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GENERALIZATION OF THE TOTAL OUTER-CONNECTED DOMINATION IN GRAPHS

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Abstract. Let $G = (V, E)$ be a graph without an isolated vertex. A set $S \subseteq V$ is a total dominating set if S is a dominating set, and the induced subgraph $G[S]$ does not contain an isolated vertex. The total domination number of G is the minimum cardinality of a total dominating set of G. A set $D \subseteq V$ is a total outer-connected dominating set if D is a total dominating set, and the induced subgraph $G[V - D]$ is connected. The total outer-connected domination number of G is the minimum cardinality of a total outer-connected dominating set of G. In this paper we generalize the total outer-connected domination number in graphs. Let $k \geq 1$ be an integer. A set $D \subseteq V$ is a total outer-k-connected component dominating set if D is a total dominating and the induced subgraph $G[V - D]$ has exactly k connected component(s). The total outer-k-connected component domination number of G, denoted by $\gamma_{tc}^k(G)$, is the minimum cardinality of a total outer-k-connected component dominating set of G. We obtain several general results and bounds for $\gamma_{tc}^k(G)$, and we determine exact values of $\gamma_{tc}^k(G)$ for some special classes of graphs G.

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1. Introduction

For notation and terminology in general we follow [4]. Let $G = (V, E)$ be a simple graph of order $n = |V(G)| = |V|$ and size $e = |E(G)| = |E|$. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$ or just $N(v)$, and its *closed neighborhood* by $N_G[v] = N[v]$. For a vertex set $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$. The *degree* $\deg(x)$ of a vertex x denotes the number of neighbors of x in G. The *maximum degree* and *minimum degree* of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The *eccentricity* of a vertex is the greatest distance between it and any other vertex. The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity among all vertices of G. A set of vertices S in G is a *dominating set*, if $N[S] = V$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. If S is a subset of V then we denote by $G[S]$ the subgraph of G induced by S. A dominating set S of G is a *total dominating set* if $G[S]$ has no isolated

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vertex. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G.

Total outer-connected domination in graphs was introduced by Cyman in [1]. If G is without an isolated vertex, then a set $D \subseteq V$ is a total outer-connected dominating set (TOCDS) of G if D is a total dominating set of G and the subgraph induced by $V \setminus D$ is connected. The minimum cardinality of a total outer-connected dominating set in G is the *total outer-connected domination number* denoted $\gamma_{tc}(G)$. A minimum TOCDS of a graph G is called a $\gamma_{tc}(G)$-set. Cyman in [1], Hattingh and Joubert in [3] obtained a lower bound for the total outer-connected domination number of a tree in terms of the order of the tree, and characterized trees achieving equality. Cyman and Raczek in [2] characterized trees with equal total domination and total outer-connected domination numbers. They also gave a lower bound for the total outer-connected domination number of a tree in terms of the order and the number of leaves of the tree, and characterized extremal trees. Jiang and Kang in [5] studied Nordhaus−Gaddum Typebounds for the total outer-connected domination number of a graph.

We generalize the total outer-connected domination number of a graph. Let G be a graph with no isolated vertex. For an integer $k \geq 1$, a subset S of the vertices of G is a *total outer-k-connected component dominating set*, or just TOkCDS, if S is a total dominating set of G and $G[V - S]$ has k connected components. The *total outer-k-connected component domination number* of G, denoted by $\gamma_{tc}^k(G)$, is the minimum cardinality of a TOkCDS of G. In the case that there is no TOkCDS of G, we define $\gamma_{tc}^k(G) = 0$. We also refer a $\gamma_{tc}^k(G)$-set in a graph G as a TOkCDS of cardinality $\gamma_{tc}^k(G)$. Note that a TOCDS S is a TO1CDS if $|S| < |V|$, and thus the concept of total outer-k-connected component domination is a generalization of the concept of total outer-connected domination.

In Section 2, we present some general results and bounds for the total outer-k-connected component domination number of graphs. In Section 3, we determine exact values of the total outer-k-connected component domination number for some special classes of graphs.

All graphs we consider in this paper are without isolated vertices and have at least three vertices. We recall that a leaf in a graph is a vertex of degree one, and a support vertex is one that is adjacent to a leaf. A pendant edge is an edge which at least one of its end-points is a leaf. We denote by $L(G)$ and $S(G)$ the set of all leaves and all support vertices of G, respectively.

With K_n we denote the *complete graph* on n vertices, with P_n the *path* on n vertices, with C_n the *cycle* of length n, and with W_n the *wheel* with $n + 1$ vertices. A *bipartite graph* is a graph whose vertex set can be partitioned into two sets of pair-wise non-adjacent vertices. We denote by $K_{m,n}$ the *complete bipartite graph* which one partite set has cardinality m and the other partite set has cardinality n. The *corona* $cor(G)$ of a graph G is the graph obtained from G by adding a pendant edge to any vertex of G. By $\alpha(G)$ we denote the *independence number* of a graph G.

2. General results and bounds

We begin with the following observation.

Observation 2.1. *Let $k \geq 1$ be an integer, and let G be a graph without isolated vertices. If $0 < \gamma_{tc}^k(G) < n$, then $\alpha(G) \geq k$, and $\delta(G) \leq n - k$.*

Proof. Assume that $0 < \gamma_{tc}^k(G) < n$ for some integer k. Let S be a $\gamma_{tc}^k(G)$-set, and $G_1, G_2, \dots, G_k$ be the components of $G[V - S]$. Let x_i be a vertex in $V(G_i)$ for $i = 1, 2, \dots, k$. Then clearly $\{x_1, x_2, \dots, x_k\}$ is an independent set, implying that $\alpha(G) \geq k$. To complete the proof, note that, since x_1 is not adjacent to any x_i, $i = 2, 3, \dots, k$, then $\delta(G) \leq \deg(x_1) \leq (n - 1) - (k - 1) = n - k$. □

Lemma 2.2. *If $\gamma_{tc}^k(G) = 0$ for some integer k, then for every $m > k$, $\gamma_{tc}^m(G) = 0$.*

Proof. Let $\gamma_{tc}^k(G) = 0$ for some integer k and $m > k$ be an integer. Suppose to the contrary that $\gamma_{tc}^m(G) \neq 0$. Let S be a $\gamma_{tc}^m(G)$-set, and let $G_1, G_2, \dots,$ and G_m be m connected components of $G[V - S]$. It is obvious that

$S_1 = S \cup V(G_{k+1}) \cup \ldots \cup V(G_m)$ is a TOkCDS for G and $G[V - S_1]$ has k connected components. This implies that $\gamma_{tc}^k(G) > 0$, a contradiction. $\square$

Lemma 2.3. *Let k be the maximum integer such that $\gamma_{tc}^k(G) > 0$. If S is a TOkCDS, then every connected component of $G[V - S]$ is a complete graph.*

Proof. Let k be the maximum integer such that $\gamma_{tc}^k(G) > 0$, and let S be a TOkCDS. Suppose to the contrary that there is a connected component G_1 of $G[V - S]$ such that G_1 is not complete. Let x, y be two non-adjacent vertices in G_1. Then $S \cup (V(G_1) - \{x, y\})$ is a TO$(k+1)$CDS for G, a contradiction. $\square$

Lemma 2.4. *If a graph G has a TOkCDS, then it has a TOtCDS for any integer $t < k$.*

Proof. Let S be a TOkCDS for a graph G, where $k > 1$, and let $G_1, G_2, \ldots, G_k$ be the components of $G[V - S]$. Let $t < k$. Then $S \cup V(G_1) \cup V(G_2) \cup \ldots \cup V(G_{k-t})$ is a TOtCDS for G. $\square$

Lemma 2.5. *Let G be a connected graph. If k is the maximum integer such that $\gamma_{tc}^k(G) > 0$, then* $\operatorname{diam}(G) \leq 3k - 1$.

Proof. If k is the maximum integer such that $\gamma_{tc}^k(G) > 0$, then $\gamma_{tc}^r(G) = 0$ for each $r \geq k + 1$. Suppose to the contrary that $\operatorname{diam}(G) \geq 3k$. Let $x_0x_1x_2 \ldots x_d$ be a diametrical path in G such that $d = 3p + t$ with an integer $0 \leq t \leq 2$, and let L_i be the set of leaves of G adjacent to x_i for $1 \leq i \leq d - 1$. Let B be the subset of vertices x_{3i} such that $|L_{3i}| = 0$ for $i = 1, 2, \ldots, p - 1$, and define the set A by

$$A = \{x_0, x_3, \ldots, x_{3(p-1)}, x_d\} \bigcup_{i=1}^{p-1} L_{3i} \setminus B.$$

Then $S = V \setminus A$ is a TO$(p+1)$CDS for G. Since $p + 1 \geq k + 1$, we obtain a contradiction to the hypothesis, and the proof is complete. $\square$

Theorem 2.6. *Let G be a connected graph G of order $n \geq 3$. Then $\gamma_{tc}^2(G) = 0$ if and only if $G \in \{P_3, C_4, C_5, K_n\}$.*

Proof. First notice that $\gamma_{tc}^1(K_n) = \gamma_{tc}^1(P_3) = \gamma_{tc}^1(C_4) = 2$, $\gamma_{tc}^1(C_5) = 3$, and $\gamma_{tc}^k(K_n) = \gamma_{tc}^k(P_3) = \gamma_{tc}^k(C_4) = \gamma_{tc}^k(C_5) = 0$ for any $k \geq 2$. Let G be a graph of order at least three and $\gamma_{tc}^2(G) = 0$. Since G is connected, we have $\gamma_{tc}^1(G) > 0$. By Lemma 2.5, $\operatorname{diam}(G) \leq 2$. If $\operatorname{diam}(G) = 1$, then clearly G is a complete graph. Thus assume that $\operatorname{diam}(G) = 2$. Let x, y be two diametrical vertices with $d(x, y) = \operatorname{diam}(G) = 2$.

Assume first that $\deg(x) \geq 3$. We show that $G[N(x)]$ is complete. Assume that there are two non-adjacent vertices a, b in $N(x)$. Since $V - \{a, b\}$ is not a TO2CDS for G, we obtain that there is a vertex z such that $N(z) \subseteq \{a, b\}$. If $z \neq y$, then $V - \{y, z\}$ is a TO2CDS for G, a contradiction. So $z = y$. Let $c \in N(x) - \{a, b\}$. Then $V - \{y, c\}$ is a TOkCDS for some $k \geq 2$, and by Lemma 2.4, G has a TO2CDS, a contradiction. We deduce that $G[N(x)]$ is complete. Now $N(x)$ is a TO2CDS for G, a contradiction. Thus $\deg(x) \leq 2$. We also have $\deg(y) \leq 2$. First assume that $\deg(x) = 1$. Let $w \in N(x)$. If $\deg(w) \geq 3$, then $V - \{x, y\}$ is a TO2CDS for G, a contradiction. Thus $\deg(w) = 2$, and so $G = P_3$. Assume thus that $\deg(x) = 2$ and $\deg(y) = 2$. Let $N(x) = \{a, w\}$, where $w \in N(y)$. If $a \in N(w)$, then $V - \{x, y\}$ is a TO2CDS for G, a contradiction. So $a \notin N(w)$. If there is a vertex $z \in N(a) - \{x, y\}$ such that $z \notin N(y)$, then $V - \{y, z\}$ is a TO2CDS for G, a contradiction. Thus each vertex of $N(a) - \{x, y\}$ is adjacent to y. Similarly, each vertex of $N(w) - \{x, y\}$ is adjacent to y. If $|N(a) - \{x, y\}| \geq 2$ or $|N(w) - \{x, y\}| \geq 2$, then $V - \{x, z\}$ is a TO2CDS for G, where $z \in N(a) - \{x, y\}$ or $z \in N(w) - \{x, y\}$, a contradiction. Thus $|N(a) - \{x, y\}| \leq 1$ and $|N(w) - \{x, y\}| \leq 1$. Let $N(a) - \{x, y\} = \{z\}$. If $a \in N(y)$, then $V - \{x, z\}$ is a TO2CDS for G, a contradiction. So assume that $a \notin N(y)$. If $w \in N(z)$, then $V - \{x, y\}$ is a TO2CDS for G, a contradiction. Thus assume now that $w \notin N(z)$. Then $G = C_5$ or $N(w) - \{x, y\} = \{z_1\}$ with $z_1 \neq z$. However, then $V - \{x, y\}$ is a TO2CDS for G, a contradiction. Since $\operatorname{diam}(G) = 2$ we deduce that $a \in N(y)$. If $N(w) - \{x, y\} = \{z\}$, then we observe that then $V - \{x, z\}$ is a TO2CDS for G, a contradiction. Thus $|N(w) - \{x, y\}| = 0$. Hence $G = C_4$. $\square$

In the following we obtain the total outer-k-connected component domination number of a disconnected graph G in terms of the total outer-k-connected component domination numbers of its components. For this purpose we define $\gamma_{tc}^0(G) = |V|$.

Theorem 2.7. *Let G be a disconnected graph with m connected components $G_1, G_2, \ldots, G_m$, and let $k \geq m$. Then*

$$\gamma_{tc}^k(G) = \min_{\sum l_i = k} \sum_{i=1}^{m} \gamma_{tc}^{l_i}(G_i)$$

where $l_i \in \{0, 1, 2, \ldots, k\}$.

Proof. Let G be a disconnected graph with m connected components $G_1, G_2, \ldots, G_m$, and let $k \geq m$. Let $S_i^{l_i}$ be a $\gamma_{tc}^{l_i}(G_i)$-set for $i = 1, 2, \ldots, m$ if G_i has a TOl_iCDS, where $0 \leq l_i \leq k - m + 1$ and $\sum_{i=1}^m l_i = k$. It is obvious that $\bigcup_{i=1}^m S_i^{l_i}$ is a TOkCDS for G. This implies that

$$\gamma_{tc}^k(G) \leq \min_{\sum l_i = k} \sum_{i=1}^{m} \gamma_{tc}^{l_i}(G_i).$$

On the other hand let S be a TOkCDS for G. Let $S_i = S \cap V(G_i)$ for $i = 1, 2, \ldots, m$. If l_i is the number of components of $G_i - S_i$, then S_i is a TOl_iCDS for G_i. This completes the proof. □

We next obtain lower bounds for the total outer-k-connected component domination number of a graph G.

Theorem 2.8. *Let G be a graph of order n and size e, and let $k \geq 2$. If $\gamma_{tc}^k(G) > 0$, then*

$$\gamma_{tc}^k(G) \geq \frac{2e - (n-k+1)(n-k)}{2(k-1)}.$$

Proof. Let S be a $\gamma_{tc}^k(G)$-set of cardinality s. If $G_1, G_2, \ldots, G_k$ are the components of $G[V - S]$ such that $|V(G_i)| = n_i$ for $i = 1, 2, .., k$, then

$$e \leq \sum_{i=1}^{k} \frac{n_i(n_i - 1)}{2} + \frac{s(s-1)}{2} + \sum_{i=1}^{k} s n_i.$$

The right hand side of this inequality becomes maximum when $n_1 = n_2 = \ldots = n_{k-1} = 1$ and $n_k = n - s - (k-1)$. Therefore we obtain

$$\begin{aligned} e &\leq \frac{(n-s-k+1)(n-s-k)}{2} + \frac{s(s-1)}{2} + s(n-s) \\ &= \frac{(n-k+1)(n-k)}{2} + s(k-1), \end{aligned}$$

and this leads to the desired bound immediately. □

Let k, p, s be integers such that $p \geq 1$ and $k, s \geq 2$. Now let the graph H consist of the disjoint union of K_s, K_p and $k - 1$ isolated vertices $v_1, v_2, \ldots, v_{k-1}$ such that all vertices of K_s are adjacent to all vertices of K_p and $v_1, v_2, \ldots, v_{k-1}$ are adjacent to all vertices of K_s. Then it is straighforward to verify that

$$\gamma_{st}^k(H) = s = \frac{2e(H) - (n(H) - k + 1)(n(H) - k)}{2(k-1)}.$$

This family of examples show that the bound of Theorem 2.8 is sharp. Since $\alpha(H) = k$, we see that the bound $\alpha(G) \geq k$ in Observation 2.1 is sharp too.

Theorem 2.9. *For a graph G of order n, size e and $\gamma_{tc}^k(G) > 0$,*

$$\gamma_{tc}^k(G) \geq \left\lceil \frac{4n - 2k - 2e}{3} \right\rceil.$$

Proof. Let S be a $\gamma_{tc}^k(G)$-set of cardinality s, and $G_1, G_2, \ldots, G_k$ be the connected components of $G[V - S]$. Suppose that $|V(G_i)| = n_i$ for $1 \leq i \leq k$. Since S is a dominating set of G, any vertex in G_i has at least one neighbor in S for $1 \leq i \leq k$. On the other hand G_i is connected and so has at least $n_i - 1$ edges for $i = 1, 2, \ldots, k$. Also $G[S]$ has no isolated vertex. Thus, we obtain

$$e \geq \sum (n_i - 1) + \sum n_i + \frac{s}{2}.$$

Since $\sum n_i = n - s$ we have $e \geq 2n - \frac{3s}{2} - k$. This implies that $s \geq \frac{4n-2k-2e}{3}$, and the proof is complete. □

An immediate consequence of Theorem 2.9 with $k = 1$ is the following corollary for trees which is a main result of [1].

Corollary 2.10 [1]**.** *For a tree T of order n, $\gamma_{tc}(T) \geq \frac{2n}{3}$.*

It is obvious that $\gamma_{tc}^k(G) \leq n - k$. To characterize graphs achieving equality for the upper bound of the above inequality, we need to introduce a family of graphs. For $k > 1$, let $\mathcal{G}_k$ be the class of all graphs G such that $G \in \mathcal{G}_k$ if and only if $V = A \cup B$ such that $|A| = n - k$, $G[A]$ has no isolated vertex, $G[B] = \overline{K_k}$, and no subset $S \subseteq A \cup B$ with $|S| < n - k$ is a total outer-k-connected component dominating set for G. The following is a characterization for graphs G with $\gamma_{tc}^k(G) = n - k$. The proof is straightforward and is omitted.

Theorem 2.11. *For a connected graph G of order n, $\gamma_{tc}^k(G) = n - k$ if and only if $G \in \mathcal{G}_k$.*

3. Exact values

In this section we determine the total outer-k-connected component domination number for some special classes of graphs.

Proposition 3.1. *For $n \geq 3$,* $\gamma_{tc}^k(K_n) = \begin{cases} 2 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2. \end{cases}$

Proof. Let $n \geq 3$. If S is a TOkCDS in K_n, then $k = 1$, since $K_n[V - S]$ contains exactly one connected component. Thus $\gamma_{tc}^k(K_n) = 0$ if $k \geq 2$. Now it is obvious that $\gamma_{tc}^1(K_n) = \gamma_t(K_n) = 2$. □

Proposition 3.2. *For $2 \leq m \leq n$,* $\gamma_{tc}^k(K_{m,n}) = \begin{cases} 0 & \text{if } n < k \\ 2 & \text{if } k = 1 \\ m + n - k & \text{if } n \geq k, k \geq 2. \end{cases}$

Proof. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the two partite sets of $K_{m,n}$. Assume that $\gamma_{tc}^k(K_{m,n}) > 0$. So $k \leq n$. If $k = 1$, then $\gamma_{tc}^1(K_{m,n}) = \gamma_t(K_{m,n}) = 2$. So we assume that $k \geq 2$. Let S be a $\gamma_{tc}^k(K_{m,n})$-set. Since $K_{m,n}[X \cup Y - S]$ is disconnected, it follows that either $X \subseteq S$ or $Y \subseteq S$. Therefore $K_{m,n}[X \cup Y - S]$ consists of isolated vertices. As $K_{m,n}[X \cup Y - S]$ has exactly k connected components, we deduce that $|S| \geq m + n - k$. On the other hand $X \cup \{y_1, y_2, \ldots, y_{n-k}\}$ is a TOkCDS for $K_{m,n}$ of cardinality $m + n - k$. This completes the proof. □

For $n \geq 3$, we have the following.

Theorem 3.3. $\gamma_{tc}^k(P_n) = \begin{cases} 0 & \text{if } n < 3k-2 \\ 2k-2 & \text{if } 3k-2 \leq n \leq 4k-4 \\ 2k-1 & \text{if } n = 4k-3 \\ 2k & \text{if } 4k-2 \leq n \leq 4k-1 \\ 2k+1 & \text{if } n = 4k \\ 2k+2 & \text{if } n = 4k+1 \\ n-2k & \text{if } n \geq 4k+2. \end{cases}$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, where v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$. Assume that $\gamma_{tc}^k(P_n) > 0$. Let S be a $\gamma_{tc}^k(P_n)$-set, and let $G_1, G_2, \ldots, G_k$ be the connected components of $G - S$. Then $G[S]$ has at least $k-1$ components. Since any component of $G[S]$ has at least two vertices, we obtain $n \geq k + 2(k-1) = 3k-2$. We deduce, in particular, that $\gamma_{tc}^k(P_n) = 0$ if $n < 3k-2$.

Assume that $n \geq 4k+2$. Any component of $G[V-S]$ has at most two vertices, so $|V-S| \leq 2k$. This implies that $|S| \geq n-2k$. On the other hand $\{v_{4i+1}, v_{4i+2} : 0 \leq i \leq k-1\} \cup \{v_j : j \geq 4k+1\}$ is a TO$(n-2k)$CDS for P_n, and thus $\gamma_{tc}^k(P_n) = n-2k$.

Next we assume that $3k-2 \leq n \leq 4k-4$. It is obvious that $G[S]$ has at least $k-1$ components, and each component of $G[S]$ has at least two vertices. Thus $|S| \geq 2(k-1) = 2k-2$. Let $D = \{v_{3i+2}, v_{3i+3} : 0 \leq i \leq k-2\}$. Then D is a TOkCDS for P_{3k-2} of cardinality $2k-2$. If $t = n-3k+2$, then we subdivide the edges $v_{3i+3}v_{3i+4}$ for $i = 1, 2, \ldots, t$ to obtain a path P_n from P_{3k-2}. Then D is still a TOkCDS for P_n. Thus $\gamma_{tc}^k(P_n) \leq 2k-2$ and the result follows.

Assume next that $n = 4k-3$. Suppose that $|S| \leq 2k-2$. For S to dominate maximum number of vertices, without loss of generality, we may assume that each component of $G[S]$ is K_2, and each component of $G[S]$ dominates two vertices of $G[V-S]$. Then $|N[S]| \leq 2(\frac{2k-2}{2}) + |S| < n$, a contradiction. Thus $|S| \geq 2k-1$. On the other hand $\{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-2\} \cup \{v_{n-1}\}$ is a TOkCDS for P_n of cardinality $2k-1$. Thus $\gamma_{tc}^k(P_{4k-3}) = 2k-1$.

Next assume that $4k-2 \leq n \leq 4k-1$. Suppose that $|S| \leq 2k-1$. If each component of $G[S]$ is a K_2, then $|S| \leq 2k-2$ and S dominates at most $4\frac{|S|}{2} \leq 4k-4 < n$ vertices of P_n, a contradiction. Thus $G[S]$ has a component with more than two vertices. For S to dominate maximum number of vertices, without loss of generality, we may assume that a component of $G[S]$ is P_3, and the other components are K_2. Furthermore, the P_3 component of $G[S]$ dominates at most five vertices of G, while any K_2-component of $G[S]$ dominates at most four vertices of G. We deduce that $|N[S]| \leq 5 + 4(\frac{2k-1-3}{2}) < n$, a contradiction. Thus $|S| \geq 2k$. On the other hand $\{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-2\} \cup \{v_n, v_{n-1}\}$ is a TOkCDS for P_n of cardinality $2k$. Thus $\gamma_{tc}^k(P_n) = 2k$.

The proof for $n \in \{4k, 4k+1\}$ is similar. □

The following theorem can be proved in a similar manner as in the proof of Theorem 3.3, and so we omit the proof.

Theorem 3.4. $\gamma_{tc}^k(C_n) = \begin{cases} 0 & \text{if } n < 3k \\ 2k & \text{if } 3k \leq n \leq 4k \\ n-2k & \text{if } n \geq 4k+1. \end{cases}$

For a wheel and an integer $k > 1$ the center of the wheel is in any TOkCDS. So the following is easily verified.

Theorem 3.5. *Let $k \geq 2$ be a positive integer, and let W_n be a wheel with $n \geq 3$. Then*

$$\gamma_{tc}^k(W_n) = \begin{cases} 0 & \text{if } n < 2k \\ k+1 & \text{if } n \geq 2k. \end{cases}$$

We close with the following problems.

Problem 1. Find sharp upper and lower bounds for the total outer-k-connected component domination number of a graph.

Problem 2. Determine the complexity issue of the total outer-k-connected component domination number.

Problem 3. Determine the total outer-k-connected component domination number in grid graphs.

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