RAIRO-Oper. Res. 51 (2017) 577–590
DOI: 10.1051/ro/2016044

RAIRO Operations Research
www.rairo-ro.org

ANALYSING THE SOLUTION OF PRODUCTION-INVENTORY OPTIMAL CONTROL SYSTEMS BY NEURAL NETWORKS *

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Abstract. In this paper, a general production-inventory optimal control system is proposed which can be used in most cases that might arise in the theory of production-inventory control. The proposed general form is considered and approximately solved using neural networks. Since the obtained solutions are achieved based on neural networks, they have several advantages in practice. One of the important advantages is that the solutions can be easily used for post optimality and sensitivity analyses. The solutions of this model are compared with those of other existing methods and some illustrating notes are presented.

Mathematics Subject Classification. 49J15, 90B05, 90B30.

Received November 9, 2015. Accepted June 20, 2016.

1. Introduction

The management of production-inventory systems and solving production planning problems have received considerable attention in the literature. Most of the available studies have considered a constant demand rate, while the demand is time variant and time in reality is not discrete. This concept can be more serious while facing several dynamic aspects like trends, seasonal behavior, life cycle patterns in demand for products, returns and global multiple sales opportunities. Many mathematical models of (continuous time) production planning problems can be posed as optimal control problems. In last decades, the use of optimal control theory in practical problems arising in economy and management sciences had a fast growth. Some authors such as Kistner and Dobos [10], Dobos [4], Sethi [19], *etc.* introduced optimal control models for the primal problems of inventory and production planning. In recent years, the control of production inventories of deteriorating items has attracted a lot of attention in inventory analysis. This is due to this fact that most of the physical goods deteriorate over time (for example see [7, 8, 15, 20], *etc.*).

Tadj *et al.* [21] introduced an optimal control model for production inventory systems with deteriorating items and proposed a closed form of optimal control problem for which they used numerical techniques to solve. Foul *et al.* [7] introduced an optimal and self-tuning optimal control problem for a periodic-review hybrid production inventory system with single reusable products. They also used recursive least-squares method to solve

Keywords. Optimal control, production planning, production-inventory systems, neural networks.

* *This work was supported in part by: Research Deputy of Ferdowsi University of Mashhad, under Grant No. 38980 (dated OCT 28, 2015).*

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the proposed model. Li [15] introduced an optimal control model for production-inventory system with deteriorating items and also with tradable emission permits. He derived the optimality conditions for the proposed optimal control problem as a two-point boundary value problem and solved it by using numerical methods. Pan and Li [16] considered an optimal control for stochastic production-inventory system with environmental constraints. Benkherouf *et al.* [3] introduced optimal control of production planning problems with reverse logistic in a finite planning horizon inventory system. Returned items may be classified as either remanufacturing or refurbishing items. They solved the proposed continuous optimization problem with discretization and nonlinear programming techniques. Hedjar *et al.* [8] developed model predictive production planning in a three-stock reverse-logistics system with deteriorating items. They used the model predictive control method to analyze the solution of the proposed optimal control problem. Alfares [2] considered a production-inventory system with finite production rate, stock dependent demand and variable holding cost. He proposed two efficient algorithms for solving the proposed model which contains of a nonlinear programming problem.

Huang and Jiang [9] proposed a neural network observer-based optimal control for unknown nonlinear systems with control constraints. Kiumarsi *et al.* [11] presented a new reinforcement learning (RL) approach based on a new neural network model to solve the optimal tracking problem of a nonlinear discrete time-varying system via an online approach. Kmet and Kmetova [12] considered a method based on neural networks for solving optimal control problems with discrete time delays in state and control variables subject to control and state constraints. The proposed optimal control model was transcribed into a nonlinear programming problem that was implemented with feed forward adaptive critic neural networks to find the optimal control and the optimal trajectory.

Without considering optimal control theory, neural networks were applied for solving problems in the inventory-production models. Partovi and Anandarajan [17] used the ability of artificial neural networks in prediction for classifying inventory in pharmaceutical companies. They proposed two different learning algorithms and compared their approach with the multiple discriminate analysis technique. Paul and Azeem [18] developed an artificial neural network model in order to determine the optimum level of finished goods inventory as a function of product demand, setup, holding, and material costs. Aengchuan and Phruksaphanrat [1] considered inventory control models and compared some soft-computing techniques for the mentioned models. They compared the abilities of fuzzy inference systems with neural networks for prediction purposes of the inventory control problem. Thomas *et al.* [22] applied neural networks for the reduction of a product-driven system emulation model. Lee *et al.* [14] studied production quantity allocation for order fulfilment in the supply chain via a neural network approach.

Most techniques used for solving the above mentioned optimal control problems are a type of discretization of the continuous model. On the other hand, it is well-known that neural networks are universal approximators. They can estimate a nonlinear function with an arbitrary degree of accuracy. For example, Lagaris *et al.* [13] proposed a neural network method to solve both ordinary and partial differential equations. Effati and Pakdaman [5] used the artificial neural networks for estimating the solution of fuzzy differential equations. In the case of optimal control theory, Effati and Pakdaman [6] used the ability of neural networks for approximating the solution of mathematical models of optimal control problems.

The aim of this paper is to propose a neural network model that is capable of solving optimal control models arising in the theory of inventory systems and production planning. The proposed solution in neural network methodology has many advantages. Since the solutions for state and control variables are presented as differentiable functions of time (unlike other existing methods), the solution can be calculated at each arbitrary point in the time horizon. Also the proposed approximate solution is a differentiable function. Thus it can be used for other applications such as post optimality analysis. In Section 2 we mention the mathematical models of optimal control problems for inventory control. In this section we introduce the models proposed by Hedjar *et al.* [8] and also Sethi [19] and derive the optimality conditions for the inventory control models and present them as a system of differential equations. Section 3 contains the proposed approximation techniques for solving the optimal control models via the neural network method. To illustrate the proposed approximate algorithm, two problems are solved in Section 4 along with a comparison and analysis. Some remarks about the proposed method are presented in Section 5 and finally, Section 6 contains conclusions.

2. Problem formulation

A general form of an optimal control problem can be defined as follows:

$$\min J = \Psi(x(T), T) + \int_{t_0}^{T} F(x(t), u(t), t)\mathrm{d}t$$
$$\text{s.t.} \quad \dot{x} = f(x(t), u(t), t), \quad x(t_0) = x_0. \tag{P1}$$

where $T > 0$ is time horizon and $t \in [t_0, T]$, $x \in R^n$ is the vector of state variables, $u \in R^m$ is the vector of control variables and the functions $f : R^n \times R^m \times R \to R^n$, $F : R^n \times R^m \times R \to R$ and $\Psi : R^n \times R \to R$ are continuously differentiable. Here $\dot{x}$ is used for $\mathrm{d}x/\mathrm{d}t$. Other constraints may be considered for the control function $u(t)$ or the state function $x(t)$. If for (P1) we define the Hamiltonian function as $H(x, u, t) = F(x, u, t) + \lambda f(x, u, t)$ (where $\lambda \in R^n$ is the co-state vector), then the necessary optimality conditions for $u^*(t)$ to be an optimal control for (P1) are:

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial \lambda} \Rightarrow \quad \dot{x} = f(x^*, u^*, t), \;\; x^*(t_0) = x_0 \\ \dot{\lambda} = -\frac{\partial H}{\partial x}, \;\; \lambda(T) = \frac{\partial \Psi}{\partial x}(x^*(T), T) \\ \frac{\partial H}{\partial u^*} = 0 \end{cases} \tag{2.1}$$

Equations in (2.1) form a system of ordinary differential equations which show the necessary conditions for optimality.

In practical models of optimal control problems in production management and inventory control, the state function $x(t)$ and control function $u(t)$ may have several descriptions based on the dynamics of the real world models. From the inventory point of view, suppose that $I_m(t)$, $I_r(t)$ and $I_t(t)$ denote the inventory of manufacturing, remanufacturing and returned items at time t, respectively and their initial values are $I_m^0(t)$, $I_r^0(t)$ and $I_t^0(t)$. Also u_m, u_r and u_d denote the rate of manufacturing, remanufacturing and disposal at time t. In this case, we set $x(t) = [I_m(t) \;\; I_r(t) \;\; I_t(t)]^T$ and $u(t) = [u_m(t) \;\; u_r(t) \;\; u_d(t)]^T$ as the state and control variables, respectively. Thus, problem (P1) can be rewritten as follows:

$$\min J = \Psi(I_m(T), I_r(T), I_t(T), T) + \int_{t_0}^{T} F(I_m(T), I_r(T), I_t(T), u_m(t), u_r(t), u_d(t), t)\mathrm{d}t$$

$$\text{s.t.} \quad \begin{cases} \dot{I}_m(t) = f_1(I_m(T), I_r(T), I_t(T), u_m(t), u_r(t), u_d(t), t), \quad I_m(t_0) = I_m^0, \\ \dot{I}_r(t) = f_2(I_m(T), I_r(T), I_t(T), u_m(t), u_r(t), u_d(t), t), \quad I_r(t_0) = I_r^0, \\ \dot{I}_t(t) = f_3(I_m(T), I_r(T), I_t(T), u_m(t), u_r(t), u_d(t), t), \quad I_t(t_0) = I_t^0, \end{cases} \tag{P2}$$

where f_1, f_2 and f_3 determine the dynamics of the system and they can be linear or non-linear. Problem (P2) is a general form of most problems in inventory control theory. Several researchers have determined the structure of functions f_1, f_2 and f_3 for their proposed new inventory models (*e.g.* [10, 15]). We can derive the necessary optimality conditions (2.1) for (P2). In some cases, system (2.1) can be solved analytically. However, when the system is complicated, some approximate methods must be applied. In Table 1, the notations of variables in model (P2) and their descriptions are listed. Note that model (P2) does not restrict us in selecting a larger number of state and control variables. We can use fewer or more number of state and control variables with different descriptions.

Table 1. List of variables and their notations in proposed model (P2).

	State variables			Control variables		
	$I_m(t)$	$I_r(t)$	$I_t(t)$	u_m	u_r	u_d
Description	inventory of manufacturing items at time t	inventory of remanufacturing items at time t	inventory of returned items at time t	rate of manufacturing at time$t$	rate of remanufacturing at time t	rate of disposal at time$t$

To show the advantages and contributions of the proposed model and algorithm, a short review on existing methods for modelling and solving optimal control problems in the theory of production-inventory control is presented in Table 2. As it can be seen from Table 2, most of the previously proposed algorithms have presented a point to point solution and have not proposed the solution as a differentiable function of time, while the neural network approach does.

3. Approximation method

In the theory of neural networks, a basic perceptron has an architecture as presented in Figure 1.

In Figure 1, t and *out* are the input and output of the network, w and v are the weights of input and output respectively and b is the bias weight and $z = wt + b$. Here Sigmoid is the activation function of the neural network with the following structure:

$$\text{Sigmoid}\,(z) = \frac{1}{1+\mathrm{e}^{-z}}\cdot$$

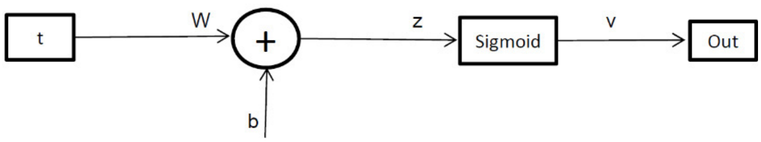

Figure 1. Basic structure of a perceptron neural network.

Instead of the Sigmoid activation function, we may use any other activation functions. Considering problem (P2), to approximate the state function $x(t) = [I_m(t)\ \ I_r(t)\ \ I_t(t)]^T$, control function $u(t) = [u_m(t)\ \ u_r(t)\ \ u_d(t)]^T$ and the co-state function$\lambda(t) = [\lambda_1(t)\ \ \lambda_2(t)\ \ \lambda_3(t)]^T$, first we propose their corresponding approximated functions respectively as follows:

$$\begin{cases} x_A(t,\phi^i) = [I_m^A(t,\phi^{im})\ \ I_r^A(t,\phi^{ir})\ \ I_t^A(t,\phi^{it})]^T, \\ u_A(t,\phi^u) = [u_m^A(t,\phi^{um})\ \ u_r^A(t,\phi^{ur})\ \ u_d^A(t,\phi^{ud})]^T, \\ \lambda_A(t,\phi^l) = [\lambda_1^A(t,\phi^1)\ \ \lambda_2^A(t,\phi^2)\ \ \lambda_3^A(t,\phi^3)]^T. \end{cases} \tag{3.1}$$

Each of the nine functions I_m^A, I_r^A, I_t^A, u_m^A, u_r^A, u_d^A, λ_1^A, λ_2^A and λ_3^A contains a neural network with its special weights (special weights for each approximate function are contained in the vector variables ϕ). The notations of the weights for each approximate function are illustrated in Table 3.

To illustrate the structure of approximate functions, we describe for example, the formulation of approximate function $I_m^A(t)$ as follows:

$$I_m^A(t,\phi^{im}) = I_{m0} + (t-t_0)N_{im}(t,\phi^{im})$$

where $N_{im}(t,\phi^{im}) = \sum_{j=1}^{N} v_j^{im} s(w_j^{im}t + b_j^{im})$, s is sigmoid transfer function and $\phi^{im} = [v^{im}\ \ w^{im}\ \ b^{im}]$. It is easy to check that $I_m^A(t)$ (which is the approximation of $I_m(t)$), satisfies the initial condition $I_m^A(t_0,\phi^{im}) = I_{m0}$. Other approximate functions have the same structure (similar to Fig. 1).

TABLE 2. A short review of existing approaches.

Topic	Author-Year	Type of problem	Solution method	Scope limitations
Optimal control in production-inventory control	Kistner and Dobos (2000)	Optimal production-inventory for a reverse logistics system	A modified forward Arrow-Karlin-type algorithm	Does not propose the solution as a function of time.
	Dobos (2003)	Optimal production-inventory for a HMMS-type reverse logistics system	Analytical solution	Sensitivity of the policy on changes of return optimal production-inventory rate are not applied
	Tadj *et al.* (2006)	Optimal control of a production inventory system with deteriorating items	Analytical and numerical solutions	It has just been examined for special exogenous functions
	Foul *et al.* (2007)	Optimal control of a periodic-review hybrid production inventory system	Recursive least-squares (RLS) method	The proposed solutions are point-wise
	Shah and Acharya (2008)	A time dependent deteriorating order level inventory model for exponentially declining demand	Study the optimal policy of the retailer when there is a decline in the demand	A simple linear objective function
	Li (2014)	Optimal control of the production-inventory system with deteriorating items and tradable emission permits	Analytical and numerical solutions	A further research direction would be needed to examine the situation when the firm introduces a new technology with tradable emission permits.
	Hedjar *et al.* (2015)	A three-stock reverse-logistics system with deteriorating items	Model predictive control approach	It does not propose the solution as a function of time
	Pan and Li (2015)	Optimal control of a stochastic production-inventory system under deteriorating items and environmental constraints	Analytical and numerical solutions	It does not propose the solution as a function of time. Also, the proposed solutions are pointwise
	Benkherouf *et al.* (2015)	Optimal control of production, remanufacturing and refurbishing activities in a finite planning horizon inventory system	Discritisation and solving a mixed-integer nonlinear program	The proposed nonlinear program is difficult to solve. Also it does not propose the solution as a function of time

Table 2. continued.

Topic	Author-Year	Type of problem	Solution method	Scope limitations
Approximate methods for optimal Control	Huang and Jiang (2015)	Neural network observer-based optimal control for unknown nonlinear systems with control constraints	Two NNs were used: a feedforward NN to constitute the NN observer which is applied to obtain the states, and a critic NN to approximate the value function	The method considered was just applied for solving the infinite horizon optimal control problem
	Kiumarsi *et al.* (2015)	Optimal tracking problem of a nonlinear discrete time-varying system via an online approach	A new reinforcement learning (RL) method which was motivated by recently discovered neurocognitive models of mechanisms in the brain,	This work is constructed for discrete time systems
	Kmet and Kmetova (2015)	Optimal control problems with discrete time delays in state and control variables subject to control and state constraints	A new network adaptive critic approach for optimal control synthesis with discrete time delay in state and control variables	The basic theory of the presented method is to discretize the continuous model
	Effati and Pakdaman (2013)	Neural network approach for solving optimal control problem	Using the capabilities of perceptron neural networks in function approximation	The objective functional had some limitations. Also terminal conditions cannot be considered for co-state variables
NN for production-inventory control	Partovi and Anandarajan (2002)	Classifying inventory in pharmaceutical companies	They used the ability of artificial neural networks in prediction	There are limitations for the number of variables for the number of variables
	Paul and Azeem (2011)	Determining the optimum level of finished goods inventory	They developed an artificial neural network model	They did not consider the reliability of the production system and the accuracy of the method was low
	Aengchuan and Phruksaphanrat (2016)	Inventory control models and some soft-computing techniques were considered	Fuzzy inference systems with neural networks for prediction purposes	They mentioned that the evaluation algorithm should also be adjusted according to the realistic situation in their future study
	Thomas *et al.* (2011)	Reduction of a product-driven system emulation model	A multilayer perceptron neural network	They did not suggest an online learning approach and did not compare the reduction system with the complete one

TABLE 3. Notations of weights for each approximate function for $j = 1, 2, \dots, N$.

Title	State variables			Control variables			Co-state variables		
Variables in original model (P2)	I_m	I_r	I_t	u_m	u_r	u_d	λ_1	λ_2	λ_3
Approximate variables	I_m^A	I_r^A	I_t^A	u_m^A	u_r^A	u_d^A	λ_1^A	λ_2^A	λ_3^A
Corresponding neural network	N_{im}	N_{ir}	N_{it}	N_{um}	N_{ur}	N_{ud}	N_1	N_2	N_3
Weights of input layer	w_j^{im}	w_j^{ir}	w_j^{it}	w_j^{um}	w_j^{ur}	w_j^{ud}	w_j^1	w_j^2	w_j^3
Bias weights	b_j^{im}	b_j^{ir}	b_j^{it}	b_j^{um}	b_j^{ur}	b_j^{ud}	b_j^1	b_j^2	b_j^3
Weights of output layer	v_j^{im}	v_j^{ir}	v_j^{it}	v_j^{um}	v_j^{ur}	v_j^{ud}	v_j^1	v_j^2	v_j^3

Based on the structures of approximate functions, we can define an approximate Hamiltonian function as follows:

$$H_A(x_A, u_A, t) = F(x_A, u_A, t) + \lambda_A f(x_A, u_A, t). \tag{3.2}$$

Since I_m^A, I_r^A, I_t^A, u_m^A, u_r^A, u_d^A, λ_1^A, λ_2^A and λ_3^A are approximate solutions of the optimal control problem (P1), they must satisfy the necessary conditions (2.1) while considering the approximate Hamiltonian function (3.2) as follows:

$$\begin{cases} \dot{x}_A = \frac{\partial H_A}{\partial \lambda_A} \Rightarrow \quad \dot{x}_A = f(x_A, u_A, t), \;\; x_A(t_0) = x_0 \\ \dot{\lambda}_A = -\frac{\partial H_A}{\partial x_A}, \;\; \lambda_A(T) = \frac{\partial \Psi}{\partial x_A}(x_A(T), T) \\ \frac{\partial H_A}{\partial u_A} = 0 \end{cases} \tag{3.3}$$

Since $\lambda_A$must satisfy a final condition in (3.3), we can choose:

$$\begin{cases} \lambda_1^A(t, \phi^1) = \frac{\partial \Psi}{\partial I_m^A}(I_m^A(T), T) + (t - T)N_1(t, \varphi^1), \\ \lambda_2^A(t, \phi^2) = \frac{\partial \Psi}{\partial I_r^A}(I_r^A(T), T) + (t - T)N_2(t, \varphi^2), \\ \lambda_3^A(t, \phi^3) = \frac{\partial \Psi}{\partial I_t^A}(I_t^A(T), T) + (t - T)N_3(t, \varphi^3). \end{cases}$$

To solve (3.3) for $t \in [t_0, T]$, we use a discretization of interval $[t_0, T]$ and define the following error minimization problem:

$$\underset{\Phi}{\text{minimize}} \quad \sum_{k=1}^{N} \left[\dot{x}_A(\Phi, t_k) - \frac{\partial H_A(\Phi, t_k)}{\partial \lambda_A}\right]^2 + \left[\dot{\lambda}_A(\Phi, t_k) + \frac{\partial H_A(\Phi, t_k)}{\partial x_A}\right]^2 + \left[\frac{\partial H_A(\Phi, t_k)}{\partial u_A}\right]^2, \tag{3.4}$$

where Φ is a weight vector containing all weights of all approximate functions. Indeed Φcontains the weight vectors of approximate state functions (weights of inventory functions *i.e.* ϕ^{im}, φ^{ir} and ϕ^{it}), the weight vectors of the control functions (weights of production functions *i.e.* ϕ^{um}, ϕ^{ur} and ϕ^{ud}) and the weight vectors of the approximate co-state functions (ϕ^1, ϕ^2 and ϕ^3). Problem (3.4) is an unconstrained optimization problem. This problem can be solved with heuristic methods such as Genetic algorithm or with classical mathematical optimization methods. By terminating the optimization step, we can replace the optimal weights Φ^* into the corresponding approximate functions (3.1).

4. Numerical examples

In this section, to show the flexibility of the proposed method, two different numerical examples are presented. As the first example, we solve a model from Sethi and Thompson [19] which has an analytical solution to verify

TABLE 4. Notations for Example 1 in comparison with notations in (P2).

Notations in Example 1. Sethi [19]	Corresponding notation in our model (P2)
I	I_m
P	u_m
$P(t)-S(t)$	f_1
$\mathrm{e}^{-\rho t}\left[\frac{h}{2}(I-\hat{I})^2+\frac{c}{2}(P-\hat{P})^2\right]$	F

the method's reliability. For the second example, based on continuous review policy of a plant and following Hedjar *et al.* [8], we solve their proposed model. Both problems are solved in Matlab 2013Rb. The number of weights for all neural network parameters is considered to be 3. Also for both problems, the time horizon is discretized into 10 equal sub-intervals.

Example 1. As the first example, we solve the problem from Sethi and Thompson [19]. Consider a factory which produces a single homogeneous good with a finished goods warehouse. The mathematical model is as follows:

$$\underset{P\geqslant 0}{\text{minimize}} \quad J=\int_0^T \mathrm{e}^{-\rho t}\left[\frac{h}{2}(I-\hat{I})^2+\frac{c}{2}(P-\hat{P})^2\right]\mathrm{d}t$$

$$\text{s.t.} \quad \frac{\mathrm{d}I}{\mathrm{d}t}=P(t)-S(t), \quad I(0)=I_0,$$

where $\hat{P}=30$, $\hat{I}=15$, $T=8$, $\rho=0$, $I(0)=10$ and $h=c=1$. Here, $\hat{I}$ and $\hat{P}$ are the goal of inventory and production levels, $\rho\geqslant 0$ is the discount rate, $h>0$ is the inventory holding cost coefficient, $c\geqslant 0$ is the production cost coefficient and $S(t)=t^3-12t^2+32t+30$ is the sales rate. The optimality conditions lead to the following two-point boundary value problem:

$$\begin{cases} \frac{\mathrm{dI}}{\mathrm{d}t}=\hat{P}+\frac{\lambda}{c}-S(t), \;\; I(0)=I_0 \\ \frac{\mathrm{d}\lambda}{\mathrm{d}t}=\rho\lambda+h(I-\hat{I}), \;\; \lambda(T)=0. \end{cases}$$

This problem has an analytical solution which is solved in [19]. In comparison with our notations (see problem P2), we can introduce the proposed notations in Table 4. This table shows that how the variables in this problem correspond with the ones in our model. Note that in this model we just have one type of inventory (I or I_m) for manufacturing items. Thus, we do not have functions f_2 and f_3.

The optimal solution for production $P(t)$ is illustrated in Figure 2. The optimal inventory $I(t)$ is plotted and compared with the exact solution in Figure 3. As it can be observed in Figures 2 and 3, the approximated solution with neural networks is very near the analytical solution with very good accuracy. The structure of the obtained inventory function is similar to (3.1).

As it can be observed in Figures 2 and 3, the proposed approximate inventory and production functions are differentiable. Also we can calculate the level of inventory as well as the level of production continuously at each arbitrary point in the time horizon [0,8]. Based on Figure 2, the final value of the production is equal to its goal level of 30.

Example 2. To illustrate the proposed method and validate it, we applied the neural network methodology to solve a numerical example from [8]. Based on the parameter selection in Hedjar *et al.* [8], suppose that $I_m(t)$, $I_r(t)$ and $I_t(t)$ denote the inventory of manufacturing, remanufacturing and returned items at time t, respectively

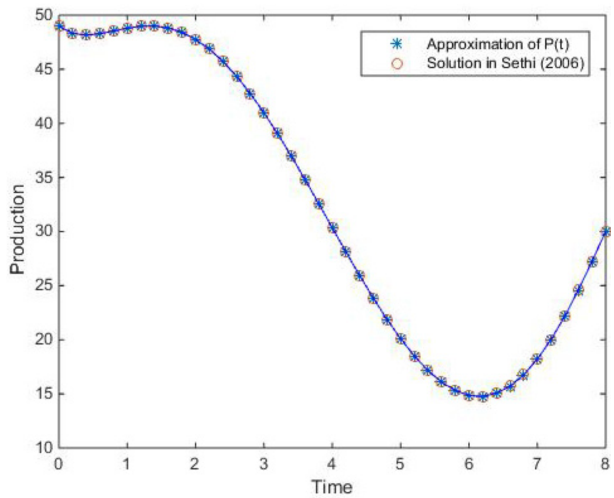

FIGURE 2. Comparison results for approximating production in Example 1.

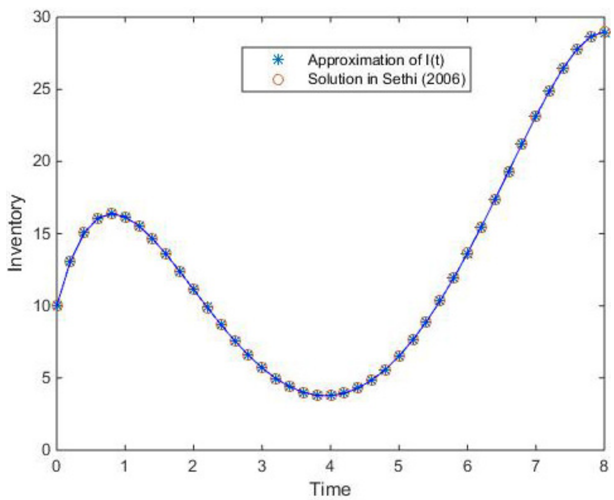

FIGURE 3. Comparison results for approximating inventory in Example 1.

and assume that their initial values are $I_m^0(t)$, $I_r^0(t)$ and $I_t^0(t)$. In addition, their goal level are denoted by $\hat{I}_m(t)$, $\hat{I}_r(t)$ and $\hat{I}_t(t)$, respectively. u_m, u_r and u_d denote the rate of manufacturing, remanufacturing and disposal at time$t$, with goal rates $\hat{u}_m$, $\hat{u}_r$ and $\hat{u}_d$, respectively. To attain the goals of the problem, Hedjar *et al.* [8] proposed the following control and state functions:

$$\begin{cases} x(t) = [\Delta I_m(t) \;\; \Delta I_r(t) \;\; \Delta I_t(t)]^T = [I_m(t) - \hat{I}_m(t) \;\, I_r(t) - \hat{I}_r(t) \;\, I_t(t) - \hat{I}_t(t)]^T, \\ u(t) = [\Delta u_m(t) \;\; \Delta u_r(t) \;\; \Delta u_d(t)]^T = [u_m(t) - \hat{u}_m(t) \;\, u_r(t) - \hat{u}_r(t) \;\, u_d(t) - \hat{u}_d(t)]^T \end{cases}$$

TABLE 5. Notations for Example 2 in comparison with notations in (P2).

Notations in Example 2	Corresponding notation in our model (P2)
$\Delta I_m(t)$	I_m
$\Delta I_r(t)$	I_r
$\Delta I_t(t)$	I_t
Δu_m	u_m
Δu_r	u_r
Δu_d	u_d

Hedjar *et al.* [8] proposed the following optimal control problem:

$$\begin{aligned} &\min \ \frac{1}{2}\int_0^T [x^T(t)Qx(t)+u^T(t)Ru(t)]\,\mathrm{d}t \\ \text{s.t.} \quad &\frac{\mathrm{d}(\Delta I_m(t))}{\mathrm{d}t} = \Delta u_m(t) - \theta_m \Delta I_m(t) \\ &\frac{\mathrm{d}(\Delta I_r(t))}{\mathrm{d}t} = \Delta u_r(t) - \theta_r \Delta I_r(t) \\ &\frac{\mathrm{d}(\Delta I_t(t))}{\mathrm{d}t} = -\Delta u_r(t) - \Delta u_d(t) \\ &\Delta I_m(0)=15,\ \Delta I_r(0)=10,\ \Delta I_t(t)=5. \end{aligned}$$

where

$$Q = \begin{bmatrix} q_m & 0 & 0 \\ 0 & q_r & 0 \\ 0 & 0 & q_t \end{bmatrix} \text{ and } R = \begin{bmatrix} r_m & 0 & 0 \\ 0 & r_r & 0 \\ 0 & 0 & r_d \end{bmatrix}.$$

This problem also agrees with our proposed model. It is enough to define the variables as shown in Table 5.

Here $x(t) = [\Delta I_m(t)\ \ \Delta I_r(t)\ \ \Delta I_t(t)]^T$ and $u(t) = [\Delta u_m(t)\ \ \Delta u_r(t)\ \ \Delta u_d(t)]^T$. In matrix notation, this problem has a linear form as follows:

$$\begin{aligned} &\min \ \frac{1}{2}\int_0^T [x^T(t)Qx(t)+u^T(t)Ru(t)]\,\mathrm{d}t \\ \text{s.t.} \quad &\dot{x}(t) = Ax(t)+Bu(t),\ x(0)=x_0 \end{aligned}$$

where

$$A = \begin{bmatrix} -\theta_m & 0 & 0 \\ 0 & -\theta_r & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \text{ and } x_0 = \begin{bmatrix} \Delta I_m(0) \\ \Delta I_r(0) \\ \Delta I_t(0) \end{bmatrix}.$$

Also q_m, q_r, q_t, r_m, r_r and r_d are the penalty parameters (see [8]). Based on Hedjar *et al.* [8], consider the initial conditions: $\Delta I_m^0 = 15$, $\Delta I_r^0 = 10$, $\Delta I_t^0 = 5$ and the following parameters:

$$T = 0.4,\ \theta_m = 0.01,\ \theta_r = 0.02,\ q_m = 1,\ q_r = 2,\ q_t = 3,\ r_m = 0.1,\ r_r = 0.2,\ r_d = 0.3.$$

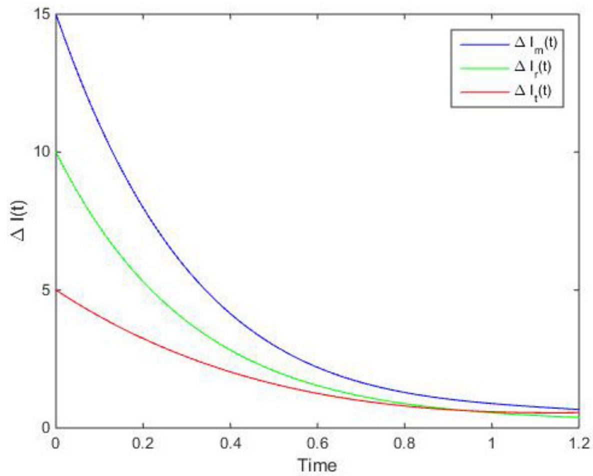

FIGURE 4. Optimal inventory functions for Example 2.

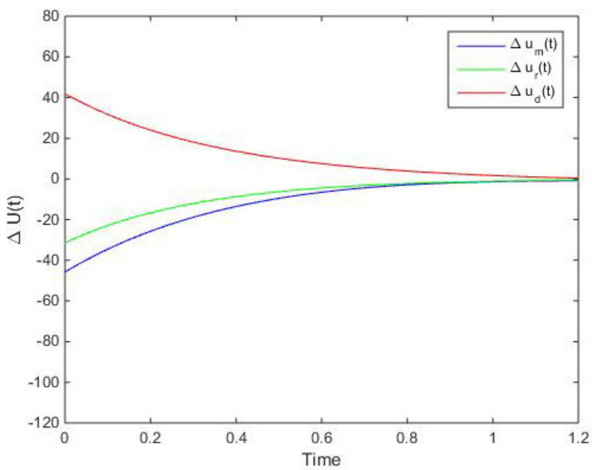

FIGURE 5. Optimal control functions for Example 2.

With these parameters in hand, we solve the optimization problem (3.4). Based on the initial conditions the proposed approximate state functions (for inventory functions) we can have the following structures:

$$\begin{cases} \Delta I_m^A(t, \phi_{im}) = 15 + t \times N_{im}(t, \phi_{im}), \\ \Delta I_r^A(t, \phi_{ir}) = 10 + t \times N_{ir}(t, \phi_{ir}), \\ \Delta I_t^A(t, \phi_{iR}) = 5 + t \times N_{it}(t, \phi_{it}). \end{cases} \tag{4.1}$$

Also, since $\Psi(x(T), T) = 0$, the structure of the approximate co-state functions can be considered as follows:

$$\begin{cases} \lambda_1^A(t, \phi^1) = (t - 0.4) \times N_1(t, \phi^1), \\ \lambda_2^A(t, \phi^2) = (t - 0.4) \times N_2(t, \phi^2), \\ \lambda_3^A(t, \phi^3) = (t - 0.4) \times N_3(t, \phi^3). \end{cases}$$

The optimal solutions are plotted in Figures 4 and 5. Similar to the results reported in [8], the solutions converge to zero. In [8], they used the model predictive control approach for solving the proposed optimal control model

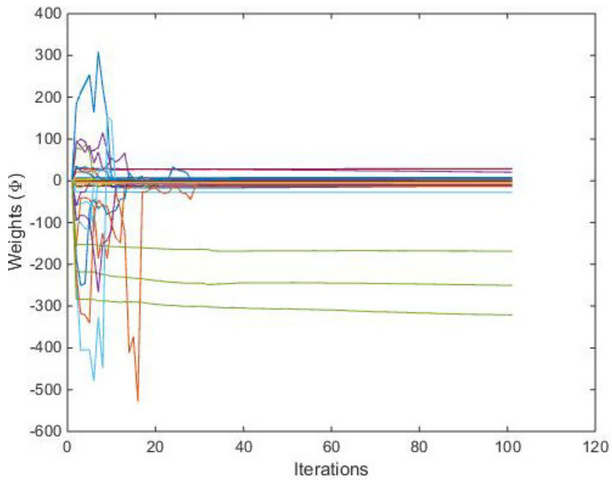

Figure 6. Convergence of the weights for Example 2.

which needs a discretization of the horizon interval. But, in the neural network method, we discretize the interval. Anyway, the final solution (for both optimal control and optimal state functions) is an approximate-analytical solution that is a differentiable function of time. The structure of the obtained inventory function is similar to (4.1). To illustrate the convergence of the weights of the proposed neural network, in each iteration, the values of all weights (vector Φ in optimization problem (3.4)) are plotted in Figure 6.

As it can be seen from both Figures 4 and 5, the differences between the goal levels of inventories and productions tend to zero. Also, as it can be observed in Figures 4 and 5, the production and inventory function are differentiable functions of time. Thus, we can calculate the difference between the goal level of inventory (production) functions and their current values at each arbitrary point in the time horizon.

5. Remarks and discussion

Considering the proposed model (P2) and the neural network-based approach, it is necessary to mention some remarks to illustrate the algorithm.

As the first remark, based on the proposed neural network method and in comparison with the other existing methods, the proposed solutions have a closed form. Indeed, the control and state variables are differentiable functions of time. Thus, we can calculate the inventory and production values at each arbitrary time in the time horizon.

The proposed model for the optimal control problem (P2) is a general form that can be considered in most cases in production-inventory models. For example, the dynamics of the system can be linear [8], time variant or time invariant. In addition, the objective functional can be quadratic or any other nonlinear model. However, in Section 4, two different problems were solved. Of course, this is not a limitation for the algorithm. Comparing Tables 4 and 5 with Table 3 helps us to define any state and control variables with different descriptions for any problem in the theory of production-inventory control. Also, the dynamics of the system can be determined by f_1, f_2 and f_3.

In comparison with Effati and Pakdaman [6], in this paper we have a different objective functional with the considered objective in [6]. In the model (P2), $\psi(x(T), T)$ denotes the salvage or scrap value which is needed so that the solution will make "good sense" at the end of the horizon (see [19]). Effati and Pakdaman [6] did not consider any salvage value of the ending state $x(T)$ at time T in their objectives. Thus, an additional condition for $\lambda(T)$ is needed (this condition is presented in (2.1)). Indeed, the proposed trial solution for co-state function must be constructed such that it satisfies this condition while in [6] they do not have this condition.

Finally the number of weights and the number of points in the time horizon can be increased to have a more precise solution. The optimization algorithm for problem (3.4) can be any mathematical optimization algorithm or a heuristic one.

6. Conclusions

In this paper, a method based on the neural networks models was proposed for solving optimal control problems arising in modelling the inventory-production systems. Based on the proposed method, the obtained results (obtained functions for inventory and production) are differentiable functions from which the value of inventory can be obtained and calculated at each point in the planning horizon. This can be important and helpful for decision makers to determine the inventory and production quantity throughout the planning horizon. The existing methods usually calculate the solution at discrete points in the planning horizon. In Example 2 we compared the method with the model predictive control method presented in [8].

Although only two different sample problems were solved, in general we can use the proposed method for solving other types of inventory-production optimal control problems. In such situations, it is enough to determine the control and state functions as presented in (P2). This method can have extensive applications for solving optimal control problems arising in the theory of production planning and inventory control. As a future work, we can apply the proposed method for solving ordinary and partial differential equations in production management as well as for multi-objective optimal control problems for inventory systems.

Acknowledgements. The authors would like to thank the referees and Editors for their valuable and constructive comments and suggestions. This work was supported in part by: Research Deputy of Ferdowsi University of Mashhad, under Grant No. 38980 (dated October 28, 2015).

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