RAIRO-Oper. Res. 52 (2018) 691–712
https://doi.org/10.1051/ro/2017039

RAIRO Operations Research
www.rairo-ro.org

AN ANALYTICAL APPROACH FOR BEHAVIORAL PORTFOLIO MODEL WITH TIME DISCOUNTING PREFERENCE *

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Abstract. This paper presents a behavioral portfolio selection model with time discounting preference. Firstly, we discuss the portfolio selection problem and then modify this model based on cumulative prospect theory (CPT) as well as considering investors' time discounting preference in psychology. Furthermore, an analytical solution with satisfying behavior is given for our proposed model, the results show that when investors' goals are very ambitious, they put a high proportion of their wealth in long-term goals and adopt aggressive investment strategies with high leverage to reach short-term goals and the overall investment strategy also displays high leverage. Finally, numerical analysis is given and it is shown that investor who tends to future bias performs adequate confidence and patience whereas investor with present bias is apt to the immediate interests.

Mathematics Subject Classification. 91B28, 91G10, 97M30.

Received November 6, 2016. Accepted May 17, 2017.

1. Introduction

Portfolio theory has become highly developed and has strong theoretical support, making it essential in economics and finance. Markowitz [26] is considered the original work of portfolio theory, later, many researchers modify and develop this theory and made great advancement in this field [7, 8, 13, 14, 27, 34, 35, 40]. Behavioral portfolio theory (BPT) is an increasingly developing branch of portfolio theory, which is goal-based theory introduced by Shefrin and Statman [37], Tversky and Kahneman [41] propose cumulative prospect theory(CPT), which makes a solid foundation for BPT's development.

There has been a growing research interest in incorporating CPT into portfolio selection. However, previous studies overwhelmingly limited to the single-period setting with emphases on qualitative properties and empirical experiments [4, 5, 15, 25]. Berkelaar *et al.* [6] propose a very specific two-piece power utility function based on prospect theory in dynamic and continuous-time environment. Hamada *et al.* [18] provide a formal treatment of risk measures based on distortion functions that are consistent with the work of [44] dual (non-expected utility) theory of choice, and set out a general layout for portfolio optimisation in this non-expected utility framework using the risk neutral computational approach. Jin and Zhou [22] formulate and study a general continuous-time

Keywords. Time discounting, portfolio choice, satisfying behavior, analytical solution.

* *This work is supported by the National Science Foundation of China (NSFC) (71171048, 71371049) and the Fundamental Research Funds for the Central Universities (WUT:2017V1010).*

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behavioral portfolio selection model under Kahneman and Tversky's (cumulative) prospect theory, featuring S-shaped utility (value) functions and probability distortions.

Recently, He and Zhou [21] propose an analytical treatment of a single-period portfolio choice model featuring a reference point in wealth, S-shaped utility (value) functions with loss aversion, and probability weighting under CPT. Jin and Zhou [23] quantify the notion of greed, and explores its connection with leverage and potential losses, in a continuous-time behavioral portfolio choice model under CPT. Rasonyi and Rodrigues [31] examine an optimal investment problem in a continuous-time complete financial market with a finite horizon. However, these studies have not considered investors' time discounting preference or simply considering a constant/linear time discounting, therefore they can not capture the difference of investors' behaviors.

Time discounting is of interest to researchers in many areas of basic and applied psychology, including behavior analysis [1, 12], consumer behavior [19, 38], health psychology [10, 11], and organizational psychology [32] *etc.* There are many experimental studies on time preference [17, 20, 29, 33, 36].

Time discounting usually refers to present bias, which is the tendency of a decision-maker in intertemporal choices that he overvalues and prefers an immediate-but-small reward to a delayed-but-large reward. Meanwhile, Takeuchi [39] examines another type of time inconsistency that is future bias, subjects tend to postpone a reward until the near future because of the reverse time inconsistency. Green and Myerson [16] and Takeuchi [39] show that the estimated discounting factor would be too small if the estimation ignored the risk averseness of subjects. Sayman and Onculer [33] and Takeuchi [39], independently, notice that an inverse S-curve time discount function captures the time inconsistent preference. Huizen and Plantenga [42] examine the effects of time preferences on job search behaviour and tests. Waegeman *et al.* [43] investigate how the neural correlates of delaying gratification during a time discounting task are associated with individual differences in self-control ability. Attema *et al.* [2] introduce a new method to measure the temporal discounting of money. Unlike preceding methods, their method requires neither knowledge nor measurement of utility.

Motivated by the above literatures, we consider portfolio selection problem with investors' time discounting preference. Although there are various studies about the behavioral portfolio model and time discounting preference, however, few of them are combined with two sides, thus our work regarding the behavioral theory is interesting and play an important role in explaining investors' psychological and strategic changes. The purpose of this study is to build portfolio choice model considering investors' time discounting preference under CPT, the main features of our study are concluded as follows:

(i) We construct behavioral portfolio model considering time discounting, behavioral portfolio theory is goal-based theory, we further modify this theory with CPT. To find investors strategy change, we adopt general time discounting functions to rebuild behavioral portfolio selection model.
(ii) Our proposed model is solved by martingale methods, we give close form solution for model with satisfying behavior and related properties, the results show that investors with very ambitious goals put a high proportion of their wealth in long-term goals, meanwhile they adopt aggressive investment strategies with high leverage to reach short-term goals, the overall investment strategy also displays high leverage.
(iii) We give comparative analysis strategy change for six form of discounting functions and illustrate interpretation in psychology, numerical results indicate that different investors have different time discounting preferences. Specifically, investor with future bias performs adequate confidence and patience whereas investor with present bias tends to the immediate interests.

The remainder of this paper is organized as follows. Section 2 introduces background about time discounting and CPT. Section 3 proposes a model with time discounting preference for portfolio choice under CPT and gives analytical results of our model. Section 4 gives the numerical analysis and interpretation. Section 5 draws conclusions. In addition, all proofs of results are in appendix.

2. Preliminaries

Before our model is introduced, we firstly introduce related preliminaries about time discounting and CPT.

2.1. Related conception of time discounting

The following example is illustrated in order to understand the conceptions of the present bias and the future bias. Suppose that there are two questions for choice.

Q_1. Which of the following reward options do you prefer?
(1) \$100 paid in 52 weeks.
(2) \$110 paid in 53 weeks.
Q_2. Which of the following reward options do you prefer?
(3) \$100 paid today.
(4) \$110 paid in a week.

Suppose that you exhibit present bias, that is, you prefer (\$100 today) to (\$110 in 1 week) in Q_1 and (\$110 in 53 weeks) to (\$100 in 52 week) in Q_2. (\$100 today) is better than (\$110 in 1 week), so you are not willing to wait one week to get \$110 instead of the immediate \$100. The one week is too long. It follows that $T(\$100; \$110) < 1$. Next, let \$z denote the present value of (\$100 in 52 week) and observe that $T(\$z; \$100) = 52$ by definition. It follows that $T(\$z; \$100) + T(\$100; \$110) < 53$. Similarly, according to your choice in Q_2, there must be $\alpha > 0$ such that $T(\$z + \alpha; \$110) = 53$ where $\$z + \alpha$ is the present value of (\$110 in 53 weeks). Consider $T(\$z; \$110)$, how long you are willing to wait to get \$110 instead of receiving \$z now. It should be longer than 53 weeks for which you are willing to wait to get \$110 instead of $\$z + \alpha$ now. Thus, $T(\$z; \$110) > 53$. Altogether with the inequality above, it follows that

$$T(\$z; \$100) + T(\$100; \$110) < T(\$z; \$110) \tag{2.1}$$

which means that T is strictly submodular[3].

For a non-parametric definition of present and future bias, Takeuchi [39] introduces an equivalent delay function T on $\{(x, x') \in R_+^2 | x \leqslant x'\}$. Suppose that a subject is indifferent between two options $(x, 0)$ and (x', T). It is denoted that $T(x, x')$ is the delay that makes these two options the same to a subject, and $D(t)$ is discounting function.

Definition 2.1. $T(x, x')$ is an equivalent delay such that $(x, 0) \sim (x', T(x, x'))$.

Present and future biases, if any, are detected in the properties of this function T. Notice that, by transitivity, $D(T(x_0, x_1), x_1) \times D(T(x_1, x_2), x_2) \equiv D(T(x_0, x_2), x_2)$ holds regardless of the form of D and T. First, the following definition is straightforward.

Definition 2.2 (Time consistency)**.**
A subject is time consistent if T is modular[4].

If T is modular, a subject will not exhibit time inconsistent preference reversal. For example, the standard exponential discount function, $D(t) = \exp(-rt)$, implies that

$$T(x_0, x_1) + T(x_1, x_2) = T(x_0, x_2) \tag{2.2}$$

Definition 2.3. (Present bias). A subject exhibits present bias if T is strictly submodular.

It is observed that hyperbolic discount functions imply $T(x_0, x_1) + T(x_1, x_2) < T(x_0, x_2)$. In fact, when utility function is continuous, this present bias is consistent with the decreasing impatience of [30].

[3] A function $f : 2^V \to R$ is submodular if: $f(A) + f(B) \geqslant f(A \cup B) + f(A \cap B) \ \ \forall A, B \in 2^V$. Similarly, f is supermodular if: $f(A) + f(B) \leqslant f(A \cup B) + f(A \cap B) \ \ \forall A, B \in 2^V$.
[4] A function $f : 2^V \to R$ is modular if: $f(A) + f(B) = f(A \cup B) + f(A \cap B) \ \ \forall A, B \in 2^V$.

TABLE 1. Deneralized Weibull Function.

θ	q	$D(t)$	Name of discounting function
0	q	$e^{-(kt)^q}$	Weibull discount function
0	1	e^{-kt}	Exponential discount function
1	q	$[1+(kt)^q]^{-1/q}$	Log-logistic
1	1	$(1+kt)^{-1}$	Hyperbolic discount function

Definition 2.4 (Future bias)**.** A subject exhibits future bias if T is strictly supermodular. Equivalently, a subject exhibits increasing impatience if for any $\delta > 0, x_2 > x_1 > 0, (x_1, t_1) \sim (x_2, t_2)$ implies $(x_2, t_2 + \delta) \sim (x_1, t_1 + \delta)$.

This paper particularly addresses the separability assumption between discounting and utility and the non-linearity assumption on the utility function. Let (x, t) denote an option that will pay x at time t. Define the discounted present value of the option $V(x, t)$ as follows:

$$V(x,t) = D(t)u(x) \tag{2.3}$$

where $D(t)$ is the discount function and $u(x)$ is the instantaneous utility of the reward. In fact, D depends not only on t but also on the reward magnitude x, though the literature always assume that x and t are separable[5]. We assume that x and t are separable consist with the literature in order to simplify our model.

As for Delay function form of D, the following form function proposed by Takeuchi [39] list as follow:

$$D(t) = \frac{1}{[1+\theta(kt)^q]^{\frac{1}{\theta}}} \tag{2.4}$$

where $\theta \in (0, 1], k \in [0, \infty)$ and $q \in [0, \infty)$. This $D(t)$ called the generalized Weibull model, is a further-generalized version of the generalized hyperbolic of (2.4). Note that, while the generalized hyperbolic form represents only decreasing impatience (present bias), the generalized Weibull function can represent increasing impatience (future bias) as well. This is the advantage of the generalized Weibull function.

In addition, consistent liminal discounting is presented in (2.5). The term "liminal" is derived from the Latin limen, meaning a "threshold". This term is explained that the decision maker's discount rate will change at some known point. At the time of making a decision, they are in the "in-between" stage. Their discount rate will change, they know when, but it remains the same for the present. It is denoted this time of change as h, for 'horizon'. The period from now to h is the liminal period; h is the liminal point.

$$D(t) = \begin{cases} \gamma^t & \text{if} \quad t \leqslant h, \\ (\gamma/\delta)^c \delta^t & \text{if} \quad t > h. \end{cases} \tag{2.5}$$

where with $c \in [0, T], \delta, \gamma \in (0, 1)$. Liminal discounting preferences coincide with exponential discounting preferences on large subsets of timed outcomes.

The most prominent alternative model of discounting (two parameter model discounting) is a generalized form described by Myerson and Green [28]:

$$D(t) = \frac{1}{(1+kt^s)} \tag{2.6}$$

where s is a free parameter that may reflect individual differences in the scaling of delay and/or amount and and k is a parameter that reflects the discounting rate. In the following subsection we introduce the basic conception about prospect theory.

[5]In the literature, [3,39] are exceptions. Their non-parametric estimation do not assume that x and t are separable and are still capable of eliciting time preference.

2.2. Behavioral portfolio theory and cumulative prospect theory

In behavioral portfolio theory, investors divide their money into many mental account layers of a portfolio pyramid corresponding to goals. A central feature in behavioral portfolio theory is the observation that investors view their portfolios not as a whole, as prescribed by mean-variance portfolio theory, but as distinct mental account layers in a pyramid of assets, where mental account layers are associated with particular goals and where attitudes toward risk vary across layers.

Different from expected theory, CPT adopts an inverse S-shaped probability weighting a concave-shaped utility function (UF) for gains and convex for losses, the distortion is typically such that low probabilities are overestimated, the concrete form of utility function is as follow:

$$\mu(x) = \begin{cases} x^{\alpha} & \text{if} \quad x \geqslant 0, \\ -\lambda(-x)^{\beta} & \text{if} \quad x < 0. \end{cases} \tag{2.7}$$

where λ denotes loss aversion degree and $\alpha > 0, \beta > 0$. CPT is the main extension form of the original prospect.

Consequently, CPT can be viewed as an extension to the expected utility theory with the following three modifications [41]:

(i) Replacing equation (2.7) with (2.8) relative to the reference point. As shown in (2.8), the outcomes x are converted into gains ($\Delta x \geqslant 0$) or losses ($\Delta x < 0$) relative to a reference point x_0.

$$\Delta x = x - x_0 \tag{2.8}$$

(ii) Replacing the utility function with a value function to capture individual's risk attitude. The S-shaped value function, $g(x)$ in (2.9). The parameter $\lambda > 1$ is called "loss-aversion" coefficient, indicating that individuals are more sensitive to losses than gains.

$$g(x) = \begin{cases} (x - x_0)^{\alpha} & \text{if} \quad x \geqslant x_0, \\ -\lambda(x_0 - x)^{\beta} & \text{if} \quad x < x_0. \end{cases} \tag{2.9}$$

(iii) Replacing cumulative probabilities with weighted cumulative probabilities.

Line with the discrete form value and weighting functions, the continuous form prospect value [4] of a decision with stochastic outcomes x can be written as:

$$V = \int_{-\infty}^{x_0} g(x) d[w^{-}(F(x))] - \int_{x_0}^{+\infty} g(x) d[w^{+}(1 - F(x))] \tag{2.10}$$

where V is the prospect value and $F(x)$ denotes the cumulative distribution function (CDF) of the associated outcome x, and x_0 is a referent point. Equation (2.10) is a straightforward generalization of the original discrete formulation proposed by Tversky and Kahneman. Based on previous introduction, we present the following behavioral portfolio choice model.

3. Behavioral portfolio model with time discounting preference

In this section, we firstly introduce the formation behavioral portfolio model with time discounting, then our proposed dynamic model is transformed into two static subproblems. Since no explicit expression of solution, finally we further discuss the optimal solution of our model with the special case(satisfying behavior).

3.1. The formation of behavioral portfolio model considering time discounting preference

In this paper, we assume a continuous-time financial market following [24]. There are $n+1$ assets at time t for $k = 0, \ldots, n$, the investor possesses n different investment goals at n planing horizons, which can be characterized by target payoffs $\overline{W}_j$, the clients wants to obtain at time T_j, for $j = 1, \ldots, n$. At time t she allocates a fraction $w_j(t)$ of her wealth $W(t)$ to goal j and chooses goal-specific portfolios $\lambda_j(t) = (\lambda_j^1(t), \ldots, \lambda_j^n(t))'$, where $\lambda_j^k(t)$ is the fraction of wealth $W_j(t) = w_j(t)W(t)$ allocated to the asset k at time t. We put $W_0 = W(0)$ and $w_j^0 = w_j(0)$. The wealth dynamics for goal j is given as follows (see [24]):

$$\mathrm{d}W_j(t) = r(t)W_j(t)\mathrm{d}t + (\mu(t) - \mathbf{1}r(t))'\lambda_j(t)W_j(t)\mathrm{d}t + \sigma(t)'\lambda_j(t)W_j(t)\mathrm{d}B(t), W_j(0) = w_j^0 W_0. \tag{3.1}$$

The value function for goal j corresponds to the cumulative prospect theory value function

$$V_j(W;\overline{W}_j) = \int_{-\infty}^{W_j} v_j(x - \overline{W}_j)d\pi_j^-(F_W(x)) - \int_{\overline{W}_j}^{\infty} v_j(x - \overline{W}_j)d\pi_j^+(1 - F_W(x)) \tag{3.2}$$

where $v_j(x)$ is a piecewise-power value function:

$$v_j(x) = \begin{cases} \beta_j^+ x^{\alpha_j} & \text{if} \quad x > 0, \\ -\beta_j^-(-x)^{\alpha_j} & \text{if} \quad x < 0. \end{cases} \tag{3.3}$$

and π_j^+, π_j^- are non-decreasing, continuous probability weighting functions from $[0,1]$ into $[0,1]$ with $\pi^{\pm}(p) = p$ for $p = 0, 1, \pi_j^{\pm}(p) > p$ for p small and $\pi_j^{\pm}(p) < p$ for p large. Furthermore, $\beta_j^- \geqslant \beta_j^+ \geqslant 0$ and $\alpha_j \in (0,1)$. $F_W(x)$ denotes the cumulative distribution function of the random payoff W.

We use a slightly different the value function in extant literature compared with [41], the difference details are got in [6] and axiomatic foundation of preferences with satisfying behavior can be covered in [9].

Given time discounting preferences as described in the previous subsection, the investor determines at each time t how to optimally split wealth among the different investment goals and, additionally, how to optimally invest the wealth amounts allocated to the different investment goals. She solves the following decision problem:

$$\max_{\lambda_1(t),\ldots,\lambda_n(t)} \sum_{j=1}^{n} D(T_j)V_j(W_j(T_j), \overline{W}_j)$$

$$s.t. \begin{cases} \mathrm{d}W_j(t) = r(t)W_j(t)\mathrm{d}t + (\mu(t) - \mathbf{1}r(t))'\lambda_j(t)W_j(t)\mathrm{d}t + \sigma(t)', \\ \sum_{j=1}^n W_j(0) = W_0, \\ W_j(t) \geqslant 0, t \in [0, T_j], \quad j = 1, 2, \ldots, n. \end{cases} \tag{3.4}$$

$D(T)$ is the discounting function and characterizes the investor's time preferences.

To keep analytical tractability, we assume no probability weighting, *i.e.*, $\pi_j^{\pm}(p) = p$ for all $j = 1, 2, \ldots n$ and $p \in [0,1]$. This is a common assumption in behavioral finance. For the discussion and the results in this paper probability weighting is not crucial.

We apply martingale methods and rewrite the dynamic decision problem (3.4) as a static one:

$$\max_{\substack{W_1(T_1),\ldots,W_n(T_n) \\ w_1^0,\ldots,w_n^0}} \sum_{j=1}^{n} D(T_j)\mathbb{E}[v_j(W_j(T_j) - \overline{W}_j)]$$

$$s.t. \begin{cases} \mathbb{E}[\xi(T_j)W_j(T_j)] \leqslant \xi_0 w_j^0 W_0, \\ \sum_{j=1}^n w_j^0 \leqslant 1, \\ W_j(t) \geqslant 0, w_j^0 \geqslant 0, \quad j = 1, \ldots, n. \end{cases} \tag{3.5}$$

The vector $w_0 = (w_1^0, \ldots, w_n^0)'$ corresponds to the wealth's shares at time $t = 0$. At time $t = 0$ the investor decides how to split wealth among the n investment goals. Consequently, she allocates $W_j(0) = w_j^0 W_0$ to goal j, which corresponds to the budget constraint for this goal. Initial wealth shares and the corresponding goal-specific terminal wealths determine the investor's global value she obtains from the n different investment goals.

We solve problem (3.4) in two stage. First, for a given vector of initial wealth shares $w_0 = (w_1^0, \ldots, w_n^0)'$, we solve for each investment goal j the following goal-specific problem:

$$\max_{W_j(T_j)} \mathbb{E}\left[v_j(W_j(T_j) - \overline{W}_j)\right]$$

$$s.t. \begin{cases} \mathbb{E}[\xi(T_j)W_j(T_j)] \leqslant \xi_0 w_j^0 W_0, \\ W_j(T_j) \geqslant 0, \end{cases} \tag{3.6}$$

Second, given optimal terminal wealths $W_j^*(T_j)$ for all goals as function of w_0, we find the optimal vector of shares w_0^* that maximizes the investor's value functions:

$$\max_{w_1^0,\ldots,w_n^0} \sum_{j=1}^{n} D(T_j)\mathbb{E}\left[v_j(W_j^*(T_j) - \overline{W}_j)\right]$$

$$s.t. \begin{cases} \sum_{j=1}^n w_j^0 \leqslant 1, \\ w_j^0 \geqslant 0, \quad j = 1, \ldots, n. \end{cases} \tag{3.7}$$

3.2. The optimal solution for problem (15) and its related properties

The following theorem gives the solution to problem (3.6):

Theorem 3.1 [6]. *Let $w_0 = (w_0^1, \ldots, w_0^n)'$ be a vector of initial wealth shares, then for $j = 1, \ldots, n$ the optimal terminal wealth for investment goal j is given by:*

$$W_j^*(T_j) = \begin{cases} \overline{W}_j + \left(\dfrac{y_j \xi(T_j)}{\beta_j^+ \alpha}\right)^{\frac{1}{\alpha_j - 1}}, & \text{if} \quad \xi_j(T_j) < \xi_j^*(y_j), \\ 0. & \text{if} \quad \xi_j(T_j) \geqslant \xi_j^*(y_j). \end{cases} \tag{3.8}$$

where $\xi_j^*(y_j)$ solves $f_j(\xi_j^*(y_j), y_j) = 0$ and $y_j > 0$ satisfies $\mathbb{E}[\xi(T_j)W_j(T_j)] = \xi_0 w_0^j W_0$, the function f_j is defined as follow:

$$f_j(x, y) = \frac{1 - \alpha_j}{\alpha_j} \left(\frac{1}{yx}\right)^{\frac{\alpha_j}{1-\alpha_j}} (\beta_j^+ \alpha_j)^{\frac{1}{1-\alpha_j}} - \overline{W}_j yx + \beta_j^- \overline{W}_j^{\alpha_j} \tag{3.9}$$

where $x, y > 0$.

Proof. See Appendix 1. □

Optimal terminal wealths $W_j^*(T_j)$ present the following characteristic. In good states of the world at time $T_j(\xi(T_j) < \xi_j^*(y_j))$ the investor is able to reach her investment goal $\overline{W}_j$. In this case there is a strictly positive surplus $(y_j \xi(T_j)/(\beta_j^+ \alpha))^{\frac{1}{\alpha_j - 1}}$ which increases as $\xi_j(T_j)$ becomes smaller. By contrast, in bad states of the world at time $T_j(\xi(T_j) < \xi_j^*(y_j))$ the investor fails to reach her goal and her terminal wealth is zero. The probability of beating the investment goal corresponds to the probability that $\xi(T_j) < \xi_j^*(y_j)$.

In order to derive optimal initial wealth shares w_0^*, we need to understand how $W_j^*(T_j)$ depends on w_0. For sake of simplicity, we drop the index j in our discussion below, since the results apply to all investment goals. When it is not confusing we denote by w_0 the wealth share allocated to one specific investment goal. The following Lemma provides an explicit characterization of $\xi^*(y)$.

Lemma 3.2. *Let $x, y > 0$ and $f : \mathbb{R}_+^2 \to \mathbb{R}$ be defined as in equation (3.9), then for $y > 0, f(x, y) = 0$ possesses a unique solution $\xi^*(y) = \frac{a}{y}$ where $a > 0$ solves*:

$$\frac{1-\alpha}{\alpha} a^{\frac{-\alpha}{1-\alpha}} (\beta^+ \alpha)^{\frac{1}{1-\alpha}} - a\overline{W} + \beta^- \overline{W}^\alpha = 0. \tag{3.10}$$

Proof. See Appendix 2. □

We impose additional conditions on the dynamics of the stochastic discount factor ξ_t. We assume that the interest rate process r, the drift process μ and the volatility matrix are constant. Let $m = -(r + (\frac{1}{2})\|\kappa\|^2)$ and $s^2 = \|\kappa\|^2$. Then $\xi(T)$ is log-normally distributed with parameters $m_T = mT$ and $s_T = s\sqrt{T}$. Under these conditions (see [24]), we can easily determine the probability of reaching an investment goal at the time horizon T, which corresponds to $\Phi((\log(\xi^*(y) - m_T)/s_T))$. We also obtain an explicit characterization of the parameter y which satisfies the budget constraint for $W^*(T)$ in Theorem 1.

Lemma 3.3. *Let $W^*(T)$ be the optimal wealth from Theorem 1, then $y > 0$ solves*

$$\mathbb{E}[\xi(T)W(T)] = \xi_0 w_0 W_0 \tag{3.11}$$

if and only if $y > 0$ solves $g(y) = w_0$ where:

$$g(y) = b\Phi\left(\frac{\log(a/y) - m_T - s_T^2}{s_T}\right) + cy^{\frac{1}{\alpha-1}}\Phi\left(\frac{\log(a/y) - m_T - \frac{\alpha}{\alpha-1}s_T^2}{s_T}\right) \tag{3.12}$$

The constant $a > 0$ solves equation (3.10) from Lemma 1, and

$$b = \frac{\overline{W}}{\xi_0 W_0}\exp(-rT), \quad c = \frac{1}{\xi_0 W_0}(\beta^+\alpha)^{\frac{1}{1-\alpha}} \exp\left(\frac{\alpha m_T}{\alpha - 1} + \frac{1}{2}\frac{\alpha^2 s_T^2}{(\alpha-1)^2}\right).$$

Proof. See Appendix 3. □

The function h is strictly decreasing. Therefore y decreases as the initial share allocated to one investment goal increases. Consequently, ξ^* becomes larger, *i.e.*, the probability of reaching the investment goal is higher as more wealth is allocated to that investment goal. Moreover, for y and β^- fix, $h(y)$ strictly increases as β^+ increases, *i.e.*, as loss aversion decreases. Consequently, y decreases with loss aversion, *i.e.*, as loss aversion increases the probability of reaching the investment goal increases, but the surplus becomes smaller. The probability of reaching the investment goal is therefore maximal when $\beta^+ = 0$, which implies satisfying behavior.

When $h(y) = w_0$ possesses a solution, then it is unique since h is strictly decreasing. For $\beta^+ > 0$, $h(y) = w_0$ possesses a solution for all w_0. However, for $\beta^+ = 0$, $h(y) = w_0$ cannot be solved for $w_0 > b$ since $h(y) \in [0, b]$ for all $y \geqslant 0$. For the case $\beta^+ = 0$ we will impose some further conditions on b when we solve for the vector of wealth share $(w_1^0, \ldots, w_n^0)'$. If $\beta^+ = 0$ and $w_0 \leqslant b$ we obtain an explicit solution for y as function of w_0:

Corollary 3.4. *Let g be as given by equation (3.10). Let $\beta^+ = 0$ and $w_0 < b$,then $g(y) = w_0$ if and only if*

$$y = a\exp(-s_T\Phi^{-1}(w_0/b) - m_T - s_T^2), \tag{3.13}$$

where

$$b = \frac{\overline{W}}{\xi_0 W_0}\exp(-rT).$$

Proof. See Appendix 4. □

Note that for $\beta^+ = 0$, the optimal terminal wealth $W^*(T)$ does not depend on β^-. Indeed, β^- only enters into $\xi^*(y)$ through the constant a. However, since $\xi^*(y) = a/y$ we have $\xi^*(y) = \exp(s_T \Phi^{-1}(w_0/b) + m_T + s_T^2)$, which is independent from a. More generally, the following results holds:

Corollary 3.5. *The optimal terminal optimal wealth $W^*(T)$ depends on β^+ and β^-, only through the ratio β^+/β^-,i.e., the degree of loss aversion.*

Proof. See Appendix 5. □

The following Lemma gives an explicit characterization of the optimal utility level $\mathbb{E}[W^*(T)]$ as function of y:

Lemma 3.6. *We have $\mathbb{E}[W^*(T)] = k(y)$ where*

$$k(y) = \overline{W}\Phi\left(\frac{\log(a/y) - m_T - s_T^2}{s_T}\right) + dy^{\frac{1}{\alpha-1}}\Phi\left(\frac{\log(a/y) - m_T + s_T^2}{s_T}\right) \tag{3.14}$$

and $d = (\beta^+\alpha)^{\frac{1}{1-\alpha}} \exp(\frac{m_T}{\alpha-1} + \frac{1}{2}\frac{s_T^2}{(\alpha-1)^2})$. The function $k(\cdot)$ is continuous, strictly decreasing and $\lim_{y\to\infty} k(y) = 0$.

Proof. See Appendix 6. □

We now rewrite problem (3.7) as follows. Optimal wealths $W_1(T_1), \ldots, W_n(T_n)$ are given by Theorem 1 where $y_j = h^{-1}(w_j^0)$ for $j = 1, 2, \ldots, n$ and $w_0 = (w_1^0, \ldots w_n^0)'$ solves

$$\max_{w_1^0,\ldots,w_n^0} \sum_{j=1}^{n} D(T_j) k_j(h_j^{-1}(w_j^0))$$

$$s.t. \begin{cases} \sum_{j=1}^n w_j^0 = 1, \\ w_j^0 \geqslant 0, \quad j = 1, \ldots, n. \end{cases} \tag{3.15}$$

In general, Problem (3.15) must be solved numerically, since no explicit expression for h_j^{-1} is available. The following section we concentrate on discussing a special case where solutions to problem (3.15) can be derived analytically. Before discussing, we report here optimal wealths and optimal strategies for all investment goals and at any time $t \in [0, T_j]$.

3.3. The analytical solution for the case with satisfying behavior

We now consider investors who display satisfying behavior, *i.e.*, $\beta_j^+ = 0$ for all j. In this case Problem (3.15) is analytically tractable since we have an explicit expression for h_j^{-1}. As discussed before, satisfying behavior describes investors who are only concerned about reaching their investment goals, while a surplus above their target wealth does not deliver any additional value to them. In our opinion this is the typical case when investment goals have been clearly specified.

when $\beta_j^+ = 0$ for all j, we can prove the following theorem:

Theorem 3.7. *Let $\beta_j^+ = 0$ for all j. Let $b_j = \overline{W}_j \exp(-rT_j)/(\xi_0 W_0)$ and assume that $\Sigma_{j=1}^n b_j \geqslant 1$, then*

$$w_j^{*0} = \begin{cases} b_j\Phi\left(-\frac{1}{s_j}\log\left(\frac{\nu}{\xi_0 W_0}\right) - \frac{1}{s_j}\left(-\log D(T_j, j) - rT_j + \frac{1}{2}s_{T_j}^2\right)\right) & \text{if} \quad \nu > 0, \\ b_j. & \text{else.} \end{cases} \tag{3.16}$$

where ν solves $\Sigma_{j=1}^n w_j^{*0} = 1$.

Proof. See Appendix 7. □

The condition $\Sigma_{j=1}^{n} b_j \geqslant 1$ is equivalent to $\Sigma_{j=1}^{n} \overline{W}_j \exp(-rT_j) \geqslant W_0$, *i.e.*, the discounted value of all target wealths must be larger than or equal to the initial wealth. If the discounted value of all target wealths is strictly larger than the initial wealth($\Sigma_{j=1}^{n} b_j \geqslant 1$) then we have $\nu > 0$ and the investor invests some of her wealth into the risky assets. Therefore, in this case, the volatility of the market price of risk s and the inter-temporal discount function $D(T_j)$ enter into the expression for wealth shares w_j^{*0}. If $\Sigma_{j=1}^{n} \overline{W}_j \exp(-rT_j) = W_0$, then $\nu \leqslant 0$ and the investor can reach all investment goals with probability one by simply putting all her wealth into the risk-free asset. Therefore, in this case, w_j^{*0} simply corresponds to b_j, that is the ratio between the discounted value of the target wealth for goal j and the initial wealth.

The ratio b_j can be interpreted as a measures of how ambitious an investment goal is relative to the initial wealth. Obviously, we expect investors with satisfying behavior to put a higher proportion of their wealth into goals with a higher discounted target wealth. Indeed, as we discussed above, if the initial wealth is high enough, then using the risk-free strategy will ensure that all investment goals will reached with probability one, and investors put a higher proportion of their wealth into goals with higher b_j. However, when $\Sigma_{j=1}^{n} b_j > 1$, the risk-free strategy causes investors to fails some of their investment goals for sure. Therefore in this case investors might prefer having some risky assets into their portfolios and optimal wealth shares will then deviate from b_j. We also point out that when the risk-free strategy fails, then wealth shares w_j^{*0} are strictly smaller than b_j for all investment goals. This means that investors decrease the proportion of wealth put into goal j relative to b_j for all investment goals, *i.e.*, instead of using the risk-free strategy form some goals and risky strategies for others, they prefer to invests into risky strategies for all investment goals. This is due to their risk-seeking behavior, which is implied by CPT preferences.

The question now is how investors decide to split their wealth among investment goals when the risk-free strategy fails. In other words, on which goals do investors take more risk and put less wealth if we also account for how ambitious an investment goal is? In order to take into consideration the importance of one investment goal relative to the others, we consider the ratio w_j^{*0}/b_j, which only depends on the characteristics of the market and the time horizon. We therefore analyze how the ratio w_j^{*0}/b_j changes as function of the time horizon. The results are reported in the following corollary:

Corollary 3.8. *Let $\nu > 0$ such that $\Sigma_{j=1}^{n} w_j^{*0} = 1$ and w_j^{*0} is given in Theorem 3.7, assume the equation $\log D(T_j) + rT_j - \frac{1}{2} s_{T_j}^2 = 0$ exists solution T_j^* then*

$$\frac{w_j^{*0}}{b_j} = \Phi\left(-\frac{1}{s_j}\log\left(\frac{\nu}{\xi_0 W_0}\right) - \frac{1}{s_j}\left(-\log D(T_j) - rT_j + \frac{1}{2}s_{T_j}^2\right)\right) \tag{3.17}$$

and following holds:

(i) *If $\log D(T_j) + rT_j - \frac{1}{2}s_{T_j}^2 < 0$, then the ratio w_j^{*0}/b_j is maximal for $T_j = T_j^*$, increasing for $T_j < T_j^*$ and decreasing for $T_j > T_j^*$ with $\lim_{T_j \to \infty} w_j^{*0}/b_j = 0$.*

(ii) *If $\log D(T_j) + rT_j - \frac{1}{2}s_{T_j}^2 > 0$,then the ratio w_j^{*0}/b_j is minimal for $T_j = T_j^*$, strictly increasing for $T_j > T_j^*$ with $\lim_{T_j \to \infty} w_j^{*0}/b_j = 1$ and strictly decreasing for $T_j < T_j^*$.*

(iii) *If $\log D(T_j) + rT_j - \frac{1}{2}s_{T_j}^2 = 0$, then the ratio w_j^{*0}/b_j is strictly increasing for all T_j if $\log(\frac{\nu}{\xi_0 W_0}) > 0$, strictly decreasing for all T_j if $\log(\frac{\nu}{\xi_0 W_0}) < 0$ and constant if $\log(\frac{\nu}{\xi_0 W_0}) = 0$.*

(iv) *Moreover, if $\lim_{T_j \to 0} \log D(T_j) = 0$, then*

$$\lim_{T_j \to 0} \frac{w_j^{*0}}{b_j} = \begin{cases} 1 & \textit{if} \quad \nu < W_0, \\ 0.5 & \textit{if} \quad \nu = W_0, \\ 0 & \textit{if} \quad \nu > W_0. \end{cases} \tag{3.18}$$

Proof. See Appendix 8. □

We define the wealth ratio $WR_0 = W_0/\Sigma_{j=1}^n \overline{W}_j \exp(-rT_j)$ as the ratio between the initial wealth and the discounted value of all target payoffs. Before we discuss Corollary 3, we briefly present here how WR_0 is linked to ν. First, we notice that $1/WR_0 = \Sigma_{j=1}^n b_j$. Therefore, when $WR_0 = 1$ then also $\Sigma_{j=1}^n b_j = 1$, and all investment goals can be reached with probability one by simply adopting the risk-free strategy for all goals. When WR_0 is smaller than 1, then $\Sigma_{j=1}^n b_j$ is larger than 1 and optimal wealth shares differ from b_j. If WR_0 is very small, then ν must be very large in order to have $\Sigma_{j=1}^n w_j^{*0} = 1$. Indeed, w_j^{*0} strictly decreases with ν and a large ν is required in order to satisfies $\Sigma_{j=1}^n w_j^{*0} = 1$ when $\Sigma_{j=1}^n b_j$ is much larger than 1. On the other hand, when WR_0 is slightly smaller than 1, then $\Sigma_{j=1}^n b_j$ is slightly higher than 1 and a small $\nu > 0$ is sufficient to have $\Sigma_{j=1}^n w_j^{*0} = 1$. Therefore, whether $\nu > 0$ is large or small depends on how ambitious the investments goals are relative to the initial wealth, *i.e.*, on whether the wealth ratio WR_0 is very small or near to 1. We now use this observation to discuss Corollary 3.

The quantity $r - \log(D(T_j))/T_j - \frac{1}{2}s^2$ can be written as $-[-(r + (\frac{1}{2})s^2) + \log(T_j)/T_j + s^2]$, where $-(r + (\frac{1}{2})s^2)$ is the growth rate of the pricing kernel, s^2 its volatility, and $D(T)$ is the discount function. Therefore, $r - \log(D(T_j))/T_j - \frac{1}{2}s^2$ is negative (positive), when the pricing kernel displays small (high) absolute growth rate, high (low) volatility, and, additionally, the inter-temporal discount factor is high (small). When these conditions hold, long-term investing appears less (more) attractive. The results in Corollary 3 are consistent with this observation, as will become clear from the following discussion.

Let us first consider the case $r - \log(D(T_j))/T_j - \frac{1}{2}s^2 < 0$, *i.e.*, long-term investing is less attractive. When the wealth ratio WR_0 is small enough such that ν is larger than W_0, then $\log(\nu/(\xi_0 W_0))$ is strictly positive and an intermediate horizon $\hat{T}_j$ exists where the corresponding ratio w_j^0/b_j is maximal, while it decreases as the horizon increases. Note that $\hat{T}_j$ can be large when WR_0 is very small, *i.e.*, when the initial wealth is very small relative to the discounted sum of target wealths the ratio w_j^0/b_j is maximal for a long term goals. On the other hand, when WR_0 is slightly smaller than 1 such that ν is smaller than W_0, then $\log(\nu/(\xi_0 W_0))$ is strictly negative and the ratio w_j^0/b_j strictly decreases with the time horizon. This means that in this case w_j^0/b_j is maximal for very short-term goals, while it is small for long-term goals. Summarizing, when long-term investing is less attractive, investors put a higher proportion of their wealth (after accounting for how important the goal is) to long-term goals when the initial wealth is very small relative to the current value of target payoffs (goals are too ambitious), while they put a higher proportion of their wealth into short-term goals when goals are not too ambitious.

Let us now consider the case $r - \log(D(T_j))/T_j - \frac{1}{2}s^2 > 0$, *i.e.*, long-term investing is more attractive. If WR_0 is small enough such that $\nu > W_0$ and $\log(\nu/(\xi_0 W_0))$ is strictly positive, then T_j^* is negative, *i.e.*, the ratio w_j^0/b_j strictly increases with T_j. It is therefore maximal for very long-term goals. On the other hand, when WR_0 is slightly smaller than 1 such that ν is smaller than W_0 and $\log(\nu/(\xi_0 W_0))$ is negative, then T_j^* is positive. Therefore, there exists an intermediate horizon T_j^* where the ratio w_j^{*0}/b_j is minimal, while it strictly increases for $T_j > T_j^*$. Moreover, when $\nu < W_0$, the ratio is also maximal equals to 1 at $T_j = 0$. Summarizing, when long-term investing is more attractive, the ratio w_j^{*0}/b_j is maximal for very long-term goals and, when goals are not too ambitious, also for very short-term investment goals.

How does the investment strategy just discussed impact the probability of reaching the investment goals? The answer to this question is given in the following corollary:

Corollary 3.9. *Let $\nu > 0$ such that $\Sigma_{j=1}^n w_j^{*0} = 1$ and w_j^{*0} is given in Theorem 2, assume that the equation $\log D(T_j) + rT_j - \frac{1}{2}s_{T_j}^2 = 0$ exists unique solution T_j^* then*

$$\mathbb{P}_j(W_j^* \geqslant \overline{W}_j) = \Phi\left(-\frac{1}{s_j}\log\left(\frac{\nu}{\xi_0 W_0}\right) - \frac{1}{s_j}\left(-\log D(T_j) - rT_j + \frac{1}{2}s_{T_j}^2\right)\right) \tag{3.19}$$

and following holds:

(i) *If* $\log D(T_j) + rT_j - \frac{1}{2}s_{T_j}^2 < 0$, *then* $\mathbb{P}_j(T_j)$ *is maximal for* $T_j = T_j^*$, *increasing for* $T_j < T_j^*$ *and decreasing for* $T_j > T_j^*$ *with* $\lim_{T_j\to\infty} w_j^{*0}/b_j = 0$.

(ii) *If* $\log D(T_j) + rT_j - \frac{1}{2}s_{T_j}^2 > 0$, *then* $\mathbb{P}_j(T_j)$ *is minimal for* $T_j = T_j^*$, *strictly increasing for* $T_j > T_j^*$ *with* $\lim_{T_j\to\infty} w_j^{*0}/b_j = 1$ *and strictly decreasing for* $T_j < T_j^*$.

(iii) *If* $\log D(T_j)+rT_j-\frac{1}{2}s_{T_j}^2 = 0$, *then* $\mathbb{P}_j(T_j)$ *is strictly increasing for all* T_j *if* $\log(\frac{\nu}{\xi_0 W_0}) > 0$, *strictly decreasing for all* T_j *if* $\log(\frac{\nu}{\xi_0 W_0}) < 0$ *and constant if* $\log(\frac{\nu}{\xi_0 W_0}) = 0$.

(iv) *Moreover, if* $\lim_{T_j\to 0} \log D(T_j) = 0$, *then*

$$\lim_{T_j\to 0} \mathbb{P}_j(T_j) = \begin{cases} 1 & \textit{if} \quad \nu < W_0, \\ 0.5 & \textit{if} \quad \nu = W_0, \\ 0 & \textit{if} \quad \nu > W_0. \end{cases} \tag{3.20}$$

Proof. See Appendix 9. □

The quantity $r - \log(D(T_j))/T_j - \frac{1}{2}s^2$ can be written as $-[-(r + (\frac{1}{2})s^2) + \log(T_j)/T_j + s^2]$, Thus $r - \log(D(T_j))/T_j - \frac{1}{2}s^2$ is negative (positive) when the absolute growth rate of the pricing kernel is small (high) and the inter-temporal discount factor is high (small). If the absolute growth rate of the pricing kernel is small, bad states of the world are more likely to occur. Therefore, the probability to reach an investment goal decreases, especially for long-term goals. On the other hand, if the absolute growth rate of the pricing kernel is large, good states of the world are more likely to occur and the probability to reach an investment goal is higher, especially for long-term goals. Finally, we also see that when the initial wealth is high enough, (very) short-term goals will be reached almost surely. This is due to the fact that for very short horizons wealth shares almost corresponds to b_j (the risk-free strategy), as reported in Corollary 3.

4. Numerical analysis

To illustrate the implications of the behavioral model presented in the previous section, we first make some discussion about the role of loss aversion parameter and then we take a simple example to show the optimal investment strategy for an investor with cumulative prospect theory preferences and satisfying behavior.

4.1. Optimal investment strategy with numerical simulation

We assume that the investor has three investment gaols at different time horizons. We assume that $W_j = \$50\,000\exp(rT_j)$ for $j = 1, 2, 3$ where $T_1 = 1$ year (short-term), $T_2 = 5$ years (medium term) and $T_3 = 20$ years (long-term), *i.e.*, all investment goals have the same discounted value equal to \$50 000. Under this assumption, b_j in Theorem 3.7 is identical for all investment goals and thus wealth shares are not affected by the how ambitious an investment goal is relative to the others.

We specify the investor's preferences assuming $\beta_j^- = 2.25$ and $\alpha_j = 0.88$ for all $j = 1, 2, 3$ and $D(t) = exp(-rt)$, whileβ_j^+ is not and will determine the degree of loss aversion $\beta_j = \beta_j^-/\beta_j^+$. In our numerical examples we further assume that there is one risky asset, that is the market portfolio, with drift μ and volatility σ. The Sharpe ratio corresponds to $\kappa = (\mu - r)/\sigma$, where r is the risk-free rate of return.

When $\beta_j^+ \neq 0$ and the degree of loss aversion $\beta_j = \beta_j^-/\beta_j^+$ is in the range of 2−4 for all j, as calibrated by Tversky and Kahneman [41] (but also if it is much larger), we see that the investor optimally puts almost all her wealth in the long-term goal. This happens also if the degree of loss aversion is much higher for the short-term goal than for the long-term goal. This is because for cumulative prospect theory investors long-term investing is very attractive. Figure 1 displays optimal wealth shares as function of β_j^+, under the assumption that $\beta_j^- = 2.25$ for all j. For β_j^+ in the range from 0.6 to 1, which implies a degree of loss aversion from 2.25 to 3.75, almost all wealth is invested in the long-term goal. It follows that in case that the degree of loss aversion

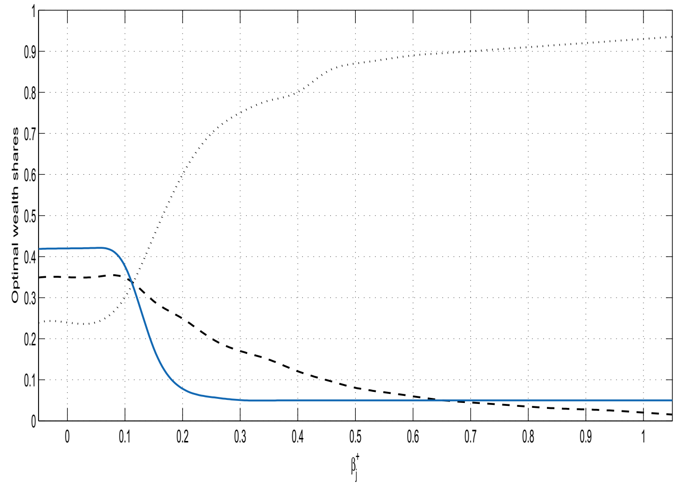

FIGURE 1. Optimal wealth shares change with β_j.

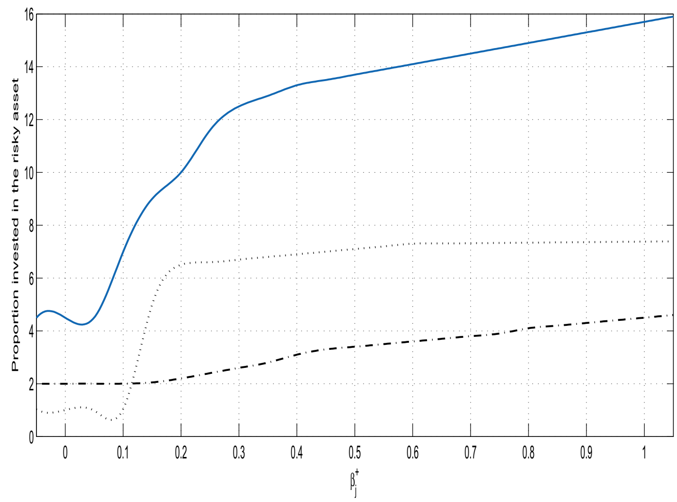

FIGURE 2. Proportion in risky asset changes with β_j.

is in the range 2−4, the overall investment strategy almost corresponds to the long-term investment strategy, as shown in Figure 2.

Figures 1 and 2 display that optimal wealth shares at time $t = 0$ as function of the of β_j^+ and optimal proportion of risky assets at time $t = 0$ as function β_j^+ respectively when the investor has three investment goals with same discounted value at time $t = 0(i.e., W_j \exp(rT_j) = \$50\,000$ for all j) at horizons 1 year (blue solid line), 5 years (dashed line) and 20 years (dotted line). Investor's preferences are characterized by $\beta_j^- = 2.25$ and $\alpha_j = 0.88$ for $j = 1, 2, 3$. We set $r = 0.03, \sigma = 0.2$ and $\kappa = 0.2$.

A numerical example based on real data from G7 counties Stock Exchange Market will be given to illustrate the validity of our proposed model. Assume that an investor wants to choose 7 risky assets for his investment, and he could reallocate his wealth at the beginning of each period. He intends to make a eight-period investment with initial wealth $W_0 = \$10\,000$, loss aversion degree $\beta_j = 2.25$, $\alpha_j = 0.88$. Original data come from the weekly

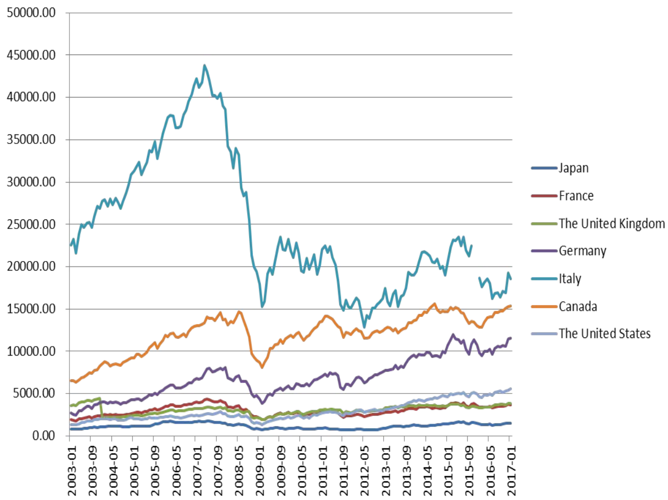

FIGURE 3. Trend of stock market in G7 countries from 2003 to 2017.

TABLE 2. Parameters of stock market data in G7 countries.

Country	μ(Mean)	σ(S.D)	max	min	Sample
US	2914.74	1136.12	5614.79	1320.91	2003.01-2017.01
UK	3095.09	534.14	4476.87	1929.75	2003.01-2017.01
Italy	24378.08	8433.53	43755.00	12874.00	2003.01-2017.01
Germany	6858.69	2334.09	11966.17	2423.87	2003.01-2017.01
France	3017.84	594.81	4354.42	1762.24	2003.01-2017.01
Japan	1180.69	309.52	1774.88	719.40	2003.01-2017.01
Canada	12006.26	2291.04	15625.73	6343.29	2003.01-2017.01

TABLE 3. The optimal investment strategies using our model.

Code	G*621	G*597	G*658	G*657	G*596	G*578	G*622	Stock code
Time	US	UK	Italy	Germany	France	Japan	Canada	Terminal wealth
2003.01	0.1944	0.1997	0.2349	0.1820	0.0678	0.0159	0.3501	\$10000.00
2005.01	0.2243	0.1819	0.0693	0.1723	0.0259	0.2372	0.0991	\$21397.20
2007.01	0.2449	0.0668	0.1993	0.2261	0.0642	0.0316	0.1671	\$33514.38
2009.01	0.2366	0.1636	0.0312	0.0160	0.0967	0.1396	0.3163	\$43689.94
2011.01	0.1576	0.1336	0.1135	0.1981	0.0547	0.0766	0.2365	\$52610.34
2013.01	0.1777	0.0205	0.0749	0.0278	0.2714	0.2409	0.1868	\$66404.49
2015.01	0.2215	0.2451	0.0057	0.1699	0.0702	0.0127	0.2749	\$76766.66
2017.01	0.1864	0.2253	0.2100	0.0831	0.0745	0.0567	0.1640	\$87537.47

where G*621 represents G002703621.

closing prices of the 7 risky assets from Jan. 2003 to Jan. 2017, the trend of market stock in G7 countries are shown in Figure 3 and the parameters of stock market data are listed in Table 2. We set 2 years as an observation to handle these historical data. Using our analytical results, the optimal strategies of investment can be obtained as shown in Table 3. From Table 3, we can find that the terminal wealth becomes larger as time increases, which implies that optimal strategy has a great impact on the portfolio selection. If the investor

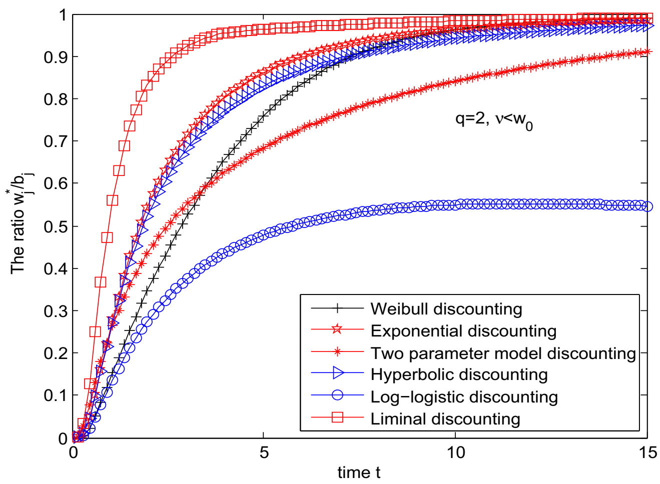

FIGURE 4. Optimal ratio change with discounting function when $q = 2, \nu < W_0$.

is not satisfied with any of the obtained investment strategy, he/she can reset the parameter values of his/her preferences. In a word, all the results above suggest that different solutions reflect investors' different investment intention.

4.2. Investors' strategies change with different discounting functions

Recall afore mentioned section, we derive the optimal investment strategy for an investor with cumulative prospect theory preferences and satisfying behavior. However, it is difficult to capture effects of investors' strategies with discounting function, therefore in this subsection, we mainly focus on how investors' strategies change with different discounting functions. Firstly, we select the following six different discounting functions: Weibull discounting, Exponential discounting, Two parameter model discounting, Hyperbolic discounting, Log-logistic discounting, Liminal discounting. Since the underlying six discounting functions can be expressed by previous section, we highlight the changes of their strategies, therefore illustrative analysis is given in Figures 4, 5 and 6.

(1) Investor strategies' changes from Figure 4. Figure 4 displays fluctuation of investors' optimal wealth ratio with time, in which preferences can be described by six discounting functions when fix parameters $q = 2, \nu < W_0$. As shown in Figure 4, the investors with liminal discounting preference hold the highest wealth ratio at the beginning. There is a sharp increase until a known point (about $t = 5$), then the ratio almost tends to 1. The wealth ratio of investors with exponential discounting rank second, the difference between the investors with liminal discounting and investors with exponential discounting is that the former ratio increases slowly at early stage and they are both approximately constant in the end. The notable point is that investors with hyperbolic discounting and exponential discounting preference almost make no difference and they rank third. The most interesting phenomenon appears that two curves with two parameter model discounting and Weibull discounting intersect, before they intersect, the ratio of two parameter model discounting is higher than Weibull discounting, but in reverse after intersection. Finally investor's wealth ratio with log-logistic discounting is the lowest. Except for hyperbolic discounting and exponential discounting, they all are future biased.

(2) Investor strategies' changes from Figures 5 and 6. Figures 5 and 6 describe the optimal wealth ratio change with six discounting functions when $q = 1, \nu < W_0$ and $q = 0.5, \nu < W_0$ respectively. The clearly noticeable

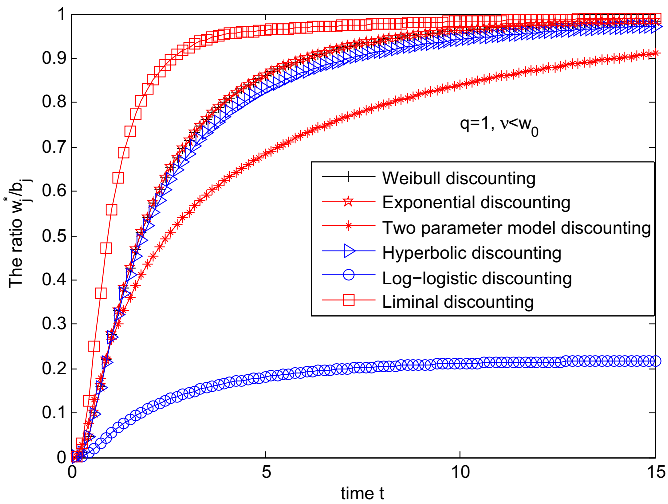

Figure 5. Optimal ratio change with discounting function when $q = 1, \nu < W_0$.

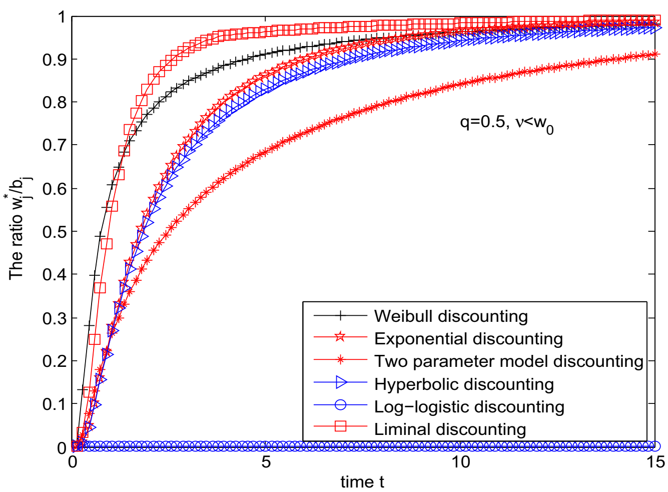

Figure 6. Optimal ratio change with discounting function $q = 0.5, \nu < W_0$.

difference between Figures 4 and 5 is that when $q = 1$ the function of exponential discounting is the same as Weibull discounting, therefore the two curves coincide the same. As shown in Figure 6, the optimal wealth ratio of investor with Log-logistic discounting function is almost zero, they don't invest in risky assets, liminal discounting investors continue to maintain leading advantages, investors with Weibull discounting rank second. If $q = 1$ and $\theta = 0$, discounting function imply time consistency and constant impatience. Investors with discounting function with $q = 0.5$ tend to present biased, they put more weight on present payment, investors with constant impatience are indifferent toward changes of strategies.

(3) Psychological accounts for future bias. There are potential interpretations and psychological accounts for future bias, which include the unreliability of own future memory and the notion of extended present. First, a subject may anticipate that she is going to forget about a delayed reward. Suppose that she thinks her

short-term memory is most likely to fade after several weeks, *i.e.*, the hazard rate of the memory loss is increasing in time during those weeks and decreasing thereafter. Assume that she is a little skeptical about the plausibility of the future payment but she still thinks the future payment will be delivered as long as she remembers it, by reclaiming it. Then, the revealed time preference results in the inverse S-shaped time discount function. It is left for further research to control this psychological factor. Secondly, but most importantly, future bias observations suggest that the present is not a single point on the time line but, it extends into the immediate future. It is also observed that the inverse S-shaped time discount function fits to this concept of the extended present.

In summary, from previous discussion, we conclude that when investors' only objective is to reach their goals and all payoffs above the reference point are considered as fully satisfactory, then any payoff (even if large) in scenarios where goals will be reached do not compensate investors for failing the reference point in other scenarios. In addition, investor with different discounting preferences make difference. In particular, investor who tends to future bias performs adequate confidence and patience while investor with present bias is apt to the immediate interests.

5. Conclusions

In this paper we apply CPT to obtain a time discounting preference portfolio selection model, where investors possess different investment goals at different time horizons. Our model assumes that investors mentally organize each investment goal as a separate account and derive optimal investment strategies for each investment goal, ignoring covariance between goal-specific portfolios. We derived optimal wealth shares allocated to each investment goal and optimal investment strategies, when investors additionally display the satisfying behavior, *i.e.*, fully satisfied when they reach their investment goals.

Based on previous presentation, we can conclude our contributions as follows:

(i) Investors mainly invest too reach short-term goals when they are not too ambitious. However, when investors are with very ambitious goals, they mainly invest to reach long-term goals and adopt aggressive investment strategies for their short-term goals. In this case, the overall investment strategy displays a high leverage. Therefore, our model explains high leverage ratios for investors, who have high incentive to reach ambitious short-term investment goals.

(ii) When investors' only objective is to reach their goals and all payoffs above the reference point are considered as fully satisfactory, then any payoff (even if large) in scenarios where goals will be reached do not compensate investors for failing the reference point in other scenarios. In addition, investor with different discounting preferences make difference.

(iii) We also consider different types preference of investors' changes towards investment strategies, including present bias and future bias, in particular, investor with present bias is apt to immediate interests, most importantly, we give other interpretations and psychological accounts for investors with future bias, which include the unreliability of own future memory and the notion of extended present.

Appendix A. Proofs

A.1. Proof of Theorem 1

See [6]. We additionally point out that $f_j(x,y)$ is a continuous, strictly decreasing function. For each $y > 0$ fix, we have $\lim_{x\to 0} f_j(x,y) = +\infty$ and $\lim_{x\to\infty} f_j(x,y) = -\infty$. So for each $y > 0$ we find a unique x such that $f_j(x,y) = 0$. For $y > 0$ fix, we denote by $\xi_j^*(y)$ the solution to $f_j(x,y) = 0$.

A.2. Proof of Lemma 1

Assume that $\xi_j^*(y) = a/y^b$ for some $a, b \in \mathbb{R}$. We have:

$$f(\xi_j^*(y), y) = \frac{1-\alpha}{\alpha}\left(\frac{y^b}{ax}\right)(\beta^+\alpha)^{1/(1-\alpha)} - \overline{W}yay^{-b} + \beta^-\overline{W}^\alpha$$
$$= \frac{1-\alpha}{\alpha}a^{-\alpha/(1-\alpha)}y^{(b-1)\alpha/(1-\alpha)}(\beta^+\alpha)^{\alpha/(1-\alpha)} - \alpha\overline{W}y^{1-b} + \beta^-\overline{W}^\alpha. \tag{A.1}$$

This expression is constant if and only if $b = 1$. In this case we have:

$$f(\xi_j^*(y), y) = \frac{1-\alpha}{\alpha}a^{-\alpha/(1-\alpha)}(\beta^+\alpha)^{\alpha/(1-\alpha)} - \alpha\overline{W} + \beta^-\overline{W}^\alpha \tag{A.2}$$

and thus $f(\xi_j^*(y), y) = 0$ if and only if

$$\frac{1-\alpha}{\alpha}a^{-\alpha/(1-\alpha)}(\beta^+\alpha)^{\alpha/(1-\alpha)} - \alpha\overline{W} + \beta^-\overline{W}^\alpha = 0. \tag{A.3}$$

We still have to show that $a > 0$ and is unique. Note that

$$\frac{1-\alpha}{\alpha}a^{-\alpha/(1-\alpha)}(\beta^+\alpha)^{\alpha/(1-\alpha)} - \alpha\overline{W} + \beta^-\overline{W}^\alpha = f(a, 1) \tag{A.4}$$

and $f(a, 1)$ is strictly decreasing, $\lim_{a\to 0} f(a, 1) = 1$ and $\lim_{a\to\infty} f(a, 1) = -\infty$. So, $f(a, 1) = 0$ possesses a unique solution $a > 0$ and the statement in the Lemma follows.

A.3. Proof of Lemma 2

From Theorem 1, it follows:

$$\mathbb{E}[\xi(T)W^*(T)] = \mathbb{E}\left[\overline{W}\xi(T)\mathbf{1}_{\xi(T)\leqslant\xi^*(y)} + \left(\frac{y}{\beta^+\alpha}\right)^{\frac{1}{\alpha-1}}\xi(T)^{\frac{\alpha}{1-\alpha}}\mathbf{1}_{\xi(T)\leqslant\xi^*(y)}\right]$$
$$= \overline{W}\mathbb{E}[\xi(T)\mathbf{1}_{\xi(T)\leqslant\xi^*(y)}] + \left(\frac{y}{\beta^+\alpha}\right)^{\frac{1}{\alpha-1}}\mathbf{E}\left[\xi(T)^{\frac{\alpha}{1-\alpha}}\mathbf{1}_{\xi(T)\leqslant\xi^*(y)}\right]. \tag{A.5}$$

Since $\xi(T)$ is log-normally distributed with parameters $m_T = mT$ and $s_T = s\sqrt{T}$, then

$$\mathbb{E}[\xi(T)\mathbf{1}_{\xi(T)\leqslant\xi^*(y)}] = \exp(m_T + \frac{1}{2}s_T^2)\Phi\left(\frac{\log(\xi^*(y)) - m_T - s_T^2}{s_T}\right) \tag{A.6}$$

Moreover, $\xi(T)^{\alpha/(\alpha-1)}$ is also log-normally distributed with parameters $\alpha m_T/(\alpha-1)$ and $\alpha s_T/(1-\alpha)$. It follows

$$\mathbb{E}[\xi(T)\mathbf{1}_{\xi(T)\leqslant\xi^*(y)}] = \exp\left(\frac{\alpha m_T}{\alpha-1} + \frac{1}{2}\frac{\alpha^2 s_T^2}{(\alpha-1)^2}\right)\Phi\left(\frac{\log(\xi^*(y)) - m_T - s_T^2}{s_T}\right) \tag{A.7}$$

Let $b = \frac{\overline{W}}{\xi_0 W_0}\exp(m_T + \frac{1}{2}s_T^2)$ and $c = \frac{1}{\xi_0 W_0}(\beta^+\alpha)^{\frac{1}{1-\alpha}}\exp(\frac{\alpha m_T}{\alpha-1} + \frac{1}{2}\frac{\alpha^2 s_T^2}{(\alpha-1)^2})$ then

$$\frac{\mathbb{E}[\xi(T)W^*(T)]}{\xi_0 W_0} = b\Phi\left(\frac{\log(a/y) - m_T - s_T^2}{s_T}\right) + cy^{\frac{1}{\alpha-1}}\Phi\left(\frac{\log(a/y) - m_T - \frac{\alpha}{\alpha-1}s_T^2}{s_T}\right) \tag{A.8}$$

Now let

$$g(y) = b\Phi\left(\frac{\log(a/y) - m_T - s_T^2}{s_T}\right) + cy^{\frac{1}{\alpha-1}}\Phi\left(\frac{\frac{\log(a/y) - m_T - \frac{\alpha}{\alpha-1}s_T^2}{1-\alpha}}{s_T}\right) \tag{A.9}$$

then

$$\mathbb{E}[\xi(T)W^*(T)] = \xi_0 w_0 W_0 \Longleftrightarrow g(y) = w_0. \tag{A.10}$$

Putting $\xi(y) = a/y$ from Lemma 1, the statement in the Lemma follows.

A.4. Proof of Corollary 1

Follows directly from equation (3.12) when $\beta^+ = 0$.

A.5. Proof of Corollary 2

Let a and y be solutions to equation (3.10) and $g(y) = w_0$, respectively, given parameters β^+ and β^-. Let $\tilde{\beta}^+ = u\beta^+$ and $\tilde{\beta}^- = u\beta^-$, while the other parameters are fix. Denote by $\tilde{a}$ and $\tilde{y}$ the solutions to equation (3.10) and $g(y) = w_0$, respectively, given parameters $\tilde{\beta}^+$ and $\tilde{\beta}^-$. It can be easily shown that $\tilde{a} = ua$ and $\tilde{y} = uy$. Therefore, $\xi^*(\tilde{y}) = \tilde{a}/\tilde{y} = a/y = \xi^*(y)$. Moreover, since the surplus in the good scenario depends on the ratio y/β^+, it is also independent from u.

A.6. Proof of Lemma 3

From Theorem 1, it follows:

$$\begin{aligned}\mathbb{E}[W^*(T)] &= \mathbb{E}\left[\overline{W}\xi(T)\mathbf{1}_{\xi(T)\leqslant\xi^*(y)} + \left(\frac{y}{\beta^+\alpha}\right)^{\frac{1}{\alpha-1}}\xi(T)^{\frac{\alpha}{1-\alpha}}\mathbf{1}_{\xi(T)\leqslant\xi^*(y)}\right]\\ &= \overline{W}\mathbb{P}[\xi(T)\leqslant\xi^*(y)] + \left(\frac{y}{\beta^+\alpha}\right)^{\frac{1}{\alpha-1}}\mathbf{E}\left[\xi(T)^{\frac{\alpha}{1-\alpha}}\mathbf{1}_{\xi(T)\leqslant\xi^*(y)}\right].\end{aligned} \tag{A.11}$$

Since $\xi(T)$ is log-normally distributed with parameters $m_T = mT$ and $s_T = s\sqrt{T}$, then

$$\mathbb{P}[\xi(T)\leqslant\xi^*(y)] = \Phi\left(\frac{\log(\xi^*(y)) - m_T}{s_T}\right). \tag{A.12}$$

Moreover, $\xi(T)^{\alpha/(\alpha-1)}$ is also log-normally distributed with parameters $m_T/(\alpha-1)$ and $s_T/(1-\alpha)$. It follows

$$\mathbb{E}\left[\xi(T)^{\frac{1}{1-\alpha}}\mathbf{1}_{\xi(T)\leqslant\xi^*(y)}\right] = \exp\left(\frac{m_T}{\alpha-1} + \frac{1}{2}\frac{s_T^2}{(1-\alpha)^2}\right)\Phi\left(\frac{\log(\xi^*(y)) - m_T - \frac{s_T^2}{1-\alpha}}{s_T}\right). \tag{A.13}$$

Let $d = (\beta^+\alpha)^{\frac{1}{1-\alpha}}\exp\left(\frac{m_T}{\alpha-1} + \frac{1}{2}\frac{s_T^2}{(1-\alpha)^2}\right)$ then

$$\mathbb{E}(W^*(T)) = \overline{W}\Phi\left(\frac{\log(\xi^*(y)) - m_T}{s_T}\right) + dy^{\frac{1}{\alpha-1}}\Phi\left(\frac{\log(\xi^*(y)) - m_T - \frac{s_T^2}{1-\alpha}}{s_T}\right). \tag{A.14}$$

We define

$$k(y) = \overline{W}\Phi\left(\frac{\log(a/y) - m_T - s_T^2}{s_T}\right) + dy^{\frac{1}{\alpha-1}}\Phi\left(\frac{\log(a/y) - m_T + s_T^2}{s_T}\right). \tag{A.15}$$

A.7. Proof of Theorem 2

When $\beta_j^+ = 0$, we know from Corollary 1 that $y_j = a_j \exp(-s_{T_j}\Phi^{-1}(w_j^0/b_j) - mT_j - s_{T_j}^2)$ solves $g_j(y) = w_j^0$. We put y_j into k_j and obtain

$$f_j(w_j^0) = k_j(h_j^{-1}(w_j^0)) = \overline{W}_j\Phi(\Phi^{-1}(w_j^0/b_j) + s_{T_j}). \tag{A.16}$$

The function f_j is strictly increasing and strictly concave with

$$\frac{f(w_j^0)}{w_j^0} = \frac{\overline{W}_j}{b_j}\exp(-s_{T_j}\Phi^{-1}(w_j^0/b_j) - (1/2)s_{T_j}^2). \tag{A.17}$$

The Karush-Kuhn-Tucker conditions for the convex optimization Problem (3.5) with the additional constraint $w_j^0 \leqslant b_j$ for all j are as follows:

$$\eta_1^j \geqslant 0, \eta_2^j \geqslant 0, w_j^0 \geqslant 0 \tag{A.18}$$

$$\eta_1^j w_j^0 = 0, \quad \eta_2^j(w_j^0 - b_j) = 0 \tag{A.19}$$

$$\sum_{j=1}^n w_j^0 = 1 \tag{A.20}$$

$$-\frac{D(T_j)\overline{W}_j}{b_j}\exp(-s_{T_j}\Phi^{-1}(w_j^0/b_j) - (1/2)s_{T_j}^2) - \eta_1^j + \eta_2^j + \nu = 0 \tag{A.21}$$

From equations (A.18) and (A.21) we obtain:

$$\eta_1^j = \nu - \frac{D(T_j)\overline{W}_j}{b_j}\exp(-s_{T_j}\Phi^{-1}(w_j^0/b_j) - (1/2)s_{T_j}^2) + \eta_2^j \geqslant 0. \tag{A.22}$$

We multiply η_1^j with w_j^0 and obtain:

$$w_j^0\left(\nu - \frac{D(T_j)\overline{W}_j}{b_j}\exp(-s_{T_j}\Phi^{-1}(w_j^0/b_j) - (1/2)s_{T_j}^2) + \eta_2^j\right) = 0. \tag{A.23}$$

Using that $\eta_2^j(w_j^0 - b_j) = 0$ we can solve the latter equation for η_2^j and we obtain:

$$\eta_2^j = -\frac{w_j^0}{b_j}\left(\nu - \frac{D(T_j)\overline{W}_j}{b_j}\exp(-s_{T_j}\Phi^{-1}(w_j^0/b_j) - (1/2)s_{T_j}^2) + \eta_2^j\right) \geqslant 0 \tag{A.24}$$

which implies

$$\nu \leqslant \frac{D(T_j)\overline{W}_j}{b_j}\exp(-s_{T_j}\Phi^{-1}(w_j^0/b_j) - (1/2)s_{T_j}^2). \tag{A.25}$$

since $w_j^0 \geqslant 0$. Finally, using $\eta_2^j(w_j^0 - b_j) = 0$ we have

$$-\frac{w_j^0}{b_j}(w_j^0 - b_j)\left(\nu - \frac{D(T_j)\overline{W}_j}{b_j}\exp(-s_{T_j}\Phi^{-1}(w_j^0/b_j) - (1/2)s_{T_j}^2) + \eta_2^j\right) = 0. \tag{A.26}$$

If $\nu > 0$, then $w_j^0 < b_j$. Indeed, if $w_j^0 = b_j$ and $\nu > 0$ then condition (A.27) is violated. Moreover, since $w_j^0 < b_j$, then $\eta_j^2 = 0$ by the second Slater's condition in (A.19). Therefore, $w_j^0 > 0$, else condition (A.22) is violated. It follows from equation (A.26) that w_j^0 must solve

$$\nu - \frac{D(T_j)\overline{W}_j}{b_j}\exp(-s_{T_j}\Phi^{-1}(w_j^0/b_j) - (1/2)s_{T_j}^2) = 0 \tag{A.27}$$

and after some re-arrangements we obtain

$$w_j^0 = b_j \Phi\left(-\frac{1}{s_j}\log\left(\frac{\nu}{\xi_0 W_0}\right) - \frac{1}{s_j}\left(-\log D(T_j) - rT_j + \frac{1}{2}s_{T_j}^2\right)\right). \tag{A.28}$$

If $\nu < 0$, then $\eta_2^j > 0$ else condition (A.22) is violated since the inequality

$$\nu - \frac{D(T_j)\overline{W}_j}{b_j}\exp\left(-s_{T_j}\Phi^{-1}(w_j^0/b_j) - (1/2)s_{T_j}^2\right) < 0 \tag{A.29}$$

holds for all $w_j^0 \in [0, b_j]$. Therefore, $w_j^0 = b_j$ by the second Slater's condition in (A.19). If $\nu = 0$ and $w_j^0 \neq b_j$ then $w_j^0 = 0$ by equation (A.26). However, when $w_j^0 = 0$ then equation (A.22) is violated. Thus, also in this case we must have $w_j^0 = b_j$.

A.8. Proof of Corollary 3

Straightforward implication of Theorem 2.

A.9. Proof of Corollary 4

Follow directly from Theorem 2, Theorem 1 and Corollary 1.

References

[1] G. Ainslie and J.R. Monterosso, Building blocks of selfcontrol: Increased tolerance for delay with bundled rewards. *J. Experimental Anal. Behavior* **79** (2003) 37–48.

[2] A.E. Attema, H. Bleichrodt, Y. Gao, Zh. Huang and P.P. Wakker, Measuring Discounting without Measuring Utility. *Amer. Econ. Rev.* **106** (2016) 1476–1494.

[3] A.E. Attema, H. Bleichrodt, K.I.M. Rohde and P.P. Wakker, Time-Tradeoff Sequences for Analyzing Discounting and Time Inconsistency. *Manag. Sci.* **56** (2010) 2015–2030.

[4] N. Barberis and M. Huang, Stocks as lotteries: The implications of probability weighting for security prices. *Amer. Econ. Rev.* **98** (2008) 2066–2100.

[5] S. Benartzi and R.H. Thaler, Myopic loss aversion and the equity premium puzzle. *Quarterly J. Econ.* **110** (1995) 73–92.

[6] A.B. Berkelaar, R. Kouwenberg and T. Post, Optimal portfolio choice under loss aversion. *Rev. Econ. Statis.* **86** (2004) 973–987.

[7] F. Black, Beta and return. *J. Portfolio Management* **20** (1993) 8–18.

[8] M.C. Jensen Black, F. and M. Scholes, *Portfolio selection: efficient diversification of investments*. Praeger, New York (1972).

[9] D.B. Brown and M. Sim, Satisficing measures for analysis of risky positions. *Manag. Sci.* **55** (2009) 71–84.

[10] G.B. Chapman, Temporal discounting and utility for health and money. *J. Experimental Psychology-Learing Memory and Cognition* **22** (1996) 771–791.

[11] G.B. Chapman, N.T. Brewer, E.J. Coups, S. Brownlee, H. Leventhal and E.A. Leventhal, Value for the future and preventive health behavior. *J. Experimental Psychology-Appl.* **7** (2001) 235–250.

[12] T.S. Critchfield and S.H. Kollins, Temporal discounting: Basic research and the analysis of socially important behavior. *J. Appl. Behavior Anal.* **34** (2001) 101–122.

[13] E.F. Fama and K.R. French, The cross-section of expected stock returns. *J. Finance* **47** (1992) 427–465.

[14] E.F. Fama and K.R. French, The CAPM is wanted, dead or alive. *J. Finance* **51** (1996) 1947–1958.

[15] F.J. Gomes, Portfolio choice and trading volume with loss-averse investors. *J. Business* **78** (2005) 675–706.

[16] L. Green and J. Myerson, Exponential versus hyperbolic discounting of delayed outcomes: Risk and waiting time. *Amer. Zoologist* **36** (1996) 496–505.

[17] W. Gueth, M.V. Levati and M. Ploner, On the social dimension of time and risk preferences: An experimental study. *Economic Inquiry* **46** (2008) 261–272.

[18] M. Hamada, M. Sherris and J. Van Der Hoek, Dynamic portfolio allocation, the dual theory of choice and probability distortion functions. *Astin Bulletin* **36** (2006) 187–217.

[19] D.A. Hantula and K. Bryant, Delay discounting determines delivery fees in an e-commerce simulation: A behavioral economic perspective. *Psychology and Maketing* **22** (2005) 153–161.

[20] G.W. Harrison and M.L. Lau, Is the evidence for hyperbolic discounting in humans just an experimental artefact? *Behavioral and Brain Sci.* **28** (2005) 657.

[21] X. Dong He and X. Yu Zhou, Portfolio choice under cumulative prospect theory: An analytical treatment. *Manag. Sci.* **57** (2011) 315–331.

[22] H. Jin and X. Yu Zhou, Continuous-time behavioral portfolio selection. In 47th IEEE Conference on Decision and Control, CDC, *IEEE Conference on Decision and Control* (2008) 5602–5607.

[23] H. Jin and X. Yu Zhou, Greed, leverage and potential losses: A prospect theory perspective. *Math. Finance* **23** (2013) 122–142.

[24] I. Karatzas and S. Shreve, Karatzas, I. and St. E. Shreve: Brownian motion and stochastic calculus. *Statist.Papers* **30** (1989) 262.

[25] H. Levy and M. Levy, Prospect theory and mean-variance analysis. *Rev. Financial Studies* **17** (2004) 1015–1041.

[26] H.M. Markowitz, The Utility of Wealth. *J. Political Economy* **60** (1952) 151–158.

[27] R.C. Merton, An analytic derivation of the efficient portfolio frontier. *J. Financial and Quantitative Anal.* **7** (1972) 1851–1872.

[28] J. Myerson and L. Green, Discounting of delayed rewards – Models of individual choice. *J. Experimental Anal. Behavior* **64** (1995) 263–276.

[29] F. Paglieri, D. Parisi, M. Patacchiola and G. Petrosino, Investigating intertemporal choice through experimental evolutionary robotics. *Behavioural Processes* **115** (2015) 1–18.

[30] D. Prelec, Decreasing impatience: A criterion for non-stationary time preference and "Hyperbolic" discounting. *Scandinavian J. Econ.* **106** (2004) 511–532.

[31] M. Rasonyi and A.M. Rodrigues, Continuous-time portfolio optimisation for a behavioural investor with bounded utility on gains. *Electr. Commun. Probab.* **19** (2014) 1–13.

[32] B. Saunders and G. Fogarty, Time discounting in relation to career preferences. *J. Vocational Behavior* **58** (2001) 118–126.

[33] S. Sayman and A. Onculer, An investigation of time inconsistency. *Mang. Sci.* **55** (2009) 470–482.

[34] W.F. Sharpe, A simplified model for portfolio analysis. *Manag. Sci.* **9** (1963) 277–293.

[35] W.F. Sharpe, Capital asset prices: A theory of market equilibrium under conditions of risk. *J. Finance* **19** (1964) 425–442.

[36] T. Shavit and M. Rosenboim, Is there time discounting for risk premium? *J. Experimental Anal. Behavior* **103** (2015) 393–404.

[37] H. Shefrin and M. Statman, Behavioral portfolio theory. *J. Financial and Quantitative Anal.* **35** (2000) 127–151.

[38] C.L. Smith and D.A. Hantula, Pricing effects on foraging in a simulated Internet shopping mall. *J. Econ. Psychology* **24** (2003) 653–674.

[39] K. Takeuchi, Non-parametric test of time consistency: Present bias and future bias. *Games and Economic Behavior* **71** (2011) 456–478.

[40] J. Tobin, Liquidity preference as behavior towards risk. *Rev. Econ. Studies* **67** (1958) 65–86.

[41] A. Tversky and D. Kahneman, Advances in prospect theory: Cumulative representation of uncertainty. *J. Risk Uncertainty* **5** (1992) 297–323.

[42] Th. Van Huizen and J. Plantenga, Job search behaviour and time preferences: Testing exponential versus hyperbolic discounting. *Econ.-Netherlands* **162** (2014) 223–245.

[43] A. Waegeman, C.H. Declerck, Ch. Boone, W. Van Hecke and P.M. Parizel, Individual differences in self-control in a time discounting task: An fMRI study. *J. Neuroscience Psychol. Econ.* **7** (2014) 65–79.

[44] M.E. Yaari, The dual theory of choice under risk. *Econ.* **55** (1987) 99–115.