RAIRO-Oper. Res. 52 (2018) 543–559
https://doi.org/10.1051/ro/2017042

RAIRO Operations Research
www.rairo-ro.org

CONTINGENT DERIVATIVES AND NECESSARY EFFICIENCY CONDITIONS FOR VECTOR EQUILIBRIUM PROBLEMS WITH CONSTRAINTS

Do Van Luu[1,*] and Tran Van Su[2]

Abstract. We establish Fritz John necessary conditions for local weak efficient solutions of vector equilibrium problems with constraints in terms of contingent derivatives. Under suitable constraint qualifications, Karush–Kuhn–Tucker necessary conditions for those solutions are investigated.

Mathematics Subject Classification. 90C46, 90C29, 49J52.

Received May 10, 2016. Accepted May 23, 2017.

1. Introduction

Vector equilibrium problems is an important part of nonlinear analysis and have been intensively studied in recent years. They include as special cases vector variational inequality problem, vector optimization problem and many other problems. There is a lot of papers dealing with optimality conditions for vector equilibrium problems and their special cases in terms of various subdifferentials, contingent derivatives and contingent epiderivatives (see, *e.g.*, [3, 5–19, 21–23], and references therein).

It is well known that the contingent cones are nonconvex, but they contain almost every existing tangent cone, and so, they give rich informations about the local behavior of sets. The contingent derivatives of set-valued maps defined by means of contingent cones are well suited to develop optimality conditions for vector optimization problems with set-valued functions. The notion of contingent derivatives of set-valued maps introduced by Aubin [1]. It is extended in a natural way the corresponding notion for real-valued functions by Jiménez and Novo [8], where contingent derivatives for the stable real-valued functions (called also the calm functions in [9]) are studied. Note that a locally Lipschitz function is stable, but the converse are false. The notion of contingent epiderivative for real-valued functions is introduced in [2] by Aubin and Frankowska. Jiménez and Novo [8] obtained a rich caculus for contingent derivatives of stable and steady real-valued functions. In the context of vector optimization, notion of contingent derivative is useful in order to get optimality conditions for efficiency. Some necessary and sufficient conditions for multiobjective optimization problems involving inequality and equality constraints with stable and steady real-valued functions via contingent derivatives are investigated in [8]. Optimality conditions for strict minimizers in multiobjective optimization problems via contingent epiderivatives

Keywords. Vector equilibrium problems, Local weak efficient solutions, Constraint qualifications, Fritz John and Karush–Kuhn–Tucker efficiency conditions.

[1] Department of Mathematics and Informatics, Thang Long University, Hanoi, and Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi.
[*] Corresponding author: dvluu@math.ac.vn

[2] Department of Mathematics, Quang Nam University, Tam Ky, Vietnam.

Article published by EDP Sciences

and hypoderivatives of scalar functions are established by Jiménez *et al.* [9]. Some necessary and sufficient conditions for efficiency of vector equilibrium problems via contingent epiderivatives are recently obtained by Su [23].

The aim of the present paper is to develop Fritz John and Karush–Kuhn–Tucker necessary conditions for local efficient solutions of constrained vector equilibrium problems via contingent derivatives. After some preliminaries, Section 3 is devoted to developing Fritz John necessary conditions for local weak efficient solutions of vector equilibrium problems involving inequality, equality and set constraints. Section 4 deals with constraint qualifications and Karush–Kuhn–Tucker necessary conditions for local weak efficient solutions of vector equilibrium problems with constraints. The results obtained in Sections 3 and 4 are applied to vector variational inequalities and vector optimization problems with constraints in Section 5.

2. Preliminaries

Let X, Y be real normed linear spaces, and let G be a set-valued map from X to Y. The domain and the graph of G are the following sets, respectively,

$$\begin{aligned} \text{dom } G &:= \{x \in X : \; G(x) \neq \emptyset\}, \\ \text{graph } G &:= \{(x, y) \in X \times Y : \; y \in G(x), x \in \text{dom} G\}. \end{aligned}$$

Let $C \subseteq X$ and $\overline{x} \in C$. We recall that the contingent cone to C at $\overline{x}$ is

$$T(C; \overline{x}) := \{v \in X : \; \exists t_n \to 0^+, \exists v_n \to v \text{ such that } \overline{x} + t_n v_n \in C, \forall n\}.$$

The interior tangent cone to C at $\overline{x}$ is

$$\begin{aligned} IT(C; \overline{x}) := \{v \in X : \; & \exists \delta > 0 \text{ such that} \\ & \overline{x} + tu \in C, \forall t \in (0, \delta], \forall u \in B(v; \delta)\}, \end{aligned}$$

where $B(v; \delta)$ stands for the open ball of radius δ around $\overline{x}$.

Definition 2.1 ([1])**.** Let $(x, y) \in$ *graph* G. The contingent derivative of G at (x, y) is the set-valued map $DG(x, y)$ from X to Y defined by

$$v \in DG(x, y)(u) \Longleftrightarrow (u, v) \in T(\text{graph } G, (x, y)),$$

which means that

$$\text{graph } DG(x, y) = T(\text{graph } G, (x, y)).$$

Let f be a single-valued map from X to Y. For the set-valued map $x \longmapsto \{f(x)\}$, Definition 2.1 is of the following form.

Definition 2.2 ([8])**.** The contingent derivative of f at $\overline{x} \in X$ in a direction $v \in X$ is the following set defined by

$$\partial_* f(\overline{x})v := \left\{ y \in Y : \; \exists (t_n v_n) \to (0^+, v) \text{ such that } \lim_{n \to \infty} \frac{f(\overline{x} + t_n v_n) - f(\overline{x})}{t_n} = y \right\}.$$

Note that $\partial_* f(\overline{x})v$ is a closed set and the set-valued map $\partial_* f(\overline{x})(.)$ is positively homogeneous.

Recall that the Hadamard directional derivative of $f : X \to Y$ at $\overline{x}$ in the ditrection $v \in X$ is the following limit:

$$df(\overline{x}; v) := \lim_{(t,u) \to (0^+, v)} \frac{f(\overline{x} + tu) - f(\overline{x})}{t},$$

if it exists. The function f is called Hadamard differentiable at $\overline{x}$ iff $df(\overline{x};v)$ exists for all $v \in X$. If f is Hadamard differentiable at $\overline{x}$, then for all $v \in X, \partial_* f(\overline{x})v = \{df(\overline{x};v)\}$.

Given $f : X \to \mathbb{R}$, the lower (resp. upper) Hadamard directional derivative of f at $\overline{x} \in X$ in the direction v is defined by

$$\underline{d}f(\overline{x};v) := \liminf_{(t,u)\to(0^+,v)} \frac{f(\overline{x}+tu)-f(\overline{x})}{t}$$
$$\left(\text{resp. } \overline{d}f(\overline{x};v) := \limsup_{(t,u)\to(0^+,v)} \frac{f(\overline{x}+tu)-f(\overline{x})}{t}\right).$$

If $\overline{d}f(\overline{x};v) = \underline{d}f(\overline{x};v)$, then their common is Hadamard directional derivative $df(\overline{x};v)$ of f at $\overline{x}$ in the direction v.

If $Y = \mathbb{R}, \underline{d}f(\overline{x};v)$ and $\overline{d}f(\overline{x};v)$ are finite, then by Remark 3.4 [8],

$$\partial_* f(\overline{x})v \subseteq [\underline{d}f(\overline{x};v), \overline{d}f(\overline{x};v)]. \tag{2.1}$$

If f is continuous in a neighborhood of $\overline{x}$, the equality holds in (2.1), and so, $\partial_* f(\overline{x})v$ is a convex set. In case $Y = \mathbb{R}^m$, it holds that

$$\partial_* f(\overline{x})v \subseteq \prod_{i=1}^{m} \partial_* f_i(\overline{x})v. \tag{2.2}$$

The following class of stable functions provides a rich calculus for contingent derivatives.

Definition 2.3 ([8])**.** The map $f : X \to Y$ is said to be stable (call also calm in [9]) at $\overline{x}$ iff there exist numbers $\alpha > 0$ and $\delta > 0$ such that

$$\| f(x) - f(\overline{x}) \| \leqslant \alpha \| x - \overline{x} \| \quad (\forall x \in B(\overline{x};\delta)).$$

If $f : X \to \mathbb{R}^m$ is stable at $\overline{x}$, then by Remarks 3.4 [8], $\partial_* f(\overline{x})v \neq \emptyset$ and compact $(\forall v \in X)$. It should be noted here that a locally Lipschitz function is stable, but the converse does not hold. This is illustrated by the following example.

Example 2.4. Let f be a real-valued function defined on $\mathbb{R}$ as

$$f(x) = \begin{cases} x\cos\frac{\pi}{x}, & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0. \end{cases}$$

Then f is stable at $\overline{x} = 0$, but it is not locally Lipschitz at $\overline{x} = 0$. It can be seen that $\partial_* f(0)v = [-v, v]$.

If $f : X \to Y, g : X \to Z$, where Z is a normed space. Proposition 3.3 [8] shows that for each $v \in X$,

$$\partial_*(f,g)(\overline{x})v \subseteq \partial_* f(\overline{x})v \times \partial_* g(\overline{x})v. \tag{2.3}$$

If f or g is Hadamard directionally differentiable at $\overline{x}$ in the direction v, then the equality in (2.3) is true. Proposition 3.13 [8] shows that if $f, g : X \to \mathbb{R}^m, f$ or g is stable at $\overline{x}$, then

$$\partial_*(f+g)(\overline{x})v \subseteq \partial_* f(\overline{x})v + \partial_* g(\overline{x})v. \tag{2.4}$$

The equality holds in (2.4) if f or g is Hadamard directionally differentiable at $\overline{x}$ in the direction v.

The class of steady functions provides a rich calculus for contingent derivatives.

Definition 2.5 ([8])**.** The function $f : X \to Y$ is called steady at $\overline{x}$ in the direction v or steady at $(\overline{x}, v)$ iff

$$\lim_{(t,u)\to(0^+,v)} \frac{f(\overline{x}+tu) - f(\overline{x}+tv)}{t} = 0.$$

The function f is called steady at $\overline{x}$ iff it is steady at $\overline{x}$ in all directions. Proposition 3.6 [8] shows that f is steady at $(\overline{x}, 0)$ if and only if it is stable at $\overline{x}$. The class of steady functions is wider than the class of locally Lipschitz functions or the class of Hadamard differentiable functions.

For a convex cone $S \subseteq \mathbb{R}^r$, the positive polar cone to S is

$$S^* := \{\mu \in \mathbb{R}^r : \ \langle \mu, y\rangle \geqslant 0\}.$$

The normal cone to S at y_0 is $N(S; y_0) := -T(S; y_0)^*$.

3. Fritz John necessary conditions for local weak efficient solutions

Let K be a subset of $\mathbb{R}^n$, and let F be a mapping from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^m$. It can be seen that $F = (F_1, \ldots, F_m)$. Denote by $\mathbb{R}^m_+$ and $\mathbb{R}^m_{++}$ the nonnegative and positive orthants in $\mathbb{R}^m$, respectively. Let us consider the following vector equilibrium problem (VEP): Finding $\overline{x} \in K$ such that

$$F(\overline{x}, y) \notin -\mathbb{R}^m_{++} \ (\forall y \in K). \tag{3.1}$$

A vector $\overline{x}$ is called a local weak efficient solution of (VEP) iff there exists a number $\delta > 0$ such that (3.1) holds for all $y \in K \cap B(\overline{x}; \delta)$. We set $F_{\overline{x}}(y) = F(\overline{x}, y), F_{k,\overline{x}}(y) = F_k(\overline{x}, y)$ $(\forall k \in J := \{1, \ldots, m\})$. Then, Definition (3.1) is equivalent to that there is no $y \in K$ such that

$$F_{k,\overline{x}}(y) < 0 \ (\forall k \in J).$$

Let g, h be maps from $\mathbb{R}^n$ into $\mathbb{R}^r, \mathbb{R}^\ell$, respectively. Thus $g = (g_1, \ldots, g_r), h = (h_1, \ldots, h_\ell)$, where g_i, h_j are extended-real-valued functions defined on $X (i \in I := \{1, \ldots, r\}, j \in L := \{1, \ldots, \ell\})$. Let us consider the following constrained vector equilibrium problem (CVEP): Find $\overline{x} \in K$ such that

$$F(\overline{x}, y) \notin -\mathbb{R}^m_{++} \quad (\forall y \in K := \{y \in C : \ g_i \leqslant 0 (i \in I), h_j(y) = 0 (j \in L)\}).$$

We set

$$\begin{aligned}
I(\overline{x}) &:= \{i \in I : \ g_i(\overline{x}) = 0\};\\
g_{I(\overline{x})} &:= (g_i)_{i \in I(\overline{x})};\\
G &:= \{x \in \mathbb{R}^n : \ g_i(x) \leqslant 0, \ \forall i \in I\};\\
G_{I(\overline{x})} &:= \{x \in \mathbb{R}^n : \ g_i(x) \leqslant 0, \ \forall i \in I(\overline{x})\};\\
G^0_{I(\overline{x})} &:= \{x \in \mathbb{R}^n : \ g_i(x) < 0, \ \forall i \in I(\overline{x})\};\\
H &:= \{x \in \mathbb{R}^n : \ h(x) = 0\};\\
F_0 &:= \{x \in \mathbb{R}^n : \ F_{\overline{x}}(x) \notin -\mathbb{R}^m_{++}\}.
\end{aligned}$$

In this section, we shall derive Fritz John necessary conditions for a local weak efficient solution of (CVEP) in terms of contingent derivatives. To do this, we introduce the following assumptions.

Assumption 3.1. $F_{\overline{x}}(\overline{x}) = 0$; the functions $F_{\overline{x}}$, g are continuous in a neighbourhood of $\overline{x}$; the functions $h_1, \ldots, h_\ell$ are Fréchet differentiable at $\overline{x}$ with Fréchet derivatives $\nabla h_1(\overline{x}), \ldots, \nabla h_\ell(\overline{x})$ linearly independent.

A Fritz John necessary condition for local weak efficient solutions of (CVEP) can be stated as follows.

Theorem 3.2. *Let $\overline{x}$ be a local weak efficient solution of (CVEP). Assume that Assumption 3.1 holds, and the functions $F_{\overline{x}}$, g are steady at $\overline{x} \in K$. Suppose, in addition, that for every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exists $z \in \partial_* g(\overline{x})v$ such that $z_i < 0$ $(\forall i \in I(\overline{x}))$. Then, for every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$ and $(y,z) \in \partial_*(F_{\overline{x}}, g)(\overline{x})v$, there exists $(\lambda,\mu) \in \mathbb{R}^m \times \mathbb{R}^r, \lambda \geqslant 0, \mu \geqslant 0$ with $(\lambda,\mu) \neq (0,0)$ such that*

$$\langle \lambda, y\rangle + \langle \mu, z\rangle \geqslant 0,$$
$$\mu_i g_i(\overline{x}) = 0 \ (\forall i \in I).$$

Proof. By assumption, $F_{\overline{x}}(\overline{x}) = 0$, and hence, $\overline{x}$ is a local weak minimizer of the following multiobjective optimization problem:

$$\text{(MP1)} \qquad \begin{aligned} &\min F_{\overline{x}}(x),\\ \text{s.t.} \quad & g_i(x) \leqslant 0,\ i \in I,\\ & h_j(x) = 0,\ j \in J,\\ & x \in C. \end{aligned}$$

Hence, $\overline{x}$ is a local weak minimizer of the multiobjective optimization problem:

$$\text{(MP2)} \qquad \begin{aligned} &\min F_{\overline{x}}(x),\\ \text{s.t.} \quad & g_i(x) \leqslant 0,\ i \in I(\overline{x}),\\ & h_j(x) = 0,\ j \in J,\\ & x \in C. \end{aligned}$$

By assumption, for $v \in \mathrm{Ker}\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exist $z \in \partial_* g(\overline{x})v$ such that $z_i < 0$ $(\forall i \in I(\overline{x}))$. Hence, $z_{I(\overline{x})} = (z_i)_{i\in I(\overline{x})} \in \partial_* g_{I(\overline{x})}(\overline{x})v$ with $z_i < 0$ $(\forall i \in I(\overline{x}))$. Moreover, $IT(-\mathbb{R}_+^{|I(\overline{x})|}; g_{I(\overline{x})}(\overline{x})) = -\mathbb{R}_{++}^{|I(\overline{x})|}$, where $|I(\overline{x})|$ is the capacity of $I(\overline{x})$. Hence, $\partial_* g_{I(\overline{x})}(\overline{x})v \cap IT(-\mathbb{R}_+^{|I(\overline{x})|}; g_{I(\overline{x})}(\overline{x})) \neq \emptyset$.

Let us show that for every $v \in \mathrm{Ker}\nabla h(\overline{x}) \cap IT(C;\overline{x})$ and $(y, z_{I(\overline{x})}) \in \partial_*(F, g_{I(\overline{x})})(\overline{x})v$, we have $(y, z_{I(\overline{x})}) \notin -\mathbb{R}_{++}^m \times -\mathbb{R}_{++}^{|I(\overline{x})|}$. Assume the contrary, that there exist $\widetilde{v} \in \mathrm{Ker}\nabla h(\overline{x}) \cap IT(C;\overline{x})$ and $(\widetilde{y}, \widetilde{z_{I(\overline{x})}}) \in (-\mathbb{R}_{++}^m \times -\mathbb{R}_{++}^{|I(\overline{x})|}) \cap \partial_*(F_{\overline{x}}, g_{I(\overline{x})})(\overline{x})\widetilde{v}$. In view of Lemma 6.2 [8], it follows that $\widetilde{v} \in T(F_0 \cap G^0_{I(\overline{x})} \cap H;\overline{x})$. Moreover,

$$T(F_0 \cap G^0_{I(\overline{x})} \cap H;\overline{x}) \cap IT(C;\overline{x}) \subseteq T(F_0 \cap G^0_{I(\overline{x})} \cap H \cap C;\overline{x}).$$

Therefore, $\widetilde{v} \in T(F_0 \cap G^0_{I(\overline{x})} \cap H \cap C;\overline{x})$. By an argument analogous to that used for the proof of Lemma 6.3 [8] to Problem (MP2), it follows that $T(F_0 \cap G^0_{I(\overline{x})} \cap H \cap C;\overline{x}) = \emptyset$. Thus, we arrive a contradiction. Consequently, for every $v \in \mathrm{Ker}\nabla h(\overline{x}) \cap IT(C;\overline{x})$ and $(y, z_{I(\overline{x})}) \in \partial_*(F_{\overline{x}}, g_{I(\overline{x})})(\overline{x})v$, we have $(y, z_{I(\overline{x})}) \notin -\mathbb{R}_{++}^m \times -\mathbb{R}_{++}^{|I(\overline{x})|}$.

Applying a separation theorem for the disjoint convex sets $\{(y, z_{I(\overline{x})})\}$ and $-\mathbb{R}_{++}^m \times -\mathbb{R}_{++}^{|I(\overline{x})|}$ (see, *e.g.*, [4], Thm. 3.4) yields the existence of $\lambda \in \mathbb{R}_+^m, \mu_{I(\overline{x})} \in N(-\mathbb{R}_+^{|I(\overline{x})|}, g_{I(\overline{x})}(\overline{x}))$ with $(\lambda, \mu_{I(\overline{x})}) \neq (0,0)$ such that

$$\langle \lambda, y\rangle + \langle \mu_{I(\overline{x})}, z_{I(\overline{x})}\rangle \geqslant 0. \tag{3.2}$$

Note that

$$\mu_{I(\overline{x})} \in N(-\mathbb{R}_+^{|I(\overline{x})|}; g_{I(\overline{x})}(\overline{x})) \Longleftrightarrow \mu_{I(\overline{x})} \in \mathbb{R}_+^{|I(\overline{x})|} \Longleftrightarrow \mu_i \geqslant 0 \ (\forall i \in I(\overline{x})).$$

Choosing $\mu_i = 0$ $(\forall i \notin I(\overline{x}))$, we get $\mu = (\mu_i)_{i\in I} \geqslant 0$. By (3.2), it results that

$$\langle \lambda, y\rangle + \langle \mu, z\rangle \geqslant 0, \quad \mu_i g_i(\overline{x}) = 0 \ (\forall i \in I),$$

which completes the proof. ☐

Remark 3.3. Theorem 3.2 is a meaningful extension of Theorem 6.4 [8] for multiobjective optimization problems without set constraint.

Now for $\overline{x} \in K$ and $v \in IT(C;\overline{x})$, we set

$$D(\overline{x};v) := \bigcup \Bigg\{ \sum_{k\in J} \lambda_k \partial_* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in I(\overline{x})} \mu_i \partial_* g_i(\overline{x})v + \sum_{j\in L} \gamma_j \langle \nabla h_j(\overline{x}), v\rangle :$$
$$\lambda_k \geqslant 0\ (\forall k \in J), \mu_i \geqslant 0\ (\forall i \in I(\overline{x})), \gamma_j \in \mathbb{R}\ (\forall j \in L), (\lambda, \mu_{I(\overline{x})}, \gamma) \neq (0,0,0) \Bigg\},$$

where $\lambda = (\lambda_k)_{k\in J}, \mu_{I(\overline{x})} = (\mu_i)_{i\in I(\overline{x})}, \gamma = (\gamma_j)_{j\in L}$.

In the sequel, we establish a Fritz John necessary condition for local weak efficient solution of (CVEP).

Theorem 3.4. *Let $\overline{x}$ be a local weak efficient solution of (CVEP). Assume that Assumption 3.1 holds, and the functions $F_{\overline{x}}, g$ are steady at $\overline{x}$. Suppose, furthermore, that for every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exists $z \in \partial_* g(\overline{x})v$ such that $z_i < 0$ $(\forall i \in I(\overline{x}))$. Then,*

(i) *For every $v \in IT(C;\overline{x})$, there exist $\overline{\lambda}_k \geqslant 0$ $(\forall k \in J), \overline{\mu}_i \geqslant 0$ $(\forall i \in I)$, and $\overline{\gamma}_j \in \mathbb{R}$ $(\forall j \in L)$, not all zero, such that*

$$0 \in \sum_{k\in J} \overline{\lambda}_k \partial_* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in I} \overline{\mu}_i \partial_* g_i(\overline{x})v + \sum_{j\in L} \overline{\gamma}_j \langle \nabla h_j(\overline{x}), v\rangle, \tag{3.3}$$
$$\overline{\mu}_i g_i(\overline{x}) = 0\ (\forall i \in I). \tag{3.4}$$

(ii) *For every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exist $\overline{\lambda}_k \geqslant 0$ $(\forall k \in J), \overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ with $(\overline{\lambda}, \overline{\mu}) \neq (0,0)$ such that*

$$0 \in \sum_{k\in J} \overline{\lambda}_k \partial_* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in I} \overline{\mu}_i \partial_* g_i(\overline{x})v, \tag{3.5}$$
$$\overline{\mu}_i g_i(\overline{x}) = 0\ (\forall i \in I).$$

Proof.

(i) Let us show that for every $v \in IT(C;\overline{x})$,

$$0 \in \mathrm{cl}D(\overline{x};v), \tag{3.6}$$

where cl indicates the closure. Assume the contrary, that there exists $v_0 \in IT(C;\overline{x})$ such that

$$0 \notin \mathrm{cl}D(\overline{x};v_0).$$

For $F_{k,\overline{x}}(.), g_i : \mathbb{R}^n \to \mathbb{R}$, we have $\partial_* F_{k,\overline{x}}(\overline{x})v_0$ and $\partial_* g_i(\overline{x})v_0$ are nonempty convex sets. Hence, $D(\overline{x};v_0)$ is nonempty convex, and so is $\mathrm{cl}D(\overline{x};v_0)$. Applying a strong separation theorem for the disjoint closed convex sets $\mathrm{cl}D(\overline{x};v_0)$ and $\{0\}$ in $\mathbb{R}$ (see [20], Cor. 11.4.2) yields the existence of $\alpha_0 \in \mathbb{R}\setminus\{0\}$ such that

$$\alpha_0 w < 0\ (\forall w \in D(\overline{x};v_0)). \tag{3.7}$$

Let us see that

$$\langle \nabla h_j(\overline{x}), v_0\rangle = 0\ (\forall j \in L). \tag{3.8}$$

Indeed, if (3.8) were false, then $\langle \nabla h_{j_0}(\overline{x}), v_0\rangle \neq 0$ for some $j_0 \in L$. Taking $y_s \in \partial_* F_{s,\overline{x}}(\overline{x})$ $(s \in J), \lambda_s = 1, \lambda_k = 0$ $(\forall k \in J, k \neq s), \mu_i = 0$ $(\forall i \in I(\overline{x})), \gamma_j = 0$ $(\forall j \in L, j \neq j_0)$, then the vector $(0,\ldots,1,\ldots,0) \in \mathbb{R}^m_+$, where the number 1 is its sth component. It follows from (3.7) that

$$\alpha_0 y_s + \gamma_{j_0}\alpha_0 \langle \nabla h_{j_0}(\overline{x}), v_0\rangle < 0. \tag{3.9}$$

Since $F_{s,\overline{x}}$ is steady at $(\overline{x},0)$, it is stable at $\overline{x}$. Hence, $|y_s| < +\infty$. By letting γ_{j_0} being sufficiently large if $\alpha_0\langle \nabla h_{j_0}(\overline{x}), v_0\rangle > 0$, while $\gamma_{j_0} < 0$ with its absolute value being large enough if $\alpha_0\langle \nabla h_{j_0}(\overline{x}), v_0\rangle < 0$, we obtain a contradiction with (3.9), and so, (3.8) holds. Hence, $v_0 \in \mathrm{Ker}\nabla h(\overline{x}) \cap IT(C;\overline{x})$.
We invoke Theorem 3.2 to deduce that for $(y,z) \in \partial_*(F_{\overline{x}}, g)(\overline{x})v_0$, there exist $(\overline{\lambda},\overline{\mu}) \in \mathbb{R}^m_+ \times \mathbb{R}^r_+$ with $(\overline{\lambda},\overline{\mu}) \neq (0,0)$ such that

$$\langle \overline{\lambda}, y\rangle + \langle \overline{\mu}, z\rangle \geqslant 0, \tag{3.10}$$

$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I). \tag{3.11}$$

It can be seen that (3.10) and (3.11) are equivalent to the following

$$\langle \overline{\lambda}, y\rangle + \langle \overline{\mu}_{I(\overline{x})}, z_{I(\overline{x})}\rangle \geqslant 0, \tag{3.12}$$

where $\overline{\mu}_{I(\overline{x})} = (\overline{\mu}_i)_{i\in I(\overline{x})}, z_{I(\overline{x})} = (z_i)_{i\in I(\overline{x})}$. It follows from (2.2) and (2.3) that

$$(y, z_{I(\overline{x})}) \in \partial_*(F_{\overline{x}}, g_{I(\overline{x})})(\overline{x})v_0 \subseteq \prod_{k=1}^{m} \partial_* F_{k,\overline{x}}(\overline{x})v_0 \times \prod_{i\in I(\overline{x})} g_i(\overline{x})v_0. \tag{3.13}$$

Thus $y = (y_1,\ldots,y_m), z = (z_i)_{i\in I(\overline{x})}$ with $y_k \in \partial_* F_{k,\overline{x}}(\overline{x})v_0, z_i \in \partial_* g_i(\overline{x})v_0$ ($\forall k \in J, \forall i \in I(\overline{x})$). Since $v_0 \in \mathrm{Ker}\nabla h(\overline{x})$, it follows from (3.13) that

$$\langle \overline{\lambda}, y\rangle + \langle \overline{\mu}_{I(\overline{x})}, z_{I(\overline{x})}\rangle \in \sum_{k\in J} \overline{\lambda}_k \partial_* F_{k,\overline{x}}(\overline{x})v_0 + \sum_{i\in I(\overline{x})} \overline{\mu}_i \partial_* g_i(\overline{x})v_0 \in D(\overline{x}; v_0). \tag{3.14}$$

Combining (3.7), (3.12) and (3.14) yields that

$$\langle \overline{\lambda}, y\rangle + \langle \overline{\mu}_{I(\overline{x})}, z_{I(\overline{x})}\rangle > 0. \tag{3.15}$$

Taking account of (3.7), (3.14) and (3.15), we get $\alpha_0 < 0$. Therefore,

$$w > 0 \ (\forall w \in D(\overline{x}; v_0)).$$

This can not happen, since for $\lambda = 0, \mu = 0, \gamma \neq 0$, we have $0 = \sum_{k\in J} 0.y_k + \sum_{i\in I(\overline{x})} 0.z_i + \sum_{j\in L} \gamma_j \langle \nabla h_j(\overline{x}), v_0\rangle \in D(\overline{x}; v_0)$ ($y_k \in \partial_* F_{k,\overline{x}}(\overline{x})v_0, z_i \in \partial_* g_i(\overline{x})v_0$). Consequently, (3.6) holds, and so, for every $v \in IT(C;\overline{x})$, there exist $\lambda_k^{(n)} \geqslant 0, y_k^{(n)} \in \partial_* F_{k,\overline{x}}(\overline{x})v$ ($\forall k \in J$), $\mu_i^{(n)} \geqslant 0, z_i^{(n)} \in \partial_* g_i(\overline{x})$ ($\forall i \in I(\overline{x})), \gamma_j^{(n)} \in \mathbb{R}$ ($\forall j \in L$) with $(\lambda^{(n)}, \mu^{(n)}_{I(\overline{x})}, \gamma^{(n)}) \neq (0,0,0)$ such that

$$0 = \lim_{n\to\infty} \left(\sum_{k\in J} \lambda_k^{(n)} y_k^{(n)} + \sum_{i\in I(\overline{x})} \mu_i^{(n)} z_i^{(n)} + \sum_{j\in L} \gamma_j^{(n)} \langle \nabla h_j(\overline{x}), v\rangle \right), \tag{3.16}$$

where $\lambda^{(n)} = (\lambda_k^{(n)})_{k\in J}, \mu^{(n)}_{I(\overline{x})} = (\mu_i^{(n)})_{i\in I(\overline{x})}, \gamma^{(n)} = (\gamma_j^{(n)})_{j\in L}$. Since $(\lambda^{(n)}, \mu^{(n)}_{I(\overline{x})}, \gamma^{(n)}) \neq (0,0,0)$, it can be taken them so that $\| (\lambda^{(n)}, \mu^{(n)}_{I(\overline{x})}, \gamma^{(n)}) \| = 1$ ($\forall n$). Without loss of generality, it can be assumed that $(\lambda^{(n)}, \mu^{(n)}_{I(\overline{x})}, \gamma^{(n)}) \to (\overline{\lambda}, \overline{\mu}_{I(\overline{x})}, \overline{\gamma})$ with $\overline{\lambda} \geqslant 0, \overline{\mu}_{I(\overline{x})} \geqslant 0, \overline{\gamma} \in \mathbb{R}^\ell$ and $\| (\overline{\lambda}, \overline{\mu}_{I(\overline{x})}, \overline{\gamma}) \| = 1$. Since $\partial_* F_{k,\overline{x}}(\overline{x})v$ ($k \in J$) and $\partial_* g_i(\overline{x})$ ($i \in I(\overline{x})$) are closed, it follows from (3.16) that

$$\begin{aligned} 0 &\in \sum_{k\in J} \overline{\lambda}_k \mathrm{cl}\ (\partial_* F_{k,\overline{x}}(\overline{x})v) + \sum_{i\in I(\overline{x})} \overline{\mu}_i \mathrm{cl}\ (\partial^* g_i(\overline{x})v) + \sum_{j\in L} \overline{\gamma}_j \langle \nabla h_j(\overline{x}), v\rangle \\ &= \sum_{k\in J} \overline{\lambda}_k \partial_* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in I(\overline{x})} \overline{\mu}_i \partial_* g_i(\overline{x})v + \sum_{j\in L} \overline{\gamma}_j \langle \nabla h_j(\overline{x}), v\rangle. \end{aligned} \tag{3.17}$$

Note that (3.3) along with (3.4) is equivalent to (3.17).

(ii) To prove the part (ii), for $v \in \mathrm{Ker}\nabla h(\overline{x}) \cap IT(C;\overline{x})$, we consider the following set:

$$D_1(\overline{x};v) := \bigcup \Bigg\{ \sum_{k\in J} \lambda_k \partial_* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in I(\overline{x})} \mu_i \partial_* g_i(\overline{x})v : \lambda_k \geqslant 0 \ (\forall k \in J), \mu_i \geqslant 0 \ (\forall i \in I(\overline{x})), (\lambda, \mu_{I(\overline{x})}) \neq (0,0) \Bigg\}.$$

By an argument to that used for the proof of the part (i), we arrive that

$$w > 0 \ (\forall w \in D_1(\overline{x};v_0)).$$

This also can not happen. In fact, taking $\lambda_k = 0 \ (\forall k \in J), \mu_{i_0} = 1$ for some $i_0 \in I(\overline{x}), \mu_i = 0 \ (\forall i \in I(\overline{x}), i \neq i_0)$, we obtain that $w_0 = z_{i_0} \in D_1(\overline{x};v_0)$. By assumption, there exists $z \in \partial_* g(\overline{x})v_0$ such that $z_i < 0 \ (\forall i \in I(\overline{x}))$. Hence, $w_0 = z_{i_0} < 0$. It is absurd.
In the same way as in the proof of the part (i), we deduce that there exist $\overline{\lambda}_k \geqslant 0 \ (\forall k \in J), \overline{\mu}_i \geqslant 0 \ (\forall i \in I)$ with $(\overline{\lambda}, \overline{\mu}) \neq (0,0)$ such that

$$0 \in \sum_{k\in J} \overline{\lambda}_k \partial_* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in I} \overline{\mu}_i \partial_* g_i(\overline{x})v,$$
$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I).$$

The proof is complete. □

In case $C = \mathbb{R}^n$, Theorem 3.4 yields the following consequence.

Corollary 3.5. *Let $C = \mathbb{R}^n$, and let $\overline{x}$ be a local weak efficient solution of (CVEP). Assume that Assumption 3.1 holds, and the functions $F_{\overline{x}}, g$ are steady at $\overline{x} \in K$. Suppose, furthermore, that for every $v \in Ker\nabla h(\overline{x})$, there exists $z \in \partial_* g(\overline{x})v$ such that $z_i < 0 \ (\forall i \in I(\overline{x}))$. Then,*

(i) *For every $v \in \mathbb{R}^n$, there exist $\overline{\lambda}_k \geqslant 0 \ (\forall k \in J), \overline{\mu}_i \geqslant 0 \ (\forall i \in I)$, and $\overline{\gamma}_j \in \mathbb{R} \ (\forall j \in L)$, not all zero, such that*

$$0 \in \sum_{k\in J} \overline{\lambda}_k \partial_* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in I} \overline{\mu}_i \partial_* g_i(\overline{x})v + \sum_{j\in L} \overline{\gamma}_j \langle \nabla h_j(\overline{x}), v\rangle, \tag{3.18}$$
$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I). \tag{3.19}$$

(ii) *For every $v \in Ker\nabla h(\overline{x})$, there exist $\overline{\lambda}_k \geqslant 0 \ (\forall k \in J), \overline{\mu}_i \geqslant 0 \ (\forall i \in I)$ with $(\overline{\lambda}, \overline{\mu}) \neq (0,0)$ such that*

$$0 \in \sum_{k\in J} \overline{\lambda}_k \partial_* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in I} \overline{\mu}_i \partial_* g_i(\overline{x})v,$$
$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I).$$

Proof. For $C = \mathbb{R}^n$, we have $IT(C;\overline{x}) = \mathbb{R}^n$. Thus all the hypotheses of Theorem 3.4 are fulfilled. We invoke this Theorem to deduce the conclusion. □

In case the functions $F_{k,\overline{x}}(.) \ (k \in J)$ and $g_i \ (i \in I)$ are Hadamard differentiable at $\overline{x}$, we get the following direct consequence of Theorem 3.4.

Corollary 3.6. *Let $\overline{x}$ be a local weak efficient solution of (CVEP). Assume that Assumption 3.1 holds, and the functions $F_{\overline{x}}(.), g$ are Hadamard differentiable and steady at $\overline{x}$. Suppose, furthermore, that for every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x}), dg_i(\overline{x};v) < 0 \ (\forall i \in I(\overline{x}))$. Then,*

(i) *For every* $v \in IT(C;\overline{x})$, *there exist* $\overline{\lambda}_k \geqslant 0$ $(\forall k \in J), \overline{\mu}_i \geqslant 0$ $(\forall i \in I)$, *and* $\overline{\gamma}_j \in \mathbb{R}$ $(\forall j \in L)$, *not all zero, such that*

$$\sum_{k\in J}\overline{\lambda}_k dF_{k,\overline{x}}(\overline{x};v) + \sum_{i\in I}\overline{\mu}_i dg_i(\overline{x};v) + \sum_{j\in L}\overline{\gamma}_j\langle\nabla h_j(\overline{x}), v\rangle = 0, \tag{3.20}$$

$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I). \tag{3.21}$$

(ii) *For every* $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, *there exist* $\overline{\lambda}_k \geqslant 0$ $(\forall k \in J), \overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ *with* $(\overline{\lambda}, \overline{\mu}) \neq (0,0)$ *such that*

$$\sum_{k\in J}\overline{\lambda}_k dF_{k,\overline{x}}(\overline{x};v) + \sum_{i\in I}\overline{\mu}_i dg_i(\overline{x};v) = 0,$$

$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I).$$

Proof. Since the functions $F_{\overline{x}}, g$ are Hadamard differentiable at $\overline{x}$, we have $\partial_* F_{k,\overline{x}}(\overline{x})v = \{dF_{k,\overline{x}}(\overline{x};v)\}$ $(\forall k \in J), \partial_* g_i(\overline{x})v = \{dg_i(\overline{x};v)\}$ $(\forall i \in I)$. Then all the hypotheses of Theorem 3.4 are fulfilled. We can invoke Theorem 3.4 to deduce the conclusion. □

Theorem 3.4 is illustrated by the following examples.

Example 3.7. Let $X = \mathbb{R}^2, Y = \mathbb{R}^2$, $C = [0,1] \times [0,1]$, $\overline{x} = (0,0)$. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ be defined as

$$F(x,y) = \begin{cases} (\mid y_1 \mid (1-x_2), y_1 \sin\frac{\pi}{y_1} - y_2^2 + x_2 y_1), & \text{if} \quad y_1 \neq 0, \\ (0, -y_2^2), & \text{if} \quad y_1 = 0, \end{cases}$$

$$g(y) = y_1^2 - \mid y_1 \mid,$$

$$h(y) = y_1 - 2y_2,$$

where $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. Then

$$F_{\overline{x}}(y) = \begin{cases} (\mid y_1 \mid, y_1 \sin\frac{\pi}{y_1} - y_2^2), & \text{if} \quad y_1 \neq 0, \\ (0, -y_2^2), & \text{if} \quad y_1 = 0. \end{cases}$$

We have $K = \{y \in C : \ g(y) \leqslant 0, h(y) = 0\} = \{(y_1, y_2) \in \mathbb{R}^2 : \ y_1 = 2y_2, 0 \leqslant y_2 \leqslant \frac{1}{2}\}$. The point $\overline{x} = (0,0)$ is a weakly efficient solution of the vector equilibrium problem:

$$F(x,y) \notin -\mathbb{R}^2_{++} \ (\forall y \in K).$$

It is obvious that the functions $F_{\overline{x}}, g$ are steady at $\overline{x}$. It can be seen that $IT(C;\overline{x}) = \mathbb{R}^2_{++}$. For $v_1 > 0$, we have that

$$\partial_* F_{\overline{x}}(\overline{x})v = [-v_1, v_1] \times [-v_1, v_1],$$
$$\partial_* g(\overline{x})v = [-v_1, v_1] \quad (v = (v_1, v_2) \in \mathbb{R}^2).$$

Since the function h is differentiable at $\overline{x} = (0,0)$ and $\nabla h(\overline{x}) = (1,-2)$, it holds that $\mathrm{Ker}\nabla h(\overline{x}) \cap IT(C;\overline{x}) = \{(v_1, v_2) \in \mathbb{R}^2_{++} : \ v_1 = 2v_2\}$. It is easily seen that all the hypotheses of Theorem 3.2 are fulfilled. Then for $(v_1, v_2) \in \mathrm{Ker}\nabla h(\overline{x}) \cap IT(C;\overline{x})$, the necessary conditions (3.5) and (3.4) hold for $\overline{\lambda} = (1,1), \overline{\mu} = 1$.

Example 3.8. Let us consider the transportion – production problem. There is a kind of production which is producted at stations $A_1, \ldots, A_m$. It is known that the station A_i spends an expenditure $f_i(x_i)$ to product x_i production units. Productions need to transport to n consume places $B_1, \ldots, B_n$ with requirements $b_1, \ldots, b_n$, respectively. The transportion expenditure of a production unit from A_i to B_j is c_{ij}. The problem of concern is that of finding a plan of production - transportion with total of expenditures of production and transportion is minimal, which ensures that requirements of consume places are satisfied. Let x_{ij} be production bloc transporting from A_i to B_j. Then the mathematical model of this problem is as

$$\sum_{i=1}^{m} f_i(x_i) + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \longrightarrow \min,$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} x_{ij} = x_i, \quad i = 1, \ldots, m,\\
&\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n,\\
&x_{ij} \geqslant 0, \quad i = 1, \ldots, m; j = 1, \ldots, n,\\
&0 \leqslant x_i \leqslant S := \sum_{j=1}^{n} b_j.
\end{aligned}$$

We set

$$f(x) := \sum_{i=1}^{m} f_i(x_i) + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij},$$

where $x = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{m1}, \ldots, x_{mn}) \in \mathbb{R}^{mn}$, and

$$\begin{aligned}
h_j(x) &:= \sum_{i=1}^{m} x_{ij} - b_j, \quad j = 1, \ldots, n,\\
F(x, y) &:= f(y) - f(x),\\
C &:= \mathbb{R}^{mn}_{+}, K := \{x \in C : h_j(x) = 0, j = 1, \ldots, n\}.
\end{aligned}$$

Then the problem can be formulated as an equilibrium problem: Finding $\overline{x} \in K$ such that

$$F(\overline{x}, y) \geqslant 0 \quad (\forall y \in K).$$

Suppose that f_i is continuously differetiable at $\overline{x}_i$ $(i = 1, \ldots, m)$. Then the function $F(\overline{x}, .)$ is continuously differentiable at $\overline{x}$. Putting $F_{\overline{x}}(.) := F(\overline{x}, .)$, we have that $F_{\overline{x}}(.)$ is steady at $\overline{x}$, and $\partial_* F_{\overline{x}}(\overline{x})v = \{\nabla F_{\overline{x}}(\overline{x})v\}$ $(\forall v \in \mathbb{R}^{mn})$. Hence, for every $v = (v_{11}, \ldots, v_{1n}, v_{21}, \ldots, v_{2n}, \ldots, v_{m1}, \ldots, v_{mn}) \in \mathbb{R}^{mn}$,

$$\nabla F(\overline{x})v = \sum_{i=1}^{m} \nabla f_i(\overline{x}_i) \sum_{j=1}^{n} v_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}v_{ij}.$$

It can be seen that $IT(C; \overline{x}) = \mathbb{R}^{mn}_{++}$. Moreover, the functions $h_1, \ldots, h_n$ are differetiable at $\overline{x}$, and $\nabla h_1(\overline{x}), \ldots, \nabla h_n(\overline{x})$ are linearly independent. Applying Theorem 3.4 yields that for every $v \in IT(C; \overline{x}) = \mathbb{R}^{mn}_{++}$, there exist $\overline{\lambda} \geqslant 0, \overline{\gamma}_j \in \mathbb{R}$ $(j = 1, \ldots, n)$, not all zero, such that

$$\overline{\lambda}\left[\sum_{i=1}^{m} \nabla f_i(\overline{x}_i) \sum_{j=1}^{n} v_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}v_{ij}\right] + \sum_{j=1}^{n} \overline{\gamma}_j \sum_{i=1}^{m} v_{ij} = 0.$$

4. Karush–Kuhn–Tucker necessary conditions

In order to derive Karush–Kuhn–Tucker necessary conditions, we introduce the following constraint qualifications:

(CQ1) There exist $s \in J, v_0 \in IT(C;\overline{x})$ such that
(i) $y_k < 0 (\forall y_k \in \partial_* F_{k,\overline{x}}(\overline{x})v_0, \forall k \in J, k \neq s); z_i < 0 (\forall z_i \in \partial_* g_i(\overline{x})v_0, \forall i \in I(\overline{x}))$;
(ii) $\langle \nabla h_j(\overline{x}), v_0\rangle = 0$ $(\forall j \in L)$.

(CQ2) There exists $s \in J, v_0 \in IT(C;\overline{x})$ such that for every $\lambda_k \geqslant 0$ $(k \in J, k \neq s); \mu_i \geqslant 0$ $(\forall i \in I(\overline{x}))$, not all zero, and $\gamma_j \in \mathbb{R}$ $(\forall j \in L)$, we have

$$0 \notin \sum_{k\in J, k\neq s} \lambda_k \partial_* F_{k,\overline{x}}(\overline{x})v_0 + \sum_{i \in I(\overline{x})} \mu_i \partial_* g_i(\overline{x})v_0 + \sum_{j\in L} \gamma_j \langle \nabla h_j(\overline{x}), v_0\rangle.$$

Hereafter, we give a relationship between (CQ1) and (CQ2).

Proposition 4.1. (CQ1) *implies* (CQ2).

Proof. Contrary to the conclusion, suppose that (CQ1) holds, but (CQ2) is false. Hence, there exist $s \in J, v_0 \in IT(C;\overline{x})$ such that (i), (ii) hold, and there exist $\lambda_k \geqslant 0$ $(\forall k \in J, k \neq s), \mu_i \geqslant 0$ $(\forall i \in I(\overline{x}))$ with $(\lambda_{(s)}, \mu) \neq (0,0)$, where $\lambda_{(s)} = (\lambda_k)_{k\in J, k\neq s}, \mu_{I(\overline{x})} = (\mu_i)_{i \in I(\overline{x})}$, $y_k \in \partial_* F_{k,\overline{x}}(\overline{x})v_0$ $(\forall k \in J, k \neq s)$, $z_i \in \partial_* g_i(\overline{x})v_0$ $(\forall i \in I(\overline{x}))$, $\gamma_j \in \mathbb{R}$ $(\forall j \in L)$, such that

$$\begin{aligned} 0 &= \sum_{k\in J, k\neq s} \lambda_k y_k + \sum_{i\in I(\overline{x})} \mu_i z_i + \sum_{j \in L} \gamma_j \langle \nabla h_j(\overline{x}), v_0\rangle \\ &= \sum_{k\in J, k\neq s} \lambda_k y_k + \sum_{i\in I(\overline{x})} \mu_i z_i \\ &< 0, \end{aligned}$$

as $(\lambda_{(s)}, \mu_{I(\overline{x})}) \neq (0,0)$. It is absurd. Hence, the conclusion follows. □

A Karush–Kuhn–Tucker necessary condition for efficiency can be stated as follows.

Theorem 4.2. *Let $\overline{x}$ be a local weak efficient solution of (CVEP). Assume all hypotheses of Theorem 3.4 are fulfilled. Suppose also that the constraint qualification* (CQ1) *or* (CQ2) *(for some $s \in J$) holds. Then, for every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exist there exist $\overline{\lambda}_s > 0, \overline{\lambda}_k \geqslant 0$ $(\forall k \in J, k \neq s)$, $\overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ such that*

$$0 \in \sum_{k\in J} \overline{\lambda}_k \partial_* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in I} \overline{\mu}_i \partial_* g_i(\overline{x})v, \tag{4.1}$$

$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I). \tag{4.2}$$

Proof. We first suppose that (CQ2) holds. Then we invoke Theorem 3.4 (ii) to deduce that there exist $\overline{\lambda}_k \geqslant 0$ $(\forall k \in J)$, $\overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ with $(\overline{\lambda}, \overline{\mu}) \neq (0,0)$ such that (4.1), (4.2) holds. If $\overline{\lambda}_s = 0$, then it follows from (4.1), (4.2) that

$$0 \in \sum_{k\in J, k\neq s} \overline{\lambda}_k \partial_* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in \overline{I}} \overline{\mu}_i \partial_* g_i(\overline{x})v, \tag{4.3}$$

which contradicts (CQ2).

If (CQ1) holds, it follows from Proposition 4.1 that (CQ1) implies (CQ2). Hence, we also arrive a contradiction, and so, $\overline{\lambda}_s > 0$. □

In what follows, we derive a strong Karush–Kuhn–Tucker necessary condition for efficiency in which all the Lagrange multipliers corresponding to all the components of the objective are positive.

Theorem 4.3. *Let $\overline{x}$ be a local weak efficient solution of (CVEP). Assume all hypotheses of Theorem 3.4 are fulfilled. Suppose also that the constraint qualification* (CQ1) *or* (CQ2) *(for every $s \in J$) holds. Then, for every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exist there exist $\overline{\lambda}_k > 0$ $(\forall k \in J)$, $\overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ such that*

$$0 \in \sum_{k\in J} \overline{\lambda}_k \partial_* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in I} \overline{\mu}_i \partial_* g_i(\overline{x})v,$$
$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I).$$

Proof. For each $s \in J$ and $v \in IT(C;\overline{x})$, we invoke Theorem 4.2 to deduce that there exist $\lambda_s^{(s)} > 0, \lambda_k^{(s)} \geqslant 0$ $(\forall k \in J, k \neq s), \mu_i^{(s)} \geqslant 0$ $(\forall i \in I)$ such that

$$0 \in \sum_{k\in J} \lambda_k^{(s)} \partial_* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in I} \mu_i^{(s)} \partial_* g_i(\overline{x})v, \tag{4.4}$$
$$\overline{\mu}_i^{(s)} g_i(\overline{x}) = 0 \ (\forall i \in I). \tag{4.5}$$

Taking $s = 1, \dots, m$ in (4.4), (4.5) and adding up both sides of the obtained inclusions, we obtain the following

$$\begin{aligned} 0 &\in \sum_{s\in J} \left(\sum_{k\in J} \lambda_k^{(s)} \partial^* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in I(\overline{x})} \mu_i^{(s)} \partial_* g_i(\overline{x})v \right) \\ &= \sum_{k\in J} \overline{\lambda}_k \partial_* F_{k,\overline{x}}(\overline{x})v + \sum_{i\in I(\overline{x})} \overline{\mu}_i \partial_* g_i(\overline{x})v, \\ &\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I), \end{aligned}$$

where $\overline{\lambda}_k = \lambda_s^{(s)} + \sum_{s\in J, s\neq k} \lambda_k^{(s)} > 0$ $(\forall k \in J)$, $\overline{\mu}_i = \sum_{s\in J} \mu_i^{(s)} \geqslant 0$ $(\forall i \in I)$. This completes the proof. □

5. Applications to vector variational inequalities and vector optimization

Let $L(\mathbb{R}^n;\mathbb{R}^m)$ be the space of all continuous linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$, and let T be a mapping from $\mathbb{R}^n$ to $L(\mathbb{R}^n;\mathbb{R}^m)$. The vector equilibrium problem (CVEP) includes as a special case the following vector variational inequality problem (CVVI): Finding a point $x \in K$ such that

$$(Tx)(y - x) \notin -\mathbb{R}^m_{++} \ (\forall y \in K).$$

A vector $\overline{x}$ will be called a local weak efficient solution of (CVVI) iff there exists a number $\delta > 0$ such that there is no $y \in K \cap B(\overline{x};\delta)$ satisfying

$$T(\overline{x})_k(y - \overline{x}) < 0 \quad \text{for all } k \in J,$$

where $T(\overline{x}) = (T(\overline{x})_1, \dots, T(\overline{x})_m)$, $T(\overline{x})_k : \mathbb{R}^n \to \mathbb{R}$ $(k \in J)$.

If $F(x, y) = f(y) - f(x)$ $(x, y \in K)$, where $f : \mathbb{R}^n \to \mathbb{R}^m$, the vector equilibrium problem (CVEP) becomes the following vector optimization problem (CVOP):

$$\min\{f(x) : \ x \in K\}.$$

It is said that $\overline{x}$ is a local weak efficient solution of (CVOP) iff there is no $x \in K \cap B(\overline{x};\delta)$ satisfying

$$f_k(x) < f_k(\overline{x}) \ (\forall k \in J).$$

Making use of the results obtained in the previous section to (CVEP), we can derive optimality conditions for the vector variational inequality problem (CVVI). We first establish a Fritz John necessary condition for local weak efficient solution of (CVVI).

Theorem 5.1. *Let $\overline{x}$ be a local weak efficient solution of (CVVI). Assume that g is steady at $\overline{x}$ and continuous in a neighbourhood of $\overline{x}$; the functions $h_1, \ldots, h_\ell$ are Fréchet differentiable at $\overline{x}$ with Fréchet derivatives $\nabla h_1(\overline{x}), \ldots, \nabla h_\ell(\overline{x})$ linearly independent. Suppose, furthermore, that for every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exists $z \in \partial_* g(\overline{x})v$ such that $z_i < 0$ $(\forall i \in I(\overline{x}))$. Then,*

(i) *For every $v \in IT(C;\overline{x})$, there exist $\overline{\lambda}_k \geqslant 0$ $(\forall k \in J), \overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ and $\overline{\gamma}_j \in \mathbb{R}$ $(\forall j \in L)$, not all zero, such that*

$$0 \in \sum_{k\in J} \overline{\lambda}_k \langle T(\overline{x})_k, v\rangle + \sum_{i\in I} \overline{\mu}_i \partial_* g_i(\overline{x})v + \sum_{j\in L} \overline{\gamma}_j \langle \nabla h_j(\overline{x}), v\rangle,$$
$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I);$$

(ii) *For every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exist $\overline{\lambda}_k \geqslant 0$ $(\forall k \in J), \overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ with $(\overline{\lambda}, \overline{\mu}) \neq (0,0)$ such that*

$$0 \in \sum_{k\in J} \overline{\lambda}_k \langle T(\overline{x})_k, v\rangle + \sum_{i\in I} \overline{\mu}_i \partial_* g_i(\overline{x})v,$$
$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I).$$

Proof. Since the mapping $T(\overline{x})(.)$ is linear and continuous, it is strictly differentiable. Hence, the contingent derivative of $T(\overline{x})_k(.)$ at $\overline{x}$ in a direction v is $\{\langle T(\overline{x})_k, v\rangle\}$ $(\forall k \in J)$. For $F(\overline{x}, y) = T(\overline{x})(y - \overline{x})$, one gets that $F_{\overline{x}}(\overline{x}) = 0$. It can be seen that all the hypotheses of Theorem 3.2 are fulfilled. We invoke this theorem to deduce the desired conclusion. □

A Karush–Kuhn–Tucker optimality condition for (CVVI) can be stated as follows.

Theorem 5.2. *Let $\overline{x}$ be a local weak efficient solution of (CVVI). Assume that g is continuous in a neighbourhood of $\overline{x}$ and steady at $\overline{x}$; the functions $h_1, \ldots, h_\ell$ are Fréchet differentiable at $\overline{x}$ with Fréchet derivatives $\nabla h_1(\overline{x}), \ldots, \nabla h_\ell(\overline{x})$ linearly independent. Suppose, furthermore, that for every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exists $z \in \partial_* g(\overline{x})v$ such that $z_i < 0$ $(\forall i \in I(\overline{x}))$, and the constraint qualification* (CQ1) *or* (CQ2) *(for some $s \in J$) holds. Then, for every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exist there exist $\overline{\lambda}_s > 0, \overline{\lambda}_k \geqslant 0$ $(\forall k \in J, k \neq s)$, $\overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ such that*

$$0 \in \sum_{k\in J} \overline{\lambda}_k \langle T(\overline{x})_k, v\rangle + \sum_{i\in I} \overline{\mu}_i \partial_* g_i(\overline{x})v, \tag{5.1}$$
$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I). \tag{5.2}$$

Proof. It can be seen that the function $F_{\overline{x}}(y) = T(\overline{x})(y - \overline{x})$ is strictly differentiable, and $\nabla F_{\overline{x}}(\overline{x}) = T(\overline{x})$. Hence, applying Theorem 4.2 yields that for every $v \in \mathrm{Ker}\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exist $\overline{\lambda}_s > 0, \overline{\lambda}_k \geqslant 0$ $(\forall k \in J, k \neq s)$, $\overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ and $\overline{\gamma}_j \in \mathbb{R}$ $(\forall j \in L)$ such that (5.1) and (5.2) hold. □

Theorem 5.2 is illustrated by the following example.

Example 5.3. Let $X = \mathbb{R}^2, Y = \mathbb{R}^2$, $C = [0,1] \times [0,1]$, $\overline{x} = (0,0)$. Let $T : \mathbb{R}^2 \to L(\mathbb{R}^2, \mathbb{R}^2)$ be defined as

$$T(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix},$$

where $a(x), b(x), c(x), d(x) \in \mathbb{R}$ $(x \in \mathbb{R}^2)$, with $a(\overline{x}) = 1, b(\overline{x}) = \frac{1}{3}, c(\overline{x}) = 1, d(\overline{x}) = -2$. Thus, $T(\overline{x}) = \begin{pmatrix} 1 & 1/3 \\ 1 & -2 \end{pmatrix}$, and for $y = (y_1, y_2) \in \mathbb{R}^2$,

$$T(\overline{x})_1(y) = y_1 + \frac{1}{3}y_2,$$
$$T(\overline{x})_2(y) = y_1 - 2y_2.$$

Let g, h be defined as $g = (g_1, g_2)$, where

$$g_1(y) = \begin{cases} y_1\left(\sin\frac{1}{y_1} - 1\right), & \text{if} \quad y_1 \neq 0, \\ 0, & \text{if} \quad y_1 = 0, \end{cases}$$
$$g_2(y) = y_2(y_2 - 1),$$
$$h(y) = \begin{cases} y_1^2 \mid \cos\frac{\pi}{y_1} \mid + y_1 - y_2, & \text{if} \quad y_1 \neq 0, \\ -y_2, & \text{if} \quad y_1 = 0, \end{cases}$$

We have $K = \{y \in C : \ g(y) \leqslant 0, h(y) = 0\} = \{(y_1, y_2) \in (0,1] \times (0,1] : \ y_2 = y_1 + y_1^2 \mid \cos\frac{\pi}{y_1} \mid\} \cup \{(0,0)\}$. The point $\overline{x} = (0,0)$ is a weakly efficient solution of the vector variational inequality problem:

$$T(\overline{x})(y - \overline{x}) \notin -\mathbb{R}^2_{++} \ (\forall y \in K).$$

It is obvious that the functions g_1, g_2 are steady at $\overline{x}$. It can be seen that $IT(C; \overline{x}) = \mathbb{R}^2_{++}$. For $v = (v_1, v_2)$, we have

$$\partial_* g_1(\overline{x})v = [-2v_1, 0],$$
$$\partial_* g_2(\overline{x})v = \{-v_2\}.$$

Hence, $\partial_* g(\overline{x})v = [-2v_1, 0] \times \{-v_2\}$. Since the function h is differentiable at $\overline{x} = (0,0)$ and $\nabla h(\overline{x}) = (1, -1)$, and $\mathrm{Ker}\nabla h(\overline{x}) = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 = v_2\}$. Hence, $\mathrm{Ker}\nabla h(\overline{x}) \cap IT(C; \overline{x}) = \{(v_1, v_2) \in \mathbb{R}^2_{++} : \ v_1 = v_2\}$. For $(v_1, v_2) \in \mathrm{Ker}\nabla h(\overline{x}) \cap IT(C; \overline{x})$, one has $v_1 = v_2 = v > 0$. It can be seen that (CQ1) holds with $s = 2, v_0 = (v, v)$. Moreover, for $v > 0$, taking $z_i = -v \in \partial_* g_i(\overline{x})(v_1, v_2)$, we have $z_i < 0$ $(i = 1, 2)$. Thus, all hypotheses of Theorem 5.2 are satisfied, and the necessary conditions (5.1) and (5.2) hold for $\overline{\lambda}_1 = 4/3, \overline{\lambda}_2 = 1/3, \overline{\mu}_1 = 1, \overline{\mu}_2 = 1/2$:

$$0 \in 1.\left(v + \frac{1}{3}v\right) + \frac{1}{3}(v - 2v) + 1.[-2v, 0] + \frac{1}{2}(-v).$$

In case (CQ1) or (CQ2) holds for all $s \in J$, we get strong Karush–Kuhn–Tucker necessary conditions for local weak efficient solutions of (CVVI).

Theorem 5.4. *Let $\overline{x}$ be a local weak efficient solution of (CVVI). Assume that all hypotheses of Theorem 5.2 are fulfilled in which* (CQ1) *or* (CQ2) *holds for all $s \in J$. Then, for every $v \in \mathrm{Ker}\nabla h(\overline{x}) \cap IT(C; \overline{x})$, there exist $\overline{\lambda}_k > 0$ $(\forall k \in J)$, $\overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ such that* (5.1) *and* (5.2) *hold.*

Proof. Taking account of Theorem 4.3 to the function $F_{\overline{x}}(y) = T(\overline{x})(y - \overline{x})$, we get the desired conclusion. □

From Theorem 3.4 we can obtain a Fritz John necessary condition for the multiobjective optimization problem (CVOP).

Theorem 5.5. *Let $\overline{x}$ be a local weak efficient solution of (CVOP). Assume that f, g are continuous in a neighbourhood of $\overline{x}$ and steady at $\overline{x}$; the functions $h_1, \ldots, h_\ell$ are Fréchet differentiable at $\overline{x}$ with Fréchet derivatives $\nabla h_1(\overline{x}), \ldots, \nabla h_\ell(\overline{x})$ linearly independent. Suppose, furthermore, that for every $v \in \mathrm{Ker}\nabla h(\overline{x}) \cap IT(C; \overline{x})$, there exists $z \in \partial_* g(\overline{x})v$ such that $z_i < 0$ $(\forall i \in I(\overline{x}))$, and the constraint qualification* (CQ1) *or* (CQ2) *(for some $s \in J$) holds. Then,*

(i) *For every $v \in IT(C; \overline{x})$, there exist there exist $\overline{\lambda}_k \geqslant 0$ $(\forall k \in J), \overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ and $\overline{\gamma}_j \in \mathbb{R}$ $(\forall j \in L)$, not all zero, such that*

$$0 \in \sum_{k \in J} \overline{\lambda}_k \partial_* f(\overline{x})v + \sum_{i \in I} \overline{\mu}_i \partial_* g_i(\overline{x})v + \sum_{j \in L} \overline{\gamma}_j \langle \nabla h_j(\overline{x}), v\rangle,$$
$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I).$$

(ii) *For every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exist $\overline{\lambda}_k \geqslant 0$ $(\forall k \in J), \overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ with $(\overline{\lambda}, \overline{\mu}) \neq (0,0)$ such that*

$$0 \in \sum_{k\in J} \overline{\lambda}_k \partial_* f_k(\overline{x})v + \sum_{i\in I} \overline{\mu}_i \partial_* g_i(\overline{x})v,$$
$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I).$$

Proof. Observe that all the hypotheses of Theorem 3.4 are fulfilled to (CVOP) in which $F(x,y) = f(y) - f(x)$. Applying this theorem to (CVOP), we obtain the desired conclusion. □

In the sequel, we give a Karush–Kuhn–Tucker necessary condition for local weakly efficient solutions of (CVOP).

Theorem 5.6. *Let $\overline{x}$ be a local weak efficient solution of $(CVOP)$. Assume that f, g are continuous in a neighbourhood of $\overline{x}$ and steady at $\overline{x}$; the functions $h_1, \ldots, h_\ell$ are Fréchet differentiable at $\overline{x}$ with Fréchet derivatives $\nabla h_1(\overline{x}), \ldots, \nabla h_\ell(\overline{x})$ linearly independent. Suppose also that for every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exists $z \in \partial_* g(\overline{x})v$ such that $z_i < 0$ $(\forall i \in I(\overline{x}))$, and the constraint qualification* (CQ1) *or* (CQ2) *(for some $s \in J$)holds. Then, for every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exist $\overline{\lambda}_s > 0, \overline{\lambda}_k \geqslant 0$ $(\forall k \in J, k \neq s)$, $\overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ such that*

$$0 \in \sum_{k\in J} \overline{\lambda}_k \partial_* f(\overline{x})v + \sum_{i\in I} \overline{\mu}_i \partial_* g_i(\overline{x})v, \tag{5.3}$$
$$\overline{\mu}_i g_i(\overline{x}) = 0 \ (\forall i \in I). \tag{5.4}$$

Proof. Observe that for the function $F_{\overline{x}}(y) = f(y) - f(\overline{x})$, it holds that $\partial_* F_{\overline{x}}(\overline{x})v = \partial_* f(\overline{x})v$. Making use of Theorem 4.2 to the function $F_{\overline{x}}(y) = f(y) - f(\overline{x})$, we obtain the derised conclusion. □

Theorem 5.6 is illustrated by the following example.

Example 5.7. Let $X = \mathbb{R}^2, Y = \mathbb{R}^2$, $C = [0,1] \times [0,1]$, $\overline{x} = (0,0)$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $f = (f_1, f_2)$, where

$$f_1(x) = \begin{cases} x_1(\cos\frac{\pi}{x_1} - 1) + x_2^2, & \text{if} \quad x_1 \neq 0, \\ x_2^2, & \text{if} \quad x_1 = 0, \end{cases}$$
$$f_2(x) = \begin{cases} \frac{x_1}{1+e^{\frac{1}{x_1}}} + x_2^3, & \text{if} \quad x_1 \neq 0, \\ x_2^3, & \text{if} \quad x_1 = 0, \end{cases}$$

where $(x_1, x_2) \in \mathbb{R}^2$. Let g, h be defined as $g = (g_1, g_2)$,

$$g_1(x) = \begin{cases} x_1(\sin\frac{\pi}{x_1} - 1) - \frac{1}{2}x_2^2, & \text{if} \quad x_1 \neq 0, \\ -\frac{1}{2}x_2^2, & \text{if} \quad x_1 \neq 0, \end{cases}$$
$$g_2(x) = -\arcsin\frac{2x_1}{1+x_1^2} - x_2 - 1,$$
$$h(x) = 2x_1 - x_2.$$

We have $K = \{x \in C : \ g(x) \leqslant 0, h(x) = 0\} = \{(x_1, x_2) \in [0,1] \times [0,1] : x_2 = 2x_1\}$. The point $\overline{x} = (0,0)$ is a weakly efficient solution of the vector optimization problem:

$$\min\{f(x) : \ x \in K\}.$$

It is obvious that the functions f_1, f_2, g_1, g_2 are steady at $\overline{x}$, and $IT(C;\overline{x}) = \mathbb{R}^2_{++}$. For $v = (v_1, v_2)$, we have

$$\begin{aligned}
&\partial_* f_1(\overline{x})v = [-2v_1, 0],\\
&\partial_* f_2(\overline{x})v = [0, v_1],\\
&\partial_* g_1(\overline{x})v = [-2v_1, 0],\\
&\partial_* g_2(\overline{x})v = \{-2v_1 - v_2\}.
\end{aligned}$$

Hence, $\partial_* g(\overline{x})v = [-2v_1, 0] \times \{-2v_1 - v_2\}$. Since the function h is differentiable at $\overline{x} = (0,0)$ and $\nabla h(\overline{x}) = (2, -1)$, and $\mathrm{Ker}\nabla h(\overline{x}) = \{(v_1, v_2) \in \mathbb{R}^2 : v_2 = 2v_1\}$. Hence, $\mathrm{Ker}\nabla h(\overline{x}) \cap IT(C;\overline{x}) = \{(v_1, v_2) \in \mathbb{R}^2_{++} :\ v_2 = 2v_1\}$. It can be seen that (CQ1) holds with $s = 1, v_0 = (v_1, 2v_1)$, and $I(\overline{x}) = \{1\}$. Moreover, for $v_1 > 0$, taking $z_1 = -v_1 \in \partial_* g_1(\overline{x})(v)$, we have $z_1 < 0$. Thus, all hypotheses of Theorem 5.5 are satisfied, and the necessary conditions (5.3) and (5.4) hold for $\overline{\lambda}_1 = 1, \overline{\lambda}_2 = 0, \overline{\mu}_1 = 1, \overline{\mu}_2 = 0$.

A strong Karush–Kuhn–Tucker necessary condition for weakly efficient solutions of (CVOP) can be stated as follows.

Theorem 5.8. *Let $\overline{x}$ be a local weak efficient solution of* (CVOP). *Assume that all hypotheses of Theorem* 5.6 *are fulfilled in which* (CQ1) *or* (CQ2) *holds for all $s \in J$. Then, for every $v \in Ker\nabla h(\overline{x}) \cap IT(C;\overline{x})$, there exist $\overline{\lambda}_k > 0$ $(\forall k \in J), \overline{\mu}_i \geqslant 0$ $(\forall i \in I)$ such that* (5.3) *and* (5.4) *hold.*

Proof. It can be seen that all hypotheses of Theorem 4.3 are fulfilled to the function $F_{\overline{x}}(y) = f(y) - f(\overline{x})$. Taking account of this theorem, we get the desired conclusion. □

6. Conclusion

Jiménez and Novo [8] obtained a rich calculus for contingent derivatives of stable and steady real-valued functions. The notion of contingent derivative for real-valued functions is useful to get necessary optimality conditions for efficiency. We develop Fritz John necessary conditions for local weak efficient solutions of constrained vector equilibrium problems involving inequality, equality and set constraints via contingent derivatives. Karush–Kuhn–Tucker necessary conditions for local weak efficient solutions are established under suitable constraint qualifications. These results are applied to vector variational inequalities and vector optimization problems involving inequality, equality and set constraints. Necessary efficiency conditions obtained here are meaningful extensions of some results obtained in [8] for vector optimization problems with inequality and equality constraints, but without set constraint, and some others obtained in [23] for vector equilibrium problems with stable and steady real-valued functions.

Acknowledgements. The authors would like to express many thanks to the referees for their valuable comments and suggestions. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2014.61.

References

[1] J.-P. Aubin, Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions, in: *Advances in Mathematics Supplementary Studies.* Academic Press, New York (1981) 160–232.

[2] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*. Birkhauser, Boston (1990).

[3] P. Daniele, Lagrange multipliers and infinite-dimensional equilibrium problems. *J. Glob. Optim.* **40** (2008) 65–70.

[4] I.V. Girsanov, Lectures on Mathematical Theory of Extremum Problems. Springer–Verlag, Berlin, Heidenberg (1972).

[5] F. Giannessi, G. Mastroeni and L. Pellegrini, On the theory of vector optimization and variational inequalities, image space analysis and separation, in: Vector Variational Inequalities and Vector Equilibria: Mathematical Theories, edited by F. Giannessi. Kluwer, Dordrecht (2000) 153–215.

[6] X.H. Gong, Optimality conditions for vector equilibrium problems. *J. Math. Anal. Appl.* **342** (2008) 1455–1466.

[7] X.H. Gong, Scalarization and optimality conditions for vector equilibrium problems. *Nonlinear Anal.* **73** (2010) 3598–3612.

[8] B. Jiménez and V. Novo, First order optimality conditions in vector optimization involving stable functions. *Optimization* **57** (2008) 449–471.

[9] B. Jiménez, V. Novo and M. Sama, Scalarization and optimality conditions for strict minimizers in multiobjective optimization via contingent epiderivatives. *J. Math. Anal. Appl.* **352** (2009) 788–798.

[10] J. Jahn and R. Rauh, Contingent epiderivatives and set-valued optimization. *Math. Meth. Oper. Res.* **46** (1997) 193–211.

[11] D.T. Luc, Contingent derivatives of set-valued maps and applications to vector optimization. *Math. Program.* **50** (1991) 99–111.

[12] D.V. Luu, Convexificators and necessary conditions for efficiency. *Optimization* **63** (2014) 321–335.

[13] D.V. Luu, Necessary and sufficient conditions for efficiency via convexificators. *J. Optim. Theory Appl.* **160** (2014) 510–526.

[14] D.V. Luu and D.D. Hang, On optimality conditions for vector variational inequalities. *J. Math. Anal. Appl.* **412** (2014) 792–804.

[15] D.V. Luu and D.D. Hang, Efficient solutions and optimality conditions for vector equilibrium problems. *Math. Methods Oper. Res.* **79** (2014) 163–177.

[16] B.C. Ma and X.H. Gong, Optimality conditions for vctor equilibrium problems in normed spaces. *Optimization* **60** (2011) 1441–1455.

[17] P. Michel and J.P. Penot, A generalized derivative for calm and stable functions. *Diff. Integ. Equ.* **5** (1992) 433–454.

[18] J. Morgan and M. Romaniello, Scalarization and Kuhn-Tucker-like conditions for weak vector generalized quasivariational inequalities. *J. Optimiz. Theory Appl.* **130** (2006) 309–316.

[19] F. Raciti, Equilibrium conditions and vector variational inequalities: a complex relation. *J. Glob. Optimiz.* **40** (2008) 353–360.

[20] R.T. Rockafellar, Convex Analysis. Princeton University Press, Princeton (1970).

[21] L. Rodriguez-Marin and M. Sama, About contingent epiderivatives. *J. Math. Anal. Appl.* **327** (2007) 745–762.

[22] L. Rodriguez-Marin and M. Sama, Variational characterizations of the contingent epiderivatives. *J. Math. Anal. Appl.* **335** (2007) 1374–1382.

[23] T.V. Su, Optimality conditions for vector equilibrium problems in terms of contingent epiderivatives. *Numer. Funct. Anal. Optimiz.* **37** (2016) 640–665.