1}a_1\overline{a_2})$.

(iii) Let $a \in A$ and $w \in A^*$. Let $v_a = \varphi'(a) \in B^*$ and $v_w = \varphi'(w) \in B^*$ such that, in particular, $\varphi'(\overline{a}) = \overline{v_a}$. Lemma 3.19 implies

$$\mathsf{rd}_2(v_av_wv_a\overline{v_a}) = v_a = \mathsf{rd}_2(v_av_w\overline{v_a})\,.$$

Since all rules in the semi-Thue system try to move read actions from $\overline{B}$ to the left, we get

$$\mathsf{nf}_{\mathcal{L}_B}(v_av_w\overline{v_a}v_a) = \mathsf{nf}_{\mathcal{L}_B}(v_av_w\overline{v_a})\, v_a$$

and therefore

$$\mathsf{rd}_2(v_av_w\overline{v_a}v_a) = \mathsf{rd}_2(v_av_w\overline{v_a}) = \mathsf{rd}_2(v_av_wv_a\overline{v_a})\,.$$

Now $\mathsf{wrt}(v_av_w\overline{v_a}v_a) = v_av_wv_a = \mathsf{wrt}(v_av_wv_a\overline{v_a})$ and $\mathsf{rd}(v_av_w\overline{v_a}v_a) = v_a = \mathsf{wrt}(v_av_wv_a\overline{v_a})$ finish the proof of $v_av_w\overline{v_a}v_a \equiv_{\mathcal{L}_B} v_av_wv_a\overline{v_a}$. By the choice of v_a and v_w, this implies $\varphi'(awa\overline{a}) \equiv_{\mathcal{L}_B} \varphi'(aw\overline{a}a)$.

(iv) Let $a \in A$, $x \in X$, and $w \in A^*$. Since we want to prove $\varphi'(xwa\overline{a}) \equiv_{\mathcal{L}_B} \varphi'(xw\overline{a}a)$, it suffices to consider the case $w \in (A \setminus X)^*$. If $a = x$, then $\varphi'(xwa\overline{a}) \equiv_{\mathcal{L}_B} \varphi'(xw\overline{a}a)$ by the previous item. So assume from now on $a \neq x$.
Let $v_a = \varphi'(a) \in B^+$, $v_x = \varphi'(x) \in YB^*$, and $v_w = \varphi'(w) \in B^*$ such that, in particular, $\varphi'(\overline{a}) = \overline{v_a}$. Note that v_x contains $2|A|$ occurrences of the non-forgettable letter $b_1 \in Y$ and v_a contains at most $2|A|$ such occurrences. Hence Lemma 3.19 implies

$$\mathsf{rd}_2(v_xv_wv_a\overline{v_a}) = \mathsf{rd}_2(v_xv_w\overline{v_a})\,.$$

Since all rules in the semi-Thue system try to move read actions from $\overline{B}$ to the left, we get

$$\mathsf{nf}_{\mathcal{L}_B}(v_xv_w\overline{v_a}v_a) = \mathsf{nf}_{\mathcal{L}_B}(v_xv_w\overline{v_a})\, v_a$$

and therefore

$$\mathsf{rd}_2(v_xv_w\overline{v_a}v_a) = \mathsf{rd}_2(v_xv_w\overline{v_a}) = \mathsf{rd}_2(v_xv_wv_a\overline{v_a})\,.$$

Now $\mathsf{wrt}(v_x v_w \overline{v_a} v_a) = v_x v_w v_a = \mathsf{wrt}(v_x v_w v_a \overline{v_a})$ and $\mathsf{rd}(v_x v_w \overline{v_a} v_a) = v_a = \mathsf{wrt}(v_x v_w v_a \overline{v_a})$ finish the proof of $v_x v_w \overline{v_a} v_a \equiv_{\mathcal{L}_B} v_x v_w v_a \overline{v_a}$. By the choice of v_x, v_a, and v_w, this implies $\varphi'(xwa\overline{a}) \equiv_{\mathcal{L}_B} \varphi'(xw\overline{a}a)$. $\square$

Thus, we have the implication "$\Rightarrow$" in (5.1) and it remains to verify the implication "$\Leftarrow$".

Lemma 5.3. *If $u, v \in \Sigma_{\mathcal{L}_A}^*$ with $\varphi'(u) \equiv_{\mathcal{L}_B} \varphi'(v)$, then $u \equiv_{\mathcal{L}_A} v$.*

Proof. There are letters $a_i \in A$ and words $y_i \in (A \setminus (X \cup \{a_i\}))^*$ for $1 \le i \le m$ and $y_0, y_{m+1} \in A^*$ such that

$$\begin{aligned} \mathsf{nf}_{\mathcal{L}_A}(u) &= \overline{y_0}(y_1 a_1 \overline{a_1})(y_2 a_2 \overline{a_2}) \cdots (y_m a_m \overline{a_m}) y_{m+1} \\ \text{and } \mathsf{nf}_{\mathcal{L}_A}(v) &= \overline{z_0}(z_1 c_1 \overline{c_1})(z_2 c_2 \overline{c_2}) \cdots (z_n c_n \overline{c_n}) z_{n+1} \end{aligned}$$

(with $c_i \in A$ and $z_i \in (A \setminus (X \cup \{c_i\}))^*$ for $1 \le i \le n$ and $z_0, z_{n+1} \in A^*$). We first prove that the images of these two words in normal form are "almost" in normal form.

We consider the blocks $y_i a_i \overline{a_i}$ for $1 \le i \le m$: First note that y_i is a word over $A \setminus (X \cup \{a_i\})$. Consequently, $\varphi'(y_i)$ is a word over $B \setminus (Y \cup \{\varphi_1(a_i), b_1, b_2\})$. If $a_i \in A \setminus X$, then $\varphi'(a_i) \in B \setminus (Y \cup \{b_1, b_2\})$. Hence the word $\varphi'(y_i a_i \overline{a_i})$ is in normal form. Next consider the case $a_i \in X$. Then there exists j with $\varphi'(a_i) = b_1^{|A|+j} b_2 b_1^{|A|-j} b_2$ and therefore

$$\mathsf{nf}_{\mathcal{L}_B}(y_i a_i \overline{a_i}) = \varphi'(y_1)\, (b_1 \overline{b_1})^{|A|+j}\, b_2 \overline{b_2}\, (b_1 \overline{b_1})^{|A|-j}\, b_2 \overline{b_2}\,.$$

It follows that the word

$$\overline{\varphi'(y_0)}\ \mathsf{nf}_{\mathcal{L}_B}(\varphi'(y_1 a_1 \overline{a_1}))\ \mathsf{nf}_{\mathcal{L}_B}(\varphi'(y_2 a_2 \overline{a_2}))\ \ldots \mathsf{nf}_{\mathcal{L}_B}(\varphi'(y_m a_m \overline{a_m}))\ \varphi'(y_m)$$

is in normal form. Since it is equivalent to $\varphi'(u)$, it equals the normal form of $\varphi'(u)$. Similarly, the normal form of $\varphi'(v)$ equals

$$\overline{\varphi'(z_0)}\ \mathsf{nf}_{\mathcal{L}_B}(\varphi'(z_1 c_1 \overline{c_1}))\ \mathsf{nf}_{\mathcal{L}_B}(\varphi'(z_2 c_2 \overline{c_2}))\ \ldots \mathsf{nf}_{\mathcal{L}_B}(\varphi'(z_n c_n \overline{c_n}))\ \varphi'(z_n)\,.$$

Now Lemma 5.2 implies $\varphi'(\mathsf{nf}_{\mathcal{L}_A}(u)) = \varphi'(u) = \varphi'(v) = \varphi'(\mathsf{nf}_{\mathcal{L}_A}(v))$, *i.e.*, the two words above are equal. In particular, $m = n$, $\varphi'(y_0) = \varphi'(z_0)$, $\mathsf{nf}_{\mathcal{L}_B}(\varphi'(y_i a_i \overline{a_i})) = \mathsf{nf}_{\mathcal{L}_B}(\varphi'(z_i c_i \overline{c_i}))$ for all $1 \le i \le m = n$, and $\varphi'(y_{m+1}) = \varphi'(z_{n+1})$. Since φ' is injective on A^*, this implies $y_0 = z_0$ and $y_{m+1} = z_{n+1}$. The above calculation of $\mathsf{nf}_{\mathcal{L}_B}(\varphi'(y_i a_i \overline{a_i}))$ yields $y_i = z_i$ and $a_i = c_i$ for all $1 \le i \le m = n$. Hence, we get $\mathsf{nf}_{\mathcal{L}_A}(u) = \mathsf{nf}_{\mathcal{L}_A}(v)$ and therefore $u \equiv_{\mathcal{L}_A} v$. $\square$

By Lemmas 5.2 and 5.3 we can infer that φ' induces an embedding of $\mathcal{Q}(\mathcal{L}_A)$ into $\mathcal{Q}(\mathcal{L}_B)$ which we conclude in the following proposition.

Proposition 5.4. *Let $\mathcal{L}_A = (A, X)$ and $\mathcal{L}_B = (B, Y)$ be two lossiness alphabets such that Conditions (A)?(C) from Theorem 5.1 hold. Then $\mathcal{Q}(\mathcal{L}_A)$ embeds into $\mathcal{Q}(\mathcal{L}_B)$.*

5.3. Necessity in Theorem 5.1

Now we have to prove the other implication of the equivalence in Theorem 5.1. Recall the embedding φ we constructed in the proof of Proposition 5.4. In particular, it has the following properties:

(1) If $a \in A$, then $\varphi(a) \in [B^+]$ and $\varphi(\overline{a}) = \overline{\varphi(a)}$. In particular, the image of every write operation a performs write operations, only, and the image of every read operation $\overline{a}$ is the "overlined version of the image of the corresponding write operation" and therefore performs read operations, only.
(2) If $a \in A \setminus X$, then $\varphi(a) \in [B \setminus Y]$. In particular, the image of every write operation of a forgettable letter writes exactly one letter and that letter is forgettable.

(3) If $x \in X$, then $\varphi(x) \in [B^*YB^*]$. In particular, the image of every write operation of a non-forgettable letter writes at least one non-forgettable letter.

Of course, there are also embeddings that do not satisfy these three properties of φ. For example, the homomorphism obtained from $\alpha \mapsto \alpha\alpha$ for any $\alpha \in \Sigma_{\mathcal{L}}$ embeds $\mathcal{Q}(\mathcal{L})$ into $\mathcal{Q}(\mathcal{L})$ but violates (2). Hence, the proof of the necessity in Theorem 5.1 first shows that any embedding satisfies slightly weaker properties.

The following two lemmas prepare the proof of the weakenings of properties (1) and (2) in Lemma 5.7.

Lemma 5.5. *Let $\mathcal{L}_A = (A, X)$ and $\mathcal{L}_B = (B, Y)$ be two lossiness alphabets, $\varphi\colon \mathcal{Q}(\mathcal{L}_A) \hookrightarrow \mathcal{Q}(\mathcal{L}_B)$ be an embedding, and $a \in A$. Then $\mathsf{rd}(\varphi(\overline{a})) \neq \varepsilon$ and $\mathsf{wrt}(\varphi(a)) \neq \varepsilon$.*

Proof. Towards a contradiction suppose that $\varphi(\overline{a})$ performs write operations, only, *i.e.*, $\mathsf{rd}(\varphi(\overline{a})) = \varepsilon$. Lemma 3.9 implies $a\overline{a}\overline{a} \equiv_{\mathcal{L}_A} \overline{a}a\overline{a}$ and therefore $\varphi(a\overline{a})\varphi(\overline{a}) = \varphi(\overline{a}a)\varphi(\overline{a})$. Since $\mathsf{rd}(\varphi(\overline{a})) = \varepsilon$ (*i.e.*, $\varphi(\overline{a}) \in [B^*]$), Corollary 3.18 yields $\varphi(a\overline{a}) = \varphi(\overline{a}a)$. As φ is injective we have $a\overline{a} \equiv_{\mathcal{L}_A} \overline{a}a$ which contradicts Theorem 3.15. Hence we showed the first claim. The proof of the other equation is very similar to this, but starts from the equation $aa\overline{a} \equiv_{\mathcal{L}_A} a\overline{a}a$. $\square$

Recall that every word w over B is the power of some primitive word. Furthermore, if w is not empty, then this primitive word is unique and called the *primitive root* of w. The following lemma shows that the sequences of write operations in $\varphi(\alpha)$ and in $\varphi(\overline{\beta})$ are powers of the same primitive word for any $\alpha, \beta \in A$ (and similarly for the read operations).

Lemma 5.6. *Let $\mathcal{L}_A = (A, X)$ and $\mathcal{L}_B = (B, Y)$ be two lossiness alphabets, $\varphi\colon \mathcal{Q}(\mathcal{L}_A) \hookrightarrow \mathcal{Q}(\mathcal{L}_B)$ be an embedding, $\alpha, \beta \in A$, and $p \in B^+$ be primitive. Then the following hold:*

(i) If $\mathsf{wrt}(\varphi(\alpha)) \in p^+$, then $\mathsf{wrt}(\varphi(\overline{\beta})) \in p^*$.
(ii) If $\mathsf{rd}(\varphi(\alpha)) \in p^+$, then $\mathsf{rd}(\varphi(\overline{\beta})) \in p^*$.

Proof. To prove the first claim, suppose $\mathsf{wrt}(\varphi(\alpha)) \in p^+$. By Lemma 3.9 we have $\alpha\alpha\overline{\beta} \equiv_{\mathcal{L}_A} \alpha\overline{\beta}\alpha$. Since wrt and φ are homomorphisms, this implies $\mathsf{wrt}(\varphi(\alpha))\,\mathsf{wrt}(\varphi(\alpha\overline{\beta})) = \mathsf{wrt}(\varphi(\alpha))\,\mathsf{wrt}(\varphi(\overline{\beta}\alpha))$. Since this equation holds in the free monoid B^*, we get

$$\mathsf{wrt}(\varphi(\alpha))\,\mathsf{wrt}(\varphi(\overline{\beta})) = \mathsf{wrt}(\varphi(\alpha\overline{\beta})) = \mathsf{wrt}(\varphi(\overline{\beta}\alpha)) = \mathsf{wrt}(\varphi(\overline{\beta}))\,\mathsf{wrt}(\varphi(\alpha))\,.$$

In other words, $\mathsf{wrt}(\varphi(\alpha))$ and $\mathsf{wrt}(\varphi(\overline{\beta}))$ commute. Since p is the primitive root of $\mathsf{wrt}(\varphi(\alpha))$, this implies $\mathsf{wrt}(\varphi(\overline{\beta})) \in p^*$.

Replacing the homomorphism wrt by the homomorphism rd in the above argument, we get the proof of the second claim. $\square$

Now we can prove the announced weakenings of properties (1) and (2). The first statement of the following lemma is a weakening of (1) since it only says something about the letters in $\varphi(a)$ and $\varphi(\overline{a})$ but not that these two elements are dual. Similarly the second statement is a weakening of (2) since it does not say anything about the length of $\varphi(a)$ but only something about the letters occurring in $\varphi(a)$.

Lemma 5.7. *Let $\mathcal{L}_A = (A, X)$ and $\mathcal{L}_B = (B, Y)$ be two lossiness alphabets and φ be an embedding of $\mathcal{Q}(\mathcal{L}_A)$ into $\mathcal{Q}(\mathcal{L}_B)$. Then the following hold:*

(i) $\varphi(a) \in [B^+]$ and $\varphi(\overline{a}) \in [\overline{B}^+]$ for each $a \in A$.
(ii) $\varphi(a) \in [(B \setminus Y)^*]$ for each $a \in A \setminus X$.

Proof. To prove (i), let $a \in A$ and suppose $\varphi(a) \notin [B^*]$. From Lemma 5.5, we know $\varphi(a) \notin [\overline{B}^*]$. Let $p, q \in B^+$ be the primitive roots of the non-empty words $\mathsf{wrt}(\varphi(a))$ and $\mathsf{rd}(\varphi(a))$, respectively.

Since $|A| \geq 2$, there exist distinct letters $a_1, a_2 \in A$. With $\alpha = a$ and $\beta = a_i$ (for $i \in \{1, 2\}$), Lemma 5.6 implies $\mathsf{wrt}(\varphi(\overline{a_i})) \in p^*$ and $\mathsf{rd}(\varphi(\overline{a_i})) \in q^*$. Consequently, $\mathsf{wrt}(\varphi(\overline{a_1}\,\overline{a_2})) = \mathsf{wrt}(\varphi(\overline{a_1}))\,\mathsf{wrt}(\varphi(\overline{a_2})) = \mathsf{wrt}(\varphi(\overline{a_2}))\,\mathsf{wrt}(\varphi(\overline{a_1})) = \mathsf{wrt}(\varphi(\overline{a_2}\,\overline{a_1}))$ and similarly $\mathsf{rd}(\varphi(\overline{a_1}\,\overline{a_2})) = \mathsf{rd}(\varphi(\overline{a_2}\,\overline{a_1}))$. Since φ is an embedding, Theorem 4.6 implies the existence of $u \in \Sigma^*_{\mathcal{L}_A}$ with $u\overline{a_1}\,\overline{a_2}u \equiv_{\mathcal{L}_A} u\overline{a_2}\,\overline{a_1}u$. It follows from Proposition 3.12 that these two words have the same sequence of read operations and therefore in particular $a_1\, a_2 = a_2\, a_1$. But this implies $a_1 = a_2$ which contradicts our choice of these two letters. Hence, indeed, $\varphi(a) \in [B^*]$ which proves the first claim.

For the second claim, let $a \in A$ and suppose $\varphi(\overline{a}) \notin [\overline{B}^*]$. From Lemma 5.5, we know $\varphi(\overline{a}) \notin [B^*]$. Let $p, q \in B^+$ be the primitive roots of the non?empty words $\mathsf{wrt}(\varphi(\overline{a}))$ and $\mathsf{rd}(\varphi(\overline{a}))$, respectively.

Since $|A| \geq 2$, there exist distinct letters $a_1, a_2 \in A$. With $\alpha = a_i$ and $\beta = a$ (for $i \in \{1, 2\}$), the contraposition of the two claims of Lemma 5.6 imply $\mathsf{rd}(\varphi(a_i)) \in p^*$ and $\mathsf{wrt}(\varphi(a_i)) \in q^*$. Consequently, $\mathsf{rd}(\varphi(a_1\, a_2)) = \mathsf{rd}(\varphi(a_2\, a_1))$ and $\mathsf{wrt}(\varphi(a_1\, a_2)) = \mathsf{wrt}(\varphi(a_2\, a_1))$. Since φ is an embedding, Theorem 4.6 implies the existence of $u \in \Sigma_{\mathcal{L}_A}^*$ with $ua_1\, a_2 u \equiv_{\mathcal{L}_A} ua_2\, a_1 u$. It follows from Proposition 3.12 that these two words have the same sequence of write operations and therefore in particular $a_1\, a_2 = a_2\, a_1$. But this implies $a_1 = a_2$ which contradicts our choice of these two letters. Hence, indeed, $\varphi(\overline{a}) \in [\overline{B}^*]$ which proves the second claim.

Statement (ii) is shown by contradiction. Let $a \in A \setminus X$ with $\varphi(a) \notin [(B \setminus Y)^*]$. Since $|A| \geq 2$, there exists a distinct letter $b \in A \setminus \{a\}$. By (i) and the assumption on $\varphi(a)$, there exist words $u, v, w \in B^*$ and letters $y \in Y$ and $b_1, b_2, \ldots, b_n \in B$ with $n \geq 1$,

$$\varphi(a) = [uyv]\,, \quad \varphi(b) = [w]\,, \text{ and } \varphi(\bar{b}) = [\overline{b_1}\,\overline{b_2}\,\ldots\overline{b_n}].$$

We get

$$\begin{aligned}
\varphi(a^n b\bar{b}) &= [(uyv)^n\, w\, \overline{b_1}\,\overline{b_2}\,\ldots\,\overline{b_n}] \\
&= [uy\overline{b_1}v\, uy\overline{b_2}v\, uy\overline{b_3}v\,\ldots\, w] && \text{(by Lem. 3.9)} \\
&= [(uyv)^n\, \overline{b_1}\,\overline{b_2}\,\ldots\,\overline{b_n}\, w] && \text{(by Lem. 3.9 again)} \\
&= \varphi(a^n \bar{b} b)\,.
\end{aligned}$$

Due to the injectivity of φ, this implies $a^n b\bar{b} \equiv_{\mathcal{L}_A} a^n\bar{b}b$. Since these two words are in normal form, this contradicts Theorem 3.15. Thus, indeed, $\varphi(a) \in [(B \setminus Y)^*]$ for any $a \in A \setminus X$. □

We next come to property (3) that we prove for every embedding.

Lemma 5.8. *Let $\mathcal{L}_A = (A, X)$ and $\mathcal{L}_B = (B, Y)$ be two lossiness alphabets and φ an embedding of $\mathcal{Q}(\mathcal{L}_A)$ into $\mathcal{Q}(\mathcal{L}_B)$. Then we have $\varphi(x) \in [B^* Y B^*]$ for each $x \in X$.*

Proof. Let $x \in X$. Since $|A| \geq 2$, there is a distinct letter $a \in A \setminus \{x\}$. By Lemma 5.7(i), there are words $u, v, w \in B^+$ with

$$\varphi(a) = [u]\,, \quad \varphi(\overline{a}) = [\overline{v}]\,, \text{ and } \varphi(x) = [w]\,.$$

Since $x \in X \setminus \{a\}$, Lemma 3.9 implies

$$xa\overline{a} \equiv_{\mathcal{L}_A} x\overline{a}a \equiv_{\mathcal{L}_A} \overline{a}xa\,.$$

Hence we have $\varphi(xa\overline{a}) = \varphi(\overline{a}xa)$ and therefore $wu\overline{v} \equiv_{\mathcal{L}_B} \overline{v}wu$. Consequently

$$\mathsf{rd}_2(wu\overline{v}) = \mathsf{rd}_2(\overline{v}wu) = \varepsilon$$

since $\overline{v}wu \in \mathsf{NF}_{\mathcal{L}_B}$. As $a\overline{a}$ and $\overline{a}a$ are two distinct words in normal form, we have $a\overline{a} \not\equiv_{\mathcal{L}_A} \overline{a}a$ by Theorem 3.15. Due to the injectivity of φ, this implies $\varphi(a\overline{a}) \neq \varphi(\overline{a}a)$ and therefore $u\overline{v} \not\equiv_{\mathcal{L}_B} \overline{v}u$, *i.e.*, $\chi(u\overline{v}) \neq \chi(\overline{v}u)$. Since $\mathsf{wrt}(u\overline{v}) = \mathsf{wrt}(\overline{v}u)$ and $\mathsf{rd}(u\overline{v}) = \mathsf{rd}(\overline{v}u)$, we consequently get

$$\begin{aligned} \mathsf{rd}_2(u\overline{v}) &\neq \mathsf{rd}_2(\overline{v}u) && (\text{since } \chi(u\overline{v}) \neq \chi(\overline{v}u)) \\ &= \varepsilon && (\text{since } \overline{v}u \in \mathsf{NF}_{\mathcal{L}_B})\,. \end{aligned}$$

By Lemma 3.19, there exist a non?empty suffix $v' \neq \varepsilon$ of v and a prefix u' of u such that $v' \preceq_Y u'$. It follows that v' is a subword of wu', *i.e.*, $v' \preceq wu'$. Since $\mathsf{rd}_2(wu\overline{v}) = \varepsilon$, Lemma 3.19 implies $\pi_Y(wu') \neq \pi_Y(v') = \pi_Y(u')$, *i.e.*, w contains some letter from Y. Thus, we proved $\varphi(x) = [w] \in [B^*YB^*]$. □

Finally we obtain the remaining implication in Theorem 5.1.

Proposition 5.9. *Let $\mathcal{L}_A = (A, X)$ and $\mathcal{L}_B = (B, Y)$ be two lossiness alphabets such that $\mathcal{Q}(\mathcal{L}_A) \hookrightarrow \mathcal{Q}(\mathcal{L}_B)$. Then the Conditions (A)?(C) from Theorem 5.1 hold.*

Proof. First suppose $X \neq \emptyset$. Then, Lemma 5.8 implies $Y \neq \emptyset$, *i.e.*, we have (B).

Condition (A) is trivial if $A \setminus X = \emptyset$. If $A \setminus X$ is a singleton, then Lemma 5.7(ii) implies $B \setminus Y \neq \emptyset$ and therefore $|A \setminus X| \leq |B \setminus Y|$. So it remains to consider the case that $A \setminus X$ contains at least two elements. For $a \in A \setminus X$, we have $\varphi(\overline{a}) \in [\overline{B}^+]$ by Lemma 5.7(i). Hence there exist $v_a \in B^*$ and $b_a \in B$ with $\varphi(\overline{a}) = [\overline{v_a}\,\overline{b_a}]$. We will prove that the letters b_a for $a \in A \setminus X$ are mutually distinct.

So let $a_1, a_2 \in A \setminus X$ be distinct. By Lemma 5.7(ii), there exists $w \in (B \setminus Y)^+$ with $\varphi(a_1) = [w]$. Then we have

$$\begin{aligned} \mathsf{rd}_2(w\,\overline{v_{a_1}}\,\overline{b_{a_1}}) &= \mathsf{rd}_2(\varphi(a_1\,\overline{a_1})) \\ &\neq \mathsf{rd}_2(\varphi(\overline{a_1}\,a_1)) && (\text{since } \varphi \text{ is injective}) \\ &= \mathsf{rd}_2(\overline{v_{a_1}}\,\overline{b_{a_1}}\,w) \\ &= \varepsilon && (\text{since } \overline{v_{a_1}}\,\overline{b_{a_1}}\,w \in \mathsf{NF}_{\mathcal{L}_B})\,. \end{aligned}$$

By Lemma 3.19, the non-empty word $\mathsf{rd}_2(w\,\overline{v_{a_1}}\,\overline{b_{a_1}})$ is a suffix of $v_{a_1}b_{a_1}$ and a subword of w. In particular, b_{a_1} is a subword of w.

On the other hand, we get

$$\begin{aligned} \mathsf{rd}_2(w\,\overline{v_{a_2}}\overline{b_{a_2}}) &= \mathsf{rd}_2(\varphi(a_1\,\overline{a_2})) \\ &= \mathsf{rd}_2(\varphi(\overline{a_2}\,a_1)) && (\text{since } a_1 \neq a_2) \\ &= \mathsf{rd}_2(\overline{v_{a_2}}\overline{b_{a_2}}\,w) \\ &= \varepsilon && (\text{since } \overline{v_{a_2}}\overline{b_{a_2}}\,w \in \mathsf{NF}_{\mathcal{L}_B})\,. \end{aligned}$$

From Lemma 3.19, we therefore get in particular $b_{a_2} \not\preceq_Y w$. Since $w \in (B \setminus Y)^*$, this implies $b_{a_2} \not\preceq w$ and therefore $b_{a_1} \neq b_{a_2}$.

Thus, the mapping $A \setminus X \to B \setminus Y\colon a \mapsto b_a$ is injective implying $|A \setminus X| \leq |B \setminus Y|$, *i.e.*, we proved Condition (A).

To prove Condition (C), suppose $Y = \{y\}$ and $|A \setminus X| = |B \setminus Y|$. We will prove $|X| \leq 1$ by considering the last letters of $\mathsf{rd}(\varphi(\overline{x}))$ for $x \in X$. So let $x_1, x_2 \in X$. By Lemma 5.7(i), there exist $u \in B^*$ and $b \in B$ with $\varphi(\overline{x_2}) = [\overline{ub}]$. We distinguish the cases $b = y$ and $b \in B \setminus Y$.

First, let $b = y$, *i.e.*, $\varphi(\overline{x_2}) = [\overline{uy}]$. By Lemma 5.8, there exist $v \in (B \setminus Y)^*$ and $w \in B^*$ with $\varphi(x_1) = [vyw]$. Then $y \preceq_Y vyw$ implies

$$\begin{aligned}
\mathsf{rd}_2(\varphi(\overline{x_2}x_1)) &= \mathsf{rd}_2([\overline{uy}\, vyw]) \\
&= \varepsilon && \text{(since } \overline{uy}\, yvw \in \mathsf{NF}_{\mathcal{L}_B}\text{)} \\
&\neq \mathsf{rd}_2([vyw\overline{uy}]) && \text{(by Lem. 3.19 since } y \preceq_Y vyw\text{)} \\
&= \mathsf{rd}_2(\varphi(x_1\overline{x_2}))\,.
\end{aligned}$$

This implies $\varphi(\overline{x_2}x_1) \neq \varphi(x_1\overline{x_2})$. Now $\overline{x_2}x_1 \not\equiv_{\mathcal{L}_A} x_1\overline{x_2}$ follows from the injectivity of φ. Hence, Lemma 3.9 implies $x_1 = x_2$, *i.e.*, $|X| = 1$.

Finally suppose $b \in B \setminus Y$. Recall that the mapping $A \setminus X \to B \setminus Y \colon a \mapsto b_a$ from the verification of Condition (A) is injective. Since $|A \setminus X| = |B \setminus Y|$, there exists $a \in A \setminus X$ with $b = b_a$, *i.e.*, $\varphi(a) = [vb]$ for some $v \in (B \setminus Y)^*$. Note that b is a subword of vb with $\pi_Y(b) = \varepsilon = \pi_Y(vb)$, *i.e.*, $b \preceq_Y vb$. We therefore get

$$\begin{aligned}
\mathsf{rd}_2(\varphi(\overline{x_2}a)) &= \mathsf{rd}_2([\overline{ub}\, vb]) \\
&= \varepsilon && \text{(since } \overline{ub}\, bv \in \mathsf{NF}_{\mathcal{L}_B}\text{)} \\
&\neq \mathsf{rd}_2([vb\overline{ub}]) && \text{(by Lem. 3.19 since } b \preceq_Y vb\text{)} \\
&= \mathsf{rd}_2(\varphi(a\overline{x_2}))\,.
\end{aligned}$$

This implies $\varphi(\overline{x_2}a) \neq \varphi(a\overline{x_2})$. Now $\overline{x_2}a \not\equiv_{\mathcal{L}_A} a\overline{x_2}$ follows from the injectivity of φ. Thus, Lemma 3.9 implies $a = x_2$ but this contradicts $a \in A \setminus X$ and $x_2 \in X$. Hence the case $b \in B \setminus Y$ is not possible and therefore by the case above $|X| = 1$. □

6. Embeddings of trace monoids

Since, by Corollary 5.4 from [13] (alternatively, this follows from our generalization Thm. 5.1), all reliable queue monoids $\mathcal{Q}(A, A)$ with $|A| \geq 2$ embed into each other, they all have the same submonoids. Our Theorem 5.1 shows that this is not the case for all plq monoids $\mathcal{Q}(\mathcal{L})$ (*e.g.*, $\mathcal{Q}(A, A)$ and $\mathcal{Q}(A, \emptyset)$ are not submonoids of each other). This final section demonstrates a surprising similarity among all these monoids, namely the trace monoids contained in them.

These trace (or free partially commutative) monoids are used for modeling concurrent systems where the concurrency is governed by the use of joint resources (*cf.* [24]). Formally such a system is a so called *independence alphabet*, *i.e.*, a tuple (Γ, I) of a non-empty finite set Γ and a symmetric, irreflexive relation $I \subseteq \Gamma^2$, *i.e.*, (Γ, I) can be thought of as a finite, simple, undirected graph. The corresponding *dependence alphabet* (Γ, D) is the complementary graph of (Γ, I), *i.e.*, $D = \Gamma^2 \setminus I$. Given an independence alphabet (Γ, I), we define the relation $\equiv_I \subseteq (\Gamma^*)^2$ as the least congruence satisfying $ab \equiv_I ba$ for each $(a, b) \in I$. The induced *trace monoid* is $\mathbb{M}(\Gamma, I) := \Gamma^*/_{\equiv_I}$. While there is a rich theory of trace monoids (see, *e.g.*, [8, 9, 24]), here we only need the following basic characterization of the congruence $\equiv_I$:

Proposition 6.1 (Projection lemma, [5, 6])**.** *Let (Γ, I) be an independence alphabet and $u, v \in \Gamma^*$. Then $u \equiv_I v$ iff $\pi_{\{a,b\}}(u) = \pi_{\{a,b\}}(v)$ for each $(a, b) \in \Gamma^2 \setminus I$.*

We consider, as usual in trace theory, the independence alphabet (Γ, I) as a graph and therefore use graph theoretic terminology for its properties.

6.1. Large alphabets

The following theorem characterizes those trace monoids that embed into the plq monoid $\mathcal{Q}(\mathcal{L})$ provided $|A| + |X| \geq 3$. It shows that all these plq monoids contain the same trace monoids as submonoids.

Theorem 6.2. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet with $|A| + |X| \geq 3$. Furthermore let (Γ, I) be an independence alphabet. Then the following are equivalent:*

(A) $\mathbb{M}(\Gamma, I)$ *embeds into* $\mathcal{Q}(\mathcal{L})$.
(B) $\mathbb{M}(\Gamma, I)$ *embeds into* $\{a, b\}^* \times \{c, d\}^*$.
(C) One of the following conditions holds:
(C.a) All nodes in (Γ, I) *have degree* ≤ 1.
(C.b) The only non-trivial connected component of (Γ, I) *is complete bipartite.*

6.1.1. The implication "(B)⇒(A)" in Theorem 6.2

Let the independence alphabet (Γ, I) satisfy Condition (B), *i.e.*, $\mathbb{M}(\Gamma, I)$ embeds into $P = \{a, b\}^* \times \{c, d\}^*$. To show that then $\mathbb{M}(\Gamma, I)$ embeds into the plq monoid, it suffices to provide an embedding of P into the plq monoid. Note that the condition $|A| + |X| \geq 3$ is satisfied if and only if $|A| \geq 3$ or $|A| \geq 2$ and $X \neq \emptyset$. The following lemmas handle these two cases separately.

Lemma 6.3. *Let $\mathcal{L} = (A, \emptyset)$ be a lossiness alphabet with $|A| \geq 3$. Then $\{a, b\}^* \times \{c, d\}^*$ embeds into $\mathcal{Q}(\mathcal{L})$.*

Proof. Let $a_1, a_2, a_3 \in A$ be pairwise distinct letters. Then we define the homomorphism $\varphi' \colon \{a, b, c, d\}^* \to \Sigma_{\mathcal{L}}^*$ by

$$\varphi'(a) := a_1\,,\ \varphi'(b) := a_2\,,\ \varphi'(c) := \overline{a_1 a_3}\,,\ \text{and } \varphi'(d) := \overline{a_2 a_3}\,.$$

Using Lemma 3.9, one can easily verify $\varphi'(\alpha)\varphi'(\beta) \equiv_{\mathcal{L}} \varphi'(\beta)\varphi'(\alpha)$ for all $\alpha \in \{a, b\}$ and $\beta \in \{c, d\}$. Hence φ' induces a homomorphism $\varphi \colon \{a, b\}^* \times \{c, d\}^* \to \mathcal{Q}(\mathcal{L})$.

Since a_1, a_2, a_3 are pairwise distinct, the homomorphisms

$$\{a, b\}^* \to A^* \colon u \mapsto \mathsf{wrt}(\varphi(u, \varepsilon)) \text{ and } \{c, d\}^* \to A^* \colon v \mapsto \mathsf{rd}(\varphi(\varepsilon, v))$$

are injective. Hence φ is an embedding. □

Lemma 6.4. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet with $X \neq \emptyset$. Then $\{a, b\}^* \times \{c, d\}^*$ embeds into $\mathcal{Q}(\mathcal{L})$.*

Proof. Let $x \in X$ and $a_1 \in A \setminus \{x\}$. Then we define the homomorphism $\varphi' \colon \{a, b, c, d\}^* \to \Sigma_{\mathcal{L}}^*$ by

$$\varphi'(a) := x\,,\ \varphi'(b) := x a_1\,,\ \varphi'(c) := \overline{a_1}\,,\ \text{and } \varphi'(d) := \overline{x a_1 a_1}\,.$$

Note that, if $A = X$, then this is the embedding from Proposition 8.3 from [13]. Again by Lemma 3.9, we get $\varphi'(\alpha)\varphi'(\beta) \equiv_{\mathcal{L}} \varphi'(\beta)\varphi'(\alpha)$ for all $\alpha \in \{a, b\}$ and $\beta \in \{c, d\}$. Hence φ' induces a homomorphism $\varphi \colon \{a, b\}^* \times \{c, d\}^* \to \mathcal{Q}(\mathcal{L})$.

Since $a_1 \neq x$, the homomorphisms

$$\{a, b\}^* \to A^* \colon u \mapsto \mathsf{wrt}(\varphi(u, \varepsilon)) \text{ and } \{c, d\}^* \to A^* \colon v \mapsto \mathsf{rd}(\varphi(\varepsilon, v))$$

are injective. Hence φ is an embedding. □

From these two lemmas we can learn that $\{a, b\}^* \times \{c, d\}^*$ embeds into $\mathcal{Q}(\mathcal{L})$ whenever $|A| + |X| \geq 3$. Since the embedding relation $\hookrightarrow$ is transitive, each trace monoid that embeds into $\{a, b\}^* \times \{c, d\}^*$ embeds into $\mathcal{Q}(\mathcal{L})$, too. Hence we get the following statement.

Proposition 6.5. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet with $|A| + |X| \geq 3$. Furthermore let (Γ, I) be an independence alphabet such that $\mathbb{M}(\Gamma, I)$ embeds into $\{a, b\}^* \times \{c, d\}^*$. Then $\mathbb{M}(\Gamma, I)$ embeds into $\mathcal{Q}(\mathcal{L})$.*

6.1.2. *The implication "(C)⇒(B)" in Theorem 6.2*

Let (Γ, I) be an independence alphabet satisfying (C.a) or (C.b) of Theorem 6.2. We will prove that $\mathbb{M}(\Gamma, I)$ embeds into the direct product of two free monoids in both these cases. The following lemma considers the case that all nodes of (Γ, I) have degree at most one (the simpler case of (C.b) is considered in the proof of Prop. 6.7).

Lemma 6.6. *Let (Γ, I) be an independence alphabet such that all nodes in (Γ, I) have degree ≤ 1. Then $\mathbb{M}(\Gamma, I)$ embeds into the direct product of two countably infinite free monoids.*

Proof. It suffices the consider the case that all letters of (Γ, I) have degree 1, *i.e.*, $\Gamma = \{a_i, b_i \mid 1 \leq i \leq N\}$ and $I = \{(a_i, b_i), (b_i, a_i) \mid 1 \leq i \leq N\}$ for some $N \in \mathbb{N}$.

We consider the direct product

$$\mathcal{M} = \{c_i \mid 1 \leq i \leq N\}^* \times \{d_i \mid 1 \leq i \leq N\}^* .$$

Note that in this monoid (c_i, d_i) and $(c_i, d_i d_i)$ commute. Hence there is a homomorphism $\eta \colon \mathbb{M}(\Gamma, I) \to \mathcal{M}$ with $\eta(a_i) = (c_i, d_i)$ and $\eta(b_i) = (c_i, d_i d_i)$ for all $1 \leq i \leq N$.

To show that this homomorphism is injective, we use lexicographic normal forms. So let $\leq$ be a linear order on Γ with $a_i < b_i$ for all $1 \leq i \leq N$. Now let $u \in \Gamma^*$ be in lexicographic normal form wrt. $\leq$ (*i.e.*, u is the (length-)lexicographic smallest word in $[u]$). Then the word u has the form

$$u = a_{i_1}^{k_1} b_{i_1}^{\ell_1} a_{i_2}^{k_2} b_{i_2}^{\ell_2} \ldots a_{i_s}^{k_s} b_{i_s}^{\ell_s}$$

where $1 \leq i_a \leq N$, $k_a + \ell_a > 0$ for all $1 \leq a \leq s$ and $i_a \neq i_{a+1}$ for all $1 \leq a < s$.

The image of u equals

$$\eta(u) = \begin{pmatrix} c_{i_1}^{k_1+\ell_1} & c_{i_2}^{k_2+\ell_2} & \cdots & c_{i_s}^{k_s+\ell_s} \\ d_{i_1}^{k_1+2\ell_1} & d_{i_2}^{k_2+2\ell_2} & \cdots & d_{i_s}^{k_s+2\ell_s} \end{pmatrix} .$$

Next let also v be a word in lexicographic normal form:

$$v = a_{j_1}^{m_1} b_{j_1}^{n_1} a_{j_2}^{m_2} b_{j_2}^{n_2} \cdots a_{j_t}^{m_t} b_{j_t}^{n_t}$$

where $1 \leq j_a \leq N$, $m_a + n_a > 0$ for all $1 \leq a \leq t$ and $j_a \neq j_{a+1}$ for all $1 \leq a < t$.

The image of v equals

$$\eta(v) = \begin{pmatrix} c_{j_1}^{m_1+n_1} & c_{j_2}^{m_2+n_2} & \cdots & c_{j_t}^{m_t+n_t} \\ d_{j_1}^{m_1+2n_1} & d_{j_2}^{m_2+2_2} & \cdots & d_{j_t}^{m_t+2_t} \end{pmatrix} .$$

Suppose $\eta(u) = \eta(v)$. Since all the exponents of c_i and d_i in the expressions for $\eta(u)$ and for $\eta(v)$ are positive and consecutive c_i and d_i have distinct indices, we obtain $s = t$, $i_a = j_a$, $k_a + \ell_a = m_a + n_a$ and $k_a + 2\ell_a = m_a + 2n_a$ for all $1 \leq a \leq s$. Hence $k_a = m_a$ and $\ell_a = n_a$ for all $1 \leq a \leq s$ and therefore $u = v$. Hence η embeds $\mathbb{M}(\Gamma, I)$ into $\mathcal{M}$. ☐

Proposition 6.7. *Let (Γ, I) be an independence alphabet such that one of the following conditions holds:*

(i) All nodes in (Γ, I) have degree ≤ 1.
(ii) The only non-trivial connected component of (Γ, I) is complete bipartite.

Then $\mathbb{M}(\Gamma, I)$ embeds into $\{a, b\}^* \times \{c, d\}^*$.

Proof. Let (Γ, I) be such that the first condition holds, *i.e.*, all nodes in (Γ, I) have degree ≤ 1. Then by Lemma 6.6 there is an embedding of $\mathbb{M}(\Gamma, I)$ into a direct product of two finitely generated free monoids.

Now let (Γ, I) be such that the second condition holds, *i.e.*, (Γ, I) has only one non-trivial connected component and this component is complete bipartite. In other words, $\Gamma = \Gamma_1 \uplus \Gamma_2 \uplus \Gamma_3$ with $I = \Gamma_1 \times \Gamma_2 \cup \Gamma_2 \times \Gamma_1$. Then the corresponding dependence alphabet (Γ, D) can be covered by the two cliques induced by $\Gamma_1 \cup \Gamma_3$ and $\Gamma_2 \cup \Gamma_3$. Consequently, Corollary 1.4.5 from [8] (General Embedding Theorem) implies that $\mathbb{M}(\Gamma, I)$ is a submonoid of a direct product of two finitely generated free monoids.

Note that the finitely generated free monoid $\{a_i \mid 1 \leq i \leq n\}^*$ embeds into $\{a, b\}^*$ *via* $a_i \mapsto a^i b$. Hence, in any case, $\mathbb{M}(\Gamma, I)$ embeds into $\{a, b\}^* \times \{c, d\}^*$. $\square$

6.1.3. The implication "(A)⇒(C)" in Theorem 6.2

Recall that the general assumption in Theorem 6.2 was $|A| + |X| \geq 3$. We prove the implication "(A)⇒(C)" for all lossiness alphabets.

So let $\mathcal{L} = (A, X)$ be a lossiness alphabet, let (Γ, I) be an independence alphabet, and let $\varphi \colon \mathbb{M}(\Gamma, I) \hookrightarrow \mathcal{Q}(\mathcal{L})$ be an embedding. We partition the independence alphabet into the following three subsets:

$$\begin{aligned}
\Gamma_+ &:= \{\alpha \in \Gamma \mid \mathsf{rd}(\varphi(\alpha)) = \varepsilon, \mathsf{wrt}(\varphi(\alpha)) \neq \varepsilon\}\,, \\
\Gamma_- &:= \{\alpha \in \Gamma \mid \mathsf{wrt}(\varphi(\alpha)) = \varepsilon, \mathsf{rd}(\varphi(\alpha)) \neq \varepsilon\}\,, \text{ and} \\
\Gamma_\pm &:= \Gamma \setminus (\Gamma_+ \cup \Gamma_-)\,.
\end{aligned}$$

The crucial steps of this proof are to verify the following properties of the induced subgraphs of (Γ, I):

(i) $(\Gamma_+ \cup \Gamma_-, I)$ is complete bipartite with the partitions Γ_+ and Γ_- (*cf.* Lem. 6.8).
(ii) All nodes from $a \in \Gamma_+ \cup \Gamma_-$ are connected to any edge in I (*cf.* Lem. 6.9).
(iii) All nodes from $a \in \Gamma_\pm$ have degree ≤ 1 in the undirected graph (Γ, I) (*cf.* Lem. 6.11).
(iv) (Γ, I) is P_4-free, *i.e.*, the path of four vertices is no induced subgraph (*cf.* Lem. 6.12).

Afterwards we can prove that graphs satisfying these four properties, also satisfy (C) in Theorem 6.2.

Lemma 6.8. $(\Gamma_+ \cup \Gamma_-, I)$ *is complete bipartite with the partitions* Γ_+ *and* Γ_-.

Proof. We first prove the discreteness of (Γ_+, I):

Let $a, b \in \Gamma_+$ and suppose $(a, b) \in I$. Then there are words $u, v \in A^+$ with $\varphi(a) = [u]$ and $\varphi(b) = [v]$. Then $ab \equiv_I ba$ implies $uv \equiv_{\mathcal{L}} vu$ and therefore $uv = vu$ by Theorem 3.15. Hence there are a primitive word p and $m, n \in \mathbb{N}$ with $u = p^m$ and $v = p^n$. But then $u^n = v^m$ implying $\varphi(a^n) = [u^n] = [v^m] = \varphi(b^m)$. Now the injectivity of φ implies $a^n \equiv_I b^m$ and therefore $a = b$, contradicting $(a, b) \in I$. The proof of the discreteness of (Γ_-, I) uses words $u, v \in \overline{A}^+$ instead.

It remains to show $(a, b) \in I$ for arbitrary $a \in \Gamma_+$ and $b \in \Gamma_-$. There are $u, v \in A^+$ such that $\varphi(a) = [u]$ and $\varphi(b) = [\overline{v}]$. Set

$$t = \mathsf{rd}_2(u\overline{v}^{|u|+1})\,.$$

From Lemma 3.19, we obtain that t is a suffix of $v^{|u|+1}$ with $t \preceq_X u$. Consequently $|t| \leq |u| \leq |v^{|u|}|$, *i.e.*, t is even a suffix of $v^{|u|}$. Again by Lemma 3.19, this implies $t = \mathsf{rd}_2(u\overline{v}^{|u|})$.

Since all rules of the semi-Thue system $\mathcal{R}_{\mathcal{L}}$ move letters from $\overline{A}$ to the left, we get $\mathsf{nf}_{\mathcal{L}}(\overline{v}u\overline{v}^{|u|}) = \overline{v}\,\mathsf{nf}_{\mathcal{L}}(u\overline{v}^{|u|})$ and therefore

$$\mathsf{rd}_2(\overline{v}u\overline{v}^{|u|}) = \mathsf{rd}_2(u\overline{v}^{|u|}) = t = \mathsf{rd}_2(u\overline{v}^{|u|+1})\,.$$

Since $\mathsf{wrt}(\overline{v}u\overline{v}^{|u|}) = u = \mathsf{wrt}(u\overline{v}^{|u|+1})$ and $\mathsf{rd}(\overline{v}u\overline{v}^{|u|}) = v^{|u|+1} = \mathsf{rd}(u\overline{v}^{|u|+1})$, we obtain $\overline{v}u\overline{v}^{|u|} \equiv_{\mathcal{L}} u\overline{v}^{|u|+1}$. The injectivity of φ implies $bab^{|u|} \equiv_I ab^{|u|+1}$. Now Proposition 6.1 implies $(a, b) \in I$ since $a \neq b$. $\square$

Lemma 6.9. *Let $a \in \Gamma_+ \cup \Gamma_-$ and $b, c \in \Gamma$ with $(b, c) \in I$. Then $(a, b) \in I$ or $(a, c) \in I$.*

Proof. Since $\varphi(bc) = \varphi(cb)$ there are primitive words $p, q \in A^+$ and exponents $m_b, m_c, n_b, n_c \in \mathbb{N}$ with

$$\mathsf{wrt}(\varphi(b)) = p^{m_b}\,,\ \mathsf{rd}(\varphi(b)) = q^{n_b}\,,\ \mathsf{wrt}(\varphi(c)) = p^{m_c}\,,\ \text{and } \mathsf{rd}(\varphi(c)) = q^{n_c}\,.$$

According to the injectivity of φ we have $m_b + n_b \neq 0 \neq m_c + n_c$. Assume now that $a \in \Gamma_+$ (the other case is symmetric).

We first show that there are natural numbers x_b, x_c, y_b, y_c not all zero such that the following holds:

$$\left.\begin{array}{rcl} m_b \cdot x_b & = & m_c \cdot y_c \\ m_c \cdot x_c & = & m_b \cdot y_b \\ n_b \cdot x_b + n_c \cdot x_c & = & n_b \cdot y_b + n_c \cdot y_c \end{array}\right\} \tag{6.1}$$

If $m_b = 0$, then set $x_b = y_b = 1$ and $x_c = y_c = 0$. Symmetrically, if $m_c = 0$, we set $x_b = y_b = 0$ and $x_c = y_c = 1$. If $m_b n_c = m_c n_b$, then set $x_b = y_b = m_c + n_c > 0$ and $x_c = y_c = m_b + n_b > 0$.

Now consider the case $m_b \neq 0 \neq m_c$ and $m_b n_c \neq m_c n_b$. The system (6.1) has a non-trivial solution over the field $\mathbb{Q}$. Consequently, there are integers x_b, x_c, y_b, y_c (not all zero) satisfying these equations. We show $x_b > 0 \iff x_c > 0$: First note that $x_b \neq 0$ iff $y_c \neq 0$ and $x_c \neq 0$ iff $y_b \neq 0$. Since not all of the integers x_b, x_c, y_b, y_c are zero, we get $x_b \neq 0$ or $x_c \neq 0$. Furthermore, since we have a solution, we get

$$y_c = \frac{m_b}{m_c} x_b \qquad \text{and} \qquad y_b = \frac{m_c}{m_b} x_c\,.$$

Substituting these into the third equation yields

$$\left(n_b - n_c \frac{m_b}{m_c}\right) \cdot x_b = \left(n_b \frac{m_c}{m_b} - n_c\right) \cdot x_c = \left(n_b - n_c \frac{m_b}{m_c}\right) \cdot \frac{m_c}{m_b} \cdot x_c\,.$$

From $m_b n_c \neq m_c n_b$, we get $n_b - n_c \frac{m_b}{m_c} \neq 0$. Hence $x_b = \frac{m_c}{m_b} \cdot x_c$ and therefore $m_b x_b = m_c x_c$ follow. Now $m_b, m_c > 0$ imply $x_b > 0 \iff x_c > 0$. Consequently, all of x_b, x_c, y_b, y_c are non-negative or all are non-positive. Hence $|x_b|, |x_c|, |y_b|, |y_c|$ is a solution to the system (6.1) in natural numbers as required.

Now we have

$$\mathsf{wrt}(\varphi(b^{x_b} a c^{x_c})) = p^{m_b \cdot x_b} \varphi(a) p^{m_c \cdot x_c} \overset{(6.1)}{=} p^{m_c \cdot y_c} \varphi(a) p^{m_b \cdot y_b} = \mathsf{wrt}(\varphi(c^{y_c} a b^{y_b}))$$

and

$$\mathsf{rd}(\varphi(b^{x_b} a c^{x_c})) = q^{n_b \cdot x_b + n_c \cdot x_c} \overset{(6.1)}{=} q^{n_c \cdot y_c + n_b \cdot y_b} = \mathsf{rd}(\varphi(c^{y_c} a b^{y_b}))\,.$$

By Theorem 4.6, there is $z \in \Gamma^*$ with $z b^{x_b} a c^{x_c} z \equiv_I z c^{y_c} a b^{y_b} z$. Now Proposition 6.1 implies $(a, b) \in I$ or $(a, c) \in I$. □

Lemma 6.12 below will show that (Γ, I) does not contain a path on four vertices as induced subgraph, *i.e.*, that with any four distinct letters a, b, c, d with $(a, b), (b, c), (c, d) \in I$, one of the pairs (a, c), (b, d), or (a, d) belongs to the independence relation I. Note that $(a, c) \in I$ implies that (Γ, I) contains three mutually independent letters a, b, and c. The following lemma shows that this is not the case.

Lemma 6.10. *The graph (Γ, I) is triangle-free.*

Proof. Suppose, there are $a, b, c \in \Gamma$ such that $(a, b), (b, c), (c, a) \in I$. Then Lemma 6.8 implies that at least one of the letters a, b, and c belongs to $\Gamma_\pm$. Furthermore there are primitive words $p, q \in A^+$ and numbers $m_\alpha, n_\alpha \in \mathbb{N}$

with $\mathsf{wrt}(\varphi(\alpha)) = p^{m_\alpha}$ and $\mathsf{rd}(\varphi(\alpha)) = q^{n_\alpha}$ for each $\alpha \in \{a,b,c\}$. Since $\{(m_a,n_a),(m_b,n_b),(m_c,n_c)\}$ is linearly dependent, there are $x_a, x_b, x_c \in \mathbb{Q}$ not all zero such that

$$x_a \cdot \begin{pmatrix} m_a \\ n_a \end{pmatrix} + x_b \cdot \begin{pmatrix} m_b \\ n_b \end{pmatrix} + x_c \cdot \begin{pmatrix} m_c \\ n_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

We can assume $x_a, x_b \geq 0$ and $x_c < 0$ (if necessary, multiply all of x_a, x_b, x_c with -1 and exchange c with a or b). Multiplying all of x_a, x_b, x_c with a large natural number, we obtain a non-trivial natural solution of the following system:

$$\left.\begin{array}{rcl} x_a \cdot m_a + x_b \cdot m_b &=& x_c \cdot m_c \\ x_a \cdot n_a + x_b \cdot n_b &=& x_c \cdot n_c \end{array}\right\}$$

Then we have:

$$\mathsf{wrt}(\varphi(a^{x_a} b^{x_b})) = \mathsf{wrt}(\varphi(c^{x_c})) \text{ and } \mathsf{rd}(\varphi(a^{x_a} b^{x_b})) = \mathsf{rd}(\varphi(c^{x_c}))\,.$$

As in the previous proof, Theorem 4.6 implies the existence of $z \in \Gamma^*$ with $za^{x_a}b^{x_b}z \equiv_I zc^{x_c}z$. As the left-hand side of this equation contains less occurrences of c than the right-hand side, this contradicts Proposition 6.1. $\square$

Lemma 6.11. *Let $a \in \Gamma_\pm$. Then a has degree ≤ 1.*

Proof. Let $b, c \in \Gamma$ with $(a,b),(a,c) \in I$. We will prove $b = c$.

From $\varphi(ab) = \varphi(ba)$, we get $\mathsf{wrt}(\varphi(a))\mathsf{wrt}(\varphi(b)) = \mathsf{wrt}(\varphi(b))\mathsf{wrt}(\varphi(a))$. Hence there is a primitive word $p \in A^+$ with $\mathsf{wrt}(\varphi(a)), \mathsf{wrt}(\varphi(b)) \in p^*$. Since $\varphi(ac) = \varphi(ca)$, there is also a primitive word $q \in A^+$ with $\mathsf{wrt}(\varphi(a)), \mathsf{wrt}(\varphi(c)) \in q^*$. Hence $\mathsf{wrt}(\varphi(a)) \in p^* \cap q^*$. Since p and q are primitive words and since $\mathsf{wrt}(\varphi(a)) \neq \varepsilon$, we infer $p = q$. Consequently, the words $\mathsf{wrt}(\varphi(b))$ and $\mathsf{wrt}(\varphi(c))$ are both powers of this word $p = q$. Hence $\mathsf{wrt}(\varphi(bc)) = \mathsf{wrt}(\varphi(cb))$.

Similarly, we can show $\mathsf{rd}(\varphi(bc)) = \mathsf{rd}(\varphi(cb))$.

By Theorem 4.6, there is consequently $z \in \Gamma^*$ with $zbcz \equiv_I zcbz$. Now Proposition 6.1 implies $(b,c) \in I$ or $b = c$. Since the former contradicts Lemma 6.10, we have $b = c$, *i.e.*, the letter $a \in \Gamma_\pm$ has degree ≤ 1. $\square$

Recall that a graph (Γ, I) is P_4-free if the path on four vertices is not an induced subgraph. If the graph is triangle-free, this is equivalent to saying that for any four distinct vertices $a,b,c,d \in \Gamma$ with $(a,b),(b,c),(c,d) \in I$ we have $(a,d) \in I$.

Lemma 6.12. *(Γ, I) is P_4-free.*

Proof. Let $a,b,c,d \in \Gamma$ be pairwise distinct letters with $(a,b),(b,c),(c,d) \in I$. Lemma 6.11 implies $b,c \in \Gamma_+ \cup \Gamma_-$. We can assume that $b \in \Gamma_+$ and $c \in \Gamma_-$ by Lemma 6.8. Then there are primitive words $p,q \in A^+$ and numbers $m_a, m_b, n_c, n_d \in \mathbb{N}$ such that $\mathsf{wrt}(\varphi(a)) = p^{m_a}$, $\mathsf{wrt}(\varphi(b)) = p^{m_b}$, $\mathsf{rd}(\varphi(c)) = q^{n_c}$, $\mathsf{rd}(\varphi(d)) = q^{n_d}$.

First we note that there are natural numbers $x_a, x_b, x_c, x_d \in \mathbb{N}$ with $x_a, x_d \neq 0$ that satisfy the following system of linear equations

$$\left.\begin{array}{rl} x_a \cdot m_a &= x_b \cdot m_b \\ x_c \cdot n_c &= x_d \cdot n_d \end{array}\right\} \tag{6.2}$$

Therefore we get

$$\begin{aligned} \mathsf{wrt}(\varphi(c^{x_c} d a^{x_a} d^{x_d} b^{x_b})) &= \mathsf{wrt}(\varphi(d)) p^{x_a \cdot m_a} \mathsf{wrt}(\varphi(d^{x_d})) p^{x_b \cdot m_b} \\ &= \mathsf{wrt}(\varphi(d)) p^{x_b \cdot m_b} \mathsf{wrt}(\varphi(d^{x_d})) p^{x_a \cdot m_a} && \text{(by (6.2))} \\ &= \mathsf{wrt}(\varphi(d b^{x_b} d^{x_d} a^{x_a} c^{x_c}))\,, \end{aligned}$$

and

$$\begin{aligned}\mathsf{rd}(\varphi(c^{x_c}da^{x_a}d^{x_d}b^{x_b})) &= q^{x_c\cdot n_c+n_d}\mathsf{rd}(\varphi(a_a^x))q^{x_d\cdot n_d}\\ &= q^{x_d\cdot n_d+n_d}\mathsf{rd}(\varphi(a_a^x))q^{x_c\cdot n_c} && \text{(by (6.2))}\\ &= \mathsf{rd}(\varphi(db^{x_b}d^{x_d}a^{x_a}c^{x_c})).\end{aligned}$$

Hence, by Theorem 4.6, there exists $z \in \Gamma^*$ with

$$z\, c^{x_c}da^{x_a}d^{x_d}b^{x_b}\, z \equiv_I z\, db^{x_b}d^{x_d}a^{x_a}c^{x_c}\, z\,.$$

Since $x_a, x_d > 0$, Proposition 6.1 implies $(a,d) \in I$. Consequently, the letters a, b, c, d do not induce P_4 in (Γ, I). $\square$

Proposition 6.13. *Let $\mathcal{L} = (A, X)$ be a lossiness alphabet and (Γ, I) be an independence alphabet such that $\mathbb{M}(\Gamma, I)$ embeds into $\mathcal{Q}(\mathcal{L})$. Then one of the following hold:*

(i) All nodes in (Γ, I) have degree ≤ 1.
(ii) The only non-trivial connected component of (Γ, I) is complete bipartite.

Proof. Suppose (Γ, I) contains a node a of degree ≥ 2. Then, by Lemma 6.11, $a \in \Gamma_+ \cup \Gamma_-$. From Lemma 6.9, we obtain that a belongs to the only non-trivial connected component C of (Γ, I). Note that $|C| \geq 3$ since it contains a and its ≥ 2 neighbors. Hence the induced subgraph (C, I) contains at least one edge. Therefore Lemma 6.9 implies $\Gamma_+ \cup \Gamma_- \subseteq C$. Note that all nodes in $C \setminus (\Gamma_+ \cup \Gamma_-)$ have degree 1 by Lemma 6.11. Hence, by Lemma 6.8, the connected graph (C, I) is a complete bipartite graph together with some additional nodes of degree 1. It follows that (C, I) is bipartite. By Lemma 6.12, it is a connected and P_4-free graph. Hence its complementary graph (C, D) is not connected [26]. But this implies that (C, I) is complete bipartite. $\square$

6.2. The binary alphabet

In Theorem 6.2 we have only considered partially lossy queues with $|A| > 2$ or $|X| \neq 0$. For a complete picture, it remains to consider the case $|A| = 2$ and $|X| = 0$. The following theorem implies in particular that $\mathcal{Q}(\{\alpha, \beta\}, \emptyset)$ does not contain the direct product of two free 2-generated monoids, *i.e.*, it contains properly less trace monoids than $\mathcal{Q}(\mathcal{L})$ with $|A| + |X| \geq 3$.

Theorem 6.14. *Let $\mathcal{L} = (A, \emptyset)$ be a lossiness alphabet with $|A| = 2$ and (Γ, I) be an independence alphabet. Then the following are equivalent:*

(A) $\mathbb{M}(\Gamma, I)$ embeds into $\mathcal{Q}(A, \emptyset)$.
(B) One of the following conditions holds:
(B.1) All nodes in (Γ, I) have degree ≤ 1.
(B.2) The only non-trivial connected component of (Γ, I) is a star graph.

The two implications of this theorem are demonstrated separately in Propositions 6.15 and 6.18, respectively.

Proposition 6.15. *Let $\mathcal{L} = (A, \emptyset)$ be a lossiness alphabet with $|A| = 2$, (Γ, I) be an independence alphabet, and $\varphi\colon \mathbb{M}(\Gamma, I) \hookrightarrow \mathcal{Q}(A, \emptyset)$. Then one of the following hold:*

(i) All nodes in (Γ, I) have degree ≤ 1.
(ii) The only non-trivial connected component of (Γ, I) is a star graph.

Proof. Suppose (Γ, I) has a node of degree ≥ 2. Then, by Proposition 6.13, the only non-trivial connected component C of (Γ, I) is complete bipartite and has at least 3 elements. Towards a contradiction, suppose (C, I) is not a star graph. Since it is complete bipartite, all nodes of C have degree at least 2. Hence, by

Lemma 6.11, $C \subseteq \Gamma_+ \cup \Gamma_-$ and $|\Gamma_+|, |\Gamma_-| \geq 2$. Let $a, b \in \Gamma_+$ with $a \neq b$ and $c \in \Gamma_-$ such that, by Lemma 6.8, we have $(a, c), (c, b) \in I$.

There are words $u_a, u_b, u_c \in A^+$ with

$$\varphi(a) = [u_a]\,,\ \varphi(b) = [u_b]\,,\ \text{and } \varphi(c) = [\overline{u_c}]\,.$$

From $abc \equiv_I cab$, we get

$$\mathsf{rd}_2(u_a u_b \overline{u_c}) = \mathsf{rd}_2(\varphi(abc)) = \mathsf{rd}_2(\varphi(cab)) = \mathsf{rd}_2(\overline{u_c} u_a u_b) = \varepsilon$$

since the word $\overline{u_c} u_a u_b$ is in normal form. Hence, by Lemma 3.19, no non-trivial suffix of u_c is a subword of $u_a u_b$. With $\alpha \in A$ *not* the last letter of u_c, this implies $u_a, u_c \in \alpha^+$ since $|A| = 2$. But this implies

$$\varphi(ab) = [u_a u_b] = [u_b u_a] = \varphi(ba)$$

and therefore $ab \equiv_I ba$ since φ is injective. This implies $(a, b) \in I$, contradicting Lemma 6.8. □

We prove the converse direction separately for the two cases from Condition (B).

Lemma 6.16. *Let $\mathcal{L} = (A, \emptyset)$ be a lossiness alphabet with $|A| = 2$ and let (Γ, I) be an independence alphabet such that each node has degree ≤ 1. Then $\mathbb{M}(\Gamma, I)$ embeds into $\mathcal{Q}(A, \emptyset)$.*

Proof. Let $A = \{\alpha, \beta\}$. It suffices to consider the case that all nodes of (Γ, I) have degree 1. So let $\Gamma = \{a_i, b_i \mid 1 \leq i \leq n\}$ and $I = \{(a_i, b_i), (b_i, a_i) \mid 1 \leq i \leq n\}$. Then we define $w_i = \alpha^i \beta$ for $1 \leq i \leq n$.

Let $\varphi' \colon \Gamma^* \to \Sigma_{\mathcal{L}}^*$ be the homomorphism with $\varphi'(a_i) = \overline{w_i} w_i$ and $\varphi'(b_i) = \overline{w_i}\,\overline{w_i} w_i$.

It remains to be shown that $u \equiv_I v \iff \varphi'(u) \equiv_{(A,\emptyset)} \varphi'(v)$ for all $u, v \in \Gamma^*$.

For the first implication "$\Rightarrow$", it suffices to show $\varphi'(a_i b_i) \equiv_I \varphi'(b_i a_i)$ for all $1 \leq i \leq n$. From Lemma 3.19, we obtain

$$\mathsf{rd}_2(w_i \overline{w_i} \overline{w_i}) = w_i = \mathsf{rd}_2(w_i \overline{w_i})\,.$$

Since all rules of the semi-Thue system $\mathcal{R}_{(A,\emptyset)}$ try to move letters from $\overline{A}$ to the left, we also have

$$\begin{aligned}\mathsf{nf}_{(A,\emptyset)}(\overline{w_i} w_i \overline{w_i} \overline{w_i} w_i) &= \overline{w_i}\,\mathsf{nf}_{(A,\emptyset)}(w_i \overline{w_i} \overline{w_i})\, w_i \text{ and}\\ \mathsf{nf}_{(A,\emptyset)}(\overline{w_i} \overline{w_i} w_i \overline{w_i} w_i) &= \overline{w_i} \overline{w_i}\,\mathsf{nf}_{(A,\emptyset)}(w_i \overline{w_i})\, w_i\,.\end{aligned}$$

Hence

$$\varphi'(a_i b_i) = \overline{w_i} w_i \overline{w_i} \overline{w_i} w_i \equiv_{(A,\emptyset)} \overline{w_i} \overline{w_i} w_i \overline{w_i} w_i = \varphi'(b_i a_i)$$

follows from Theorem 3.15.

For the converse implication "$\Leftarrow$" suppose $\varphi'(u) \equiv_{(A,\emptyset)} \varphi'(v)$. Then, by Proposition 3.12, we have $\mathsf{wrt}(\varphi'(u)) = \mathsf{wrt}(\varphi'(v))$ and $\mathsf{rd}(\varphi'(u)) = \mathsf{rd}(\varphi'(v))$. For $1 \leq i < j \leq n$, we therefore get

$$\pi_{\{a_i,a_j\}}(u) = \pi_{\{a_i,a_j\}}(v)\,, \pi_{\{b_i,b_j\}}(u) = \pi_{\{b_i,b_j\}}(v)\,,\ \text{and } \pi_{\{a_i,b_j\}}(u) = \pi_{\{a_i,b_j\}}(v)\,.$$

Hence Proposition 6.1 implies $u \equiv_I v$. □

Lemma 6.17. *Let $\mathcal{L} = (A, \emptyset)$ be a lossiness alphabet with $|A| = 2$ and let (Γ, I) be an independence alphabet such that its only non-trivial connected component is a star graph. Then $\mathbb{M}(\Gamma, I)$ embeds into $\mathcal{Q}(A, \emptyset)$.*

Proof. Let $A = \{\alpha, \beta\}$. Let c be the center of the star graph, s_i for $1 \le i \le m$ its neighbors, and r_i for $1 \le i \le n$ the isolated nodes of (Γ, I). Then $I = \{(c, s_i), (s_i, c) \mid 1 \le i \le m\}$ and $\Gamma = \{c, s_i, r_j \mid 1 \le i \le m, 1 \le j \le n\}$.

Then we define the homomorphism $\varphi' \colon \Gamma^* \to (A \cup \overline{A})^*$ by

$$\varphi'(c) = [\alpha]\,,\ \varphi'(s_i) = [\overline{\alpha^i \beta}]\,,\ \text{and } \varphi'(r_j) := [\overline{\alpha^j \beta^2} \beta]\,.$$

The proof of $u \equiv_I v \iff \varphi'(u) \equiv_{(A,\emptyset)} \varphi'(v)$ is similar to the corresponding proof in Lemma 6.16. □

Finally we can summarize the last two lemmas:

Proposition 6.18. *Let $\mathcal{L} = (A, \emptyset)$ be a lossiness alphabet with $|A| = 2$ and (Γ, I) be an independence alphabet such that one of the following hold:*

(i) All nodes in (Γ, I) have degree ≤ 1.
(ii) The only non-trivial connected component of (Γ, I) is a star graph.

Then $\mathbb{M}(\Gamma, I)$ embeds into $\mathcal{Q}(A, \emptyset)$.

7. Further research and open problems

We think that our model of partially lossy queues helps to argue about properties of both, reliable and lossy queues, at the same time. This could result in the unification of proofs which finally are cleaner and easier to understand.

In [19], the first author gives some algorithmic properties on the rational subsets in the plq monoid. Additionally, that paper contains Kleene- and Büchi-type characterizations of the recognizable subsets and Schützenberger- and McNaughton & Papert-type characterizations of the aperiodic subsets in this monoid.

An open question is whether the plq monoid is automatic. From [13] we know that the reliable queue monoid is neither automatic in the sense of Khoussainov-Nerode [17] nor in the sense of Thurston *et al.* [3] and we think that this also holds for arbitrary plq monoids. Though it is still not clear whether the plq monoid (incl. the reliable queue monoid) is automatic in the sense of [16], *i.e.*, whether the plq monoid's Cayley graph is automatic.

Another open question concerns automata using plqs as their storage mechanism. Such automata can possibly be studied as valence automata with target [12, 25].

Acknowledgements. We like to thank the reviewers of this paper for their helpful suggestions.

References

[1] P.A. Abdulla and B. Jonsson, Verifying programs with unreliable channels. *Inf. Comput.* **127** (1996) 91–101.
[2] D. Brand and P. Zafiropulo, On communicating finite-state machines. *J. ACM* **30** (1983) 323–342.
[3] J.W. Cannon, D.B.A. Epstein, D.F. Holt, S.V.F. Levy, M.S. Paterson and W.P. Thurston, Word Processing in Groups. Jones and Barlett Publ., Boston, MA (1992).
[4] G. Cécé, A. Finkel and S. Purushotaman Iyer, Unreliable channels are easier to verify than perfect channels. *Inf. Comput.* **124** (1996) 20–31.
[5] M. Clerbout and M. Latteux, Partial commutations and faithful rational transductions. *Theor. Comput. Sci.* **34** (1984) 241–254.
[6] R. Cori and D. Perrin, Automates et commutations partielles. *RAIRO: ITA* **19** (1985) 21–32.
[7] P. Chambart and P. Schnoebelen, The ordinal recursive complexity of lossy channel systems, in *LICS'08*. IEEE Computer Society Press (2008) 205–216.
[8] V. Diekert, Combinatorics on Traces. Vol. 454 of *Lecture Notes in Computer Science*. Springer (1990).
[9] V. Diekert and G. Rozenberg, The Book of Traces. World Scientific (1995).
[10] V. Diekert, A. Muscholl and K. Reinhardt, On codings of traces, in *STACS 95*. Vol. 900 of *Lecture Notes in Computer Science*. Springer, Berlin, Heidelberg (1995) 385–396.
[11] A. Finkel, Decidability of the termination problem for completely specified protocols. *Distrib. Comput.* **7** (1994) 129–135.
[12] H. Fernau and R. Stiebe, Sequential grammars and automata with valences. *Theor. Comput. Sci.* **276** (2002) 377–405.
[13] M. Huschenbett, D. Kuske and G. Zetzsche, The monoid of queue actions. *Semigroup Forum* **95** (2017) 475–508.
[14] C. Haase, S. Schmitz and P. Schnoebelen, The power of priority channel systems. *Log. Methods Comput. Sci.* **10** (2014).

[15] M. Kambites, Formal languages and groups as memory. *Commun. Algebra* **37** (2009) 193–208.
[16] O. Kharlampovich, B. Khoussainov and A. Miasnikov, From automatic structures to automatic groups. *Groups Geometry Dyn.* **8** (2014) 157–198.
[17] B. Khoussainov and A. Nerode, Automatic presentations of structures, in Logic and Computational Complexity. Vol. 960 of *Lecture Notes in Computer Science*. Springer (1995) 367–392.
[18] C. Köcher, *Einbettungen in das Transformationsmonoid einer vergesslichen Warteschlange*. Master's Thesis, TU Ilmenau (2016).
[19] C. Köcher, Rational, recognizable, and aperiodic sets in the partially lossy queue monoid, in *STACS'18*. Vol. 96 of *LIPIcs*. Dagstuhl Publishing (2018) 45:1–45:14.
[20] C. Köcher and D. Kuske, The transformation monoid of a partially lossy queue, in *CSR'17*. Vol. 10304 of *Lecture Notes in Computer Science*. Springer (2017) 191–205.
[21] M. Kunc, Undecidability of the trace coding problem and some decidable cases. *Theor. Comput. Sci.* **310** (2004) 393–456.
[22] D. Kuske and O. Prianychnykova, The trace monoids in the queue monoid and in the direct product of two free monoids, in *DLT'16*. Vol. 9840 of *Lecture Notes in Computer Science*. Springer (2016) 256–267.
[23] B. Masson and P. Schnoebelen, On verifying fair lossy channel systems, in *MFCS'02*. Vol. 2420 of *Lecture Notes in Computer Science*. Springer (2002) 543–555.
[24] A. Mazurkiewicz, Concurrent program schemes and their interpretations. *DAIMI Rep. Ser.* **6** (1977).
[25] E. Render and M. Kambites, Rational subsets of polycyclic monoids and valence automata. *Inf. Comput.* **207** (2009) 1329–1339.
[26] D. Seinsche, On a property of the class of n-colorable graphs. *J. Comb. Theory (B)* **16** (1974) 191–193.