RAIRO-Oper. Res. 52 (2018) 95–117
https://doi.org/10.1051/ro/2017079

RAIRO Operations Research
www.rairo-ro.org

THE ANALYSIS OF DISCRETE TIME GEOM/GEOM/1 QUEUE WITH SINGLE WORKING VACATION AND MULTIPLE VACATIONS (GEOM/GEOM/1/SWV+MV)

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Abstract. In this article, we consider a discrete-time Geom/Geom/1 queue with two phase vacation policy that comprises single working vacation and multiple vacations, denoted by Geom/Geom/1/SWV+MV. For this model, we first derive the explicit expression for the stationary system size by the matrix-geometric solution method. Next, we obtain the stochastic decomposition structures of system size and the sojourn time of an arbitrary customer in steady state. Moreover, the regular busy period and busy cycle are analyzed by limiting theorem of alternative renewal process. Besides, some special cases are presented and the relationship between the Geom/Geom/1/SWV+MV queue and its continuous time counterpart is investigated. Finally, we perform several experiments to illustrate the effect of model parameters on some performance measures.

Mathematics Subject Classification. 60K25, 68M20

Received September 19, 2017. Accepted October 13, 2017.

1. Introduction

The discrete-time queueing system, which was first presented by Meisling [17], have been well studied in various forms by numerous researchers and have wide applications in digital communication and telecommunication networks, such as broad integrated services digital networks (B-ISDN) based on asynchronous transfer mode (ATM) technology, since information in B-ISDN is transported by means of discrete units. Moreover, one advantage of analyzing the discrete-time is that we can derive the results of the continuous-time counterparts in a limiting case. For more details and applications on the topic of the discrete-time queue, we can refer to the survey paper of Kobayashi *et al.* [8], the monographs of Hunter [7] and Takagi [21].

Since 1970, the vacation queueing systems has been attracted considerable attentions because it is more flexible to find the optimal service policy in queueing systems if the servers are allowed to take vacations. In the real life world, the vacation queueing system has a wide range of applications in many fields such as telecommunication networks, production managing systems, inventory systems, etc. For the details on vacation queues and their applications, the readers may refer to the survey paper by Doshi [5], the monographs of Takagi [20] and Tian and Zhang [22]. In the vacation queues, there is an underlying assumption that the server

Keywords and phrases: Discrete time queue, vacation, working vacation, matrix-geometric solution, stochastic decomposition.

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completely stops its service during the vacation period. In 2002, motivated by the performance analysis of the wavelength division multiplexing (WDM) optical access network, a class of semi-vacation policy, called working vacation (WV), was presented by Servi and Finn, such a vacation policy has the feature that server can still provide the service for the customers presenting in the system during the vacation period, but at a lower service rate. Servi and Finn [19] first analyzed the M/M/1 type queue with multiple working vacations, denoted by M/M/1/MWV, and derived transform formulaes for the distribution of stationary system size and sojourn time of an arbitrary customer. Later on, using matrix-geometric solution method, Liu *et al.* [15] demonstrated the stochastic decomposition properties of the stationary system size and the sojourn time of the M/M/1/MWV queue. The M/M/1 queue with single working vacation, denoted by M/M/1/SWV, was investigated by Tian and Zhao [23]. Subsequently, Wu and Takagi [25] generalized the work in [19] to an M/G/1 queue with general working vacations. Using the matrix analytic method, Li *et al.* [12] studied the M/G/1 queue with exponential working vacations. Extension to GI/M/1 type queue with working vacations was analyzed by Baba [1] by the matrix analysis method. Banik *et al.* [3] analyzed the GI/M/1/N queue with working vacations. Parallel to the results of the continuous time queue with working vacations, the discrete time Geom/Geom/1 queue with single working vacation (Geom/Geom/1/SWV) was studied by Li and Tian [10]. Further, Tian *et al.* [24] discussed the discrete time Geom/Geom/1 queue with multiple working vacations (Geom/Geom/1/MWV). Li *et al.* [11] and Li *et al.* [9], respectively, analyzed the discrete-time GI/Geo/1 type queue with multiple working vacations and the discrete-time GI/Geo/1 type queue with multiple working vacations and vacation interruption. Extension to the batch arrival discrete-time Geo/GI/1 queue with working vacations was studied by Li *et al.* [13].Recently, Baba [2], Gao *et al.* [6], Li *et al.* [14], Luo *et al.* [16], Yang and Wu [26] and and others considered the working vacation queueing systems with various features. More details and recent work related to working vacation, the readers can refer to the survey of Chandrasekaran *et al.* [4].

Although there are various literatures concentrated on vacation queues and working vacation queues, there has been a very few works on the queues related to the combination of vacation and working vacation. The recent work of Ye and Liu [27] presents a new class of vacation policy which comprises single working vacation and multiple vacations. During the regular busy period, if the server finds the queue empty after completing the service of a customer, the server takes a working vacation during which the server can still provide the service but at a lower rate. After the working vacation, if there are customers staying in the system, the service rate will resume to the normal service rate immediately and a regular busy period starts. Otherwise, one vacation will be taken during which the server stops its service completely. Upon return from the vacation, if it finds one or more customers waiting in the system, it takes them for service at normal service rate immediately, on the other hand, if there are no customers yet in the system, it immediately proceeds for another vacation and continues in this manner until it finds at least one customer queued for the service upon returning from a vacation. In order to describe this queue model with two phase vacation policy more clearly, we represent this system schematically in Figure 1. In the real life situation situations, the two phase vacation policy has practical cases and application, for example. (1) In several banks or service centers, the service windows can be divided into corporation business windows and personal service windows. When the customers with corporation business are relatively few, the service center/bank usually just keeps one window for corporation business in operation waiting for the possible customers, this Min Speed period can be viewed as working vacation, after this period, if there are still no customers with corporation business and the number of customers with personal service grows too large, the window for corporation business may temporarily stop its service and begin to provide service to customers with personal service, which can be seen as a vacation. In above case, we can regard that the corporation business windows take two phase vacation policy we study. (2) The escalators in some large supermarkets and metros are designed to operate at a lower service rate for a certain length when the escalators just have no passenger, we can see this low service rate period as a working vacation, after the low service period, if there are passengers coming, the escalators will switch to regular service rate. If there is still no passenger, the escalators will temporarily stop the running until the new arrival of passengers which can be seen as a vacation. In above case, we can regard that escalators take two phase vacation policy we study. (3) This type of two phase vacation policy can also be utilized to model building of server maintenance model. When the workload in the system is relatively few, the server may first enter a buffer period (working vacation)

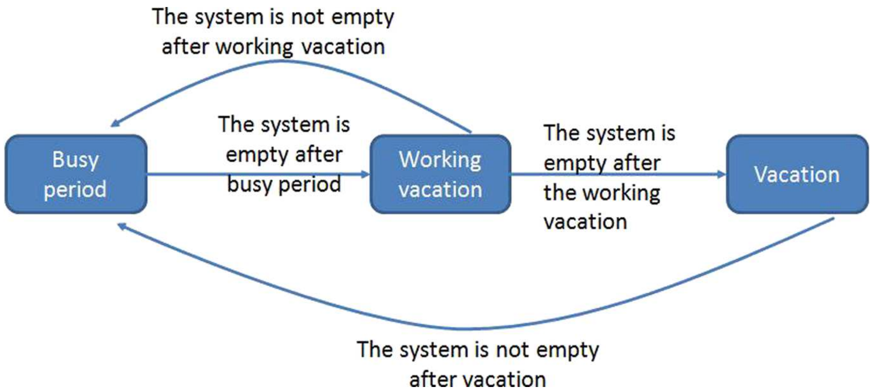

Figure 1. Schematic representation. (Color online.)

waiting for the possible tasks arriving, in which server work in a lower rate operation to economize operation cost and energy consumption. After the buffer period, the system may resume to the regular service state for the more benefit if there are tasks staying in the system. Otherwise, the server temporarily stops the running to make the maintenance.

Ye and Liu [27] first studied the M/M/1 queue with single working vacation and multiple vacations, subsequently, Ye and Liu [28] generalized the work in Ye and Liu [27] to the GI/M/1 type queue with single working vacation and multiple vacations. Inspired by classical discrete-time queues and the works of Ye and Liu [27, 28], in this paper, we aim to extend the work in Ye and Liu [27] to the discrete-time Geom/Geom/1 queue and discuss various properties for Geom/Geom/1 queue with single working vacation and multiple vacations and obtain the various properties for this model.

The rest of this paper is organized as follows. In Section 2, we give a detailed description of the Geom/Geom/1/SWV+MV queue and obtain the ergodicity condition. Section 3 is denoted to obtain the stationary distribution of system size. Section 4 demonstrates the stochastic decomposition properties of the system size and the sojourn time in steady state. In Section 5, we discuss the regular busy period and busy cycle. The relationship between the discrete-time Geom/Geom/1/SWV+MV queue and its continuous-time counterpart M/M/1/SWV+MV queue is given in Section 6. Several special cases are presented in Section 7. The effects of model parameters on the key performance measures are performed in Section 8. The last is some conclusions.

2. Model description and ergodicity condition

We consider the Geom/Geom/1 queue with two phase vacation policy that comprises single working vacation and vacations, the detailed description for this model is as follows. For mathematical convenience, we denote $\bar{x} = 1 - x$, for any real number $x \in [0, 1]$.

In the discrete-time queues, the queueing activities (the arrivals, the departures and the end of vacations) may take place at the same time. For mathematical clarity, we suppose that a potential arrival occurs in the interval (n^-, n), $n = 0, 1, 2, \ldots$, where n^- is the moment immediately before n, and a potential departure occurs in the interval (n, n^+), where n^- is the moment immediately before n, that is, the system we consider here is an early arrival system (EAS). Further, we assume that the beginning and ending of the working vacations or vacations occur in the interval (n^-, n).

Customers arrive at the system according to a geometric arrival process with parameter p, that is, the inter-arrival time T follows a geometric distribution as follows.

$$P\{T = k\} = p\bar{p}^{k-1}, \quad k \geq 1,\ 0 < p < 1.$$

The service time S_b during the regular busy period and service time S_v during the working vacation period follow the geometric distribution with parameter μ_b and μ_v, respectively. That is

$$P\{S_b = k\} = \mu_b \bar{\mu}_b^{k-1}, \quad k \geq 1,\ 0 < \mu_b < 1,$$

$$P\{S_v = k\} = \mu_v \bar{\mu}_v^{k-1}, \quad k \geq 1,\ 0 < \mu_v < 1,$$

where $\mu_v < \mu_b$.

The working vacation V_w and vacation V_v follow the geometric distribution with parameter θ_w and θ_v, respectively. That is

$$P\{V_w = k\} = \theta_w \bar{\theta}_w^{k-1}, \quad k \geq 1,\ 0 < \theta_w < 1.$$

$$P\{V_v = k\} = \theta_v \bar{\theta}_v^{k-1}, \quad k \geq 1,\ 0 < \theta_v < 1.$$

The two phase vacation policy is described explicitly as follows: During the regular busy period, the customers are served at rate μ_b, if the server after completing the service of a customer finds the queue empty, the server enters into a working vacation during which the server can still provide the service, but at lower rate μ_v. At the end of the working vacation, if there are customers staying in the system, the server switches its service rate from μ_v to μ_b immediately, and then the system will enter into the regular busy period. Otherwise the server will take a vacation during which the server stops its service completely. Upon return from a vacation if it finds one or more customers waiting, it takes them for service at a rate of μ_b immediately, on the other hand, if there are no customers yet in the system, it immediately proceeds for another vacation and continues in this manner until it finds at least one customer queued for the service upon returning from a vacation.

We assume that inter-arrival times, service times during the regular busy period, services during the working vacation, working vacation times and vacation times are mutually independent. Further, the service order is assumed to be first in first out (FIFO).

Let Q_n be the number of customers in the system at $t = n^+$ and J_n be the state of server at $t = n^+$ with

$$J_n = \begin{cases} 0, & \text{the server is in working vacation period at time } n^+, \\ 1, & \text{the server is in vacation period at time } n^+, \\ 2, & \text{the server is in regular busy period at time } n^+, \end{cases}$$

then $\{Q_n, J_n\}$ is a Markov process with the state space

$$\Omega = \{(0,0)\,,\ (0,1)\} \cup \{(k,j) : k = 1,2,\ldots,\ j = 0,1,2\}\,.$$

Using the lexicographical order for the states, *i.e.*, (0, 0), (0, 1), (1, 0), (1, 1), (1, 2),. . . , the transition probability matrix can be written as a Block-Jacobi matrix

$$\tilde{P} = \begin{bmatrix} A_{00} & A_{01} & & & \\ B_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & \vdots & \vdots & \vdots \end{bmatrix}, \tag{2.1}$$

where

$$A_{00} = \begin{bmatrix} \bar{\theta}_w \bar{p} & \theta_w \bar{p} \\ 0 & \bar{p} \end{bmatrix}, \quad A_{01} = \begin{bmatrix} \bar{\theta}_w p & 0 & \theta_w p \\ 0 & \bar{\theta}_v p & \theta_v p \end{bmatrix}, \quad B_{10} = \begin{bmatrix} \bar{\theta}_w \bar{p}\mu_v & \theta_w \bar{p}\mu_v \\ 0 & 0 \\ \bar{p}\mu_b & 0 \end{bmatrix},$$

and

$$A_0 = \begin{bmatrix} \bar{\theta}_w p\bar{\mu}_v & 0 & \theta_w p\bar{\mu}_v \\ & \bar{\theta}_v p & \theta_v p \\ & & p\bar{\mu}_b \end{bmatrix}, \quad A_2 = \begin{bmatrix} \bar{\theta}_w \bar{p}\mu_v & 0 & \theta_w \bar{p}\mu_v \\ & 0 & 0 \\ & & \bar{p}\mu_b \end{bmatrix},$$

$$A_1 = \begin{bmatrix} \bar{\theta}_w(1-p\bar{\mu}_v-\bar{p}\mu_v) & 0 & \theta_w(1-p\bar{\mu}_v-\bar{p}\mu_v) \\ & \bar{\theta}_v\bar{p} & \theta_v\bar{p} \\ & & 1-p\bar{\mu}_b-\bar{p}\mu_b \end{bmatrix}.$$

It is easily seen from the structure of transition probability matrix that $\{Q_n, J_n\}$ is a Quasi-Birth-and-Death (QBD) process, so we are ready to analyze this model by matrix geometric solution method (the details can be found in Neuts [18]). To this end, it is necessary to solve the minimal non-negative solution of the matrix quadratic equation

$$R = R^2 A_2 + R A_1 + A_0. \tag{2.2}$$

In the following lemma, we give the explicit expression for R.

Lemma 2.1. *If $\alpha = p\bar{\mu}_b/\bar{p}\mu_b < 1$, the matrix equation $R = R^2A_2 + RA_1 + A_0$ has the minimal non-negative solution*

$$R = \begin{bmatrix} r & 0 & \frac{\theta_w r}{\bar{\theta}_w \bar{p}\mu_b \bar{r}} \\ 0 & \beta & \frac{p}{\bar{p}\mu_b} \\ 0 & 0 & \alpha \end{bmatrix}, \tag{2.3}$$

where $0 < r < 1$, and

$$r = \frac{1}{2\bar{p}\mu_v}\left(\theta_w\bar{\theta}_w^{-1} + p\bar{\mu}_v + \bar{p}\mu_v - \left(\left(\theta_w\bar{\theta}_w^{-1} + p\bar{\mu}_v + \bar{p}\mu_v\right)^2 - 4p\bar{p}\mu_v\bar{\mu}_v\right)^{\frac{1}{2}}\right), \tag{2.4}$$

$$\beta = \frac{\bar{\theta}_v p}{1-\bar{\theta}_v\bar{p}}. \tag{2.5}$$

Proof. Since A_0, A_1, A_2 are all upper triangular matrices, we can assume that R has the same structure as

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{bmatrix}. \tag{2.6}$$

By substituting (2.6) in (2.2), we will have the following set of equations:

$$r_{11} = \bar{\theta}_w \bar{p}\mu_v r_{11}^2 + \bar{\theta}_w(1-\bar{p}\mu_v - p\bar{\mu}_v)r_{11} + \bar{\theta}_w p\bar{\mu}_v, \tag{2.7}$$

$$r_{12} = \bar{\theta}_v \bar{p} r_{12}, \tag{2.8}$$

$$\begin{aligned} r_{13} = {} & \theta_w \bar{p}\mu_v r_{11}^2 + \bar{p}\mu_b(r_{11}r_{13} + r_{12}r_{23} + r_{13}r_{33}) + (1 - p\bar{\mu}_b - \bar{p}\mu_b)r_{13} \\ & + \theta_v \bar{p} r_{12} + \theta_w(1 - p\bar{\mu}_v - \bar{p}\mu_v)r_{11} + \theta_w p\bar{\mu}_v, \end{aligned} \tag{2.9}$$

$$r_{22} = \bar{\theta}_v \bar{p} r_{22} + \bar{\theta}_v p, \tag{2.10}$$

$$r_{23} = \bar{p}\mu_b(r_{22}r_{23} + r_{23}r_{33}) + \theta_v \bar{p} r_{22} + (1 - p\bar{\mu}_b - \bar{p}\mu_b)r_{23} + \theta_v p, \tag{2.11}$$

$$r_{33} = \bar{p}\mu_b r_{33}^2 + (1 - p\bar{\mu}_b - \bar{p}\mu_b)r_{33} + p\bar{\mu}_b. \tag{2.12}$$

Clearly, we can know the quadratic equation (2.7) has two real roots, one is r which is given by (2.4), and the other is

$$\frac{1}{2p\bar{\mu}_v}\left(\theta_w\bar{\theta}_w^{-1} + p\bar{\mu}_v + \bar{p}\mu_v + \left(\left(\theta_w\bar{\theta}_w^{-1} + p\bar{\mu}_v + \bar{p}\mu_v\right)^2 - 4p\bar{p}\mu_v\bar{\mu}_v\right)^{\frac{1}{2}}\right). \tag{2.13}$$

Note that

$$\left(\theta_w\bar{\theta}_w^{-1} + p\bar{\mu}_v + \bar{p}\mu_v\right) - 2\bar{p}\mu_v = \theta_w\bar{\theta}_w^{-1} + p\bar{\mu}_v - \bar{p}\mu_v < \left(\left(\theta_w\bar{\theta}_w^{-1} + p\bar{\mu}_v + \bar{p}\mu_v\right)^2 - 4p\bar{p}\mu_v\bar{\mu}_v\right)^{\frac{1}{2}},$$

we can verify that $r < 1$, and the other root given by (2.13) is larger than 1. Thus, we take r_{11} as r. Note that the minimal non-negative root of quadratic equation (2.12) is $p\bar{\mu}_b/\bar{p}\mu_b$ that is denoted by α, and the other root is 1, so we take $r_{33} = \alpha$. From (2.8) and (2.10), we can directly obtain $r_{12} = 0$ and $r_{22} = \beta = \bar{\theta}_v p\left(1 - \bar{\theta}_v\bar{p}\right)^{-1}$, Substituting $r_{11} = \alpha, r_{12} = 0, r_{22} = \beta$, and $r_{33} = \alpha$ in the (2.9) and (2.11), and after some mathematical simplification, we can get

$$r_{13} = \frac{\theta_w r}{\bar{\theta}_w \bar{p}\mu_b \bar{r}}, \quad r_{23} = \frac{p}{\bar{p}\mu_b}.$$

Then the proof is completed. □

Based on Lemma 2.1, we can give the ergodicity condition of the queueing system in the following lemma

Lemma 2.2. *The process $\{Q_n, J_n\}$ is ergodic if and only if $\alpha < 1$.*

Proof. According to Theorem 1.5.1 in Neuts [18], we know that $\{Q_n, J_n\}$ is ergodic if and only if the spectral radius $SP(R)$ of matrix R is less than 1, and set of equations

$$(\pi_{00}, \pi_{01}, \pi_{10}, \pi_{11}, \pi_{12})B\,[R] = (\pi_{00}, \pi_{01}, \pi_{10}, \pi_{11}, \pi_{12}), \tag{2.14}$$

has a positive solution, where

$$B\,[R] = \begin{bmatrix} A_{00} & A_{01} \\ B_{10} & RA_2 + A_1 \end{bmatrix}. \tag{2.15}$$

Note that

$$B\,[R] = \begin{bmatrix} \bar{\theta}_w\bar{p} & \theta_w\bar{p} & \bar{\theta}_w p & 0 & \theta_w p \\ 0 & \bar{p} & 0 & \bar{\theta}_v p & \theta_v p \\ \bar{\theta}_w\bar{p}\mu_v & \theta_w\bar{p}\mu_v & \bar{\theta}_w(1-\bar{p}\mu_v) - \frac{\theta_w r}{\bar{r}} & 0 & \theta_w(1-\bar{p}\mu_v) + \frac{\theta_w r}{\bar{r}} \\ 0 & 0 & 0 & \bar{\theta}_v\bar{p} & p + \theta_v\bar{p} \\ \bar{p}\mu_b & 0 & 0 & 0 & 1-\bar{p}\mu_b \end{bmatrix}, \tag{2.16}$$

we can verify $B\,[R]$ is a stochastic matrix which ensures that set of equations with coefficient matrix $B\,[R]$ has positive solution. Since $r < 1$ and $\beta < 1$, we can find that $SP(R) = \max(r, \beta, \alpha) < 1$ is fulfilled if and only if $\alpha < 1$. Then the proof is completed. □

3. Stationary distribution of the system size

If the ergodicity condition $\alpha < 1$ is fulfilled, then its limiting probabilities exist and are positive. Define

$$\pi_{k,j} = \lim_{n\to\infty} P\{Q_n = k, J_n = j\}, \quad (k,j) \in \Omega$$

$$\pi_0 = (\pi_{00}, \pi_{01})\,, \pi_k = (\pi_{k0}, \pi_{k1}, \pi_{k2})\,, \quad k = 0, 1, 2, \ldots.$$

The stationary distribution of system size is given in the following theorem.

Theorem 3.1. *Under the condition that $\alpha < 1$, the stationary probability distribution is*

$$\begin{cases} \pi_{00} = Kp\left(\theta_w + \bar{\theta}_w\bar{p}\mu_v\bar{r}\right), & \\ \pi_{k0} = Kp^2\bar{\theta}_w\bar{r}r^{k-1}, & k \geq 1, \\ \pi_{k1} = K\bar{p}\theta_w\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right)\beta^k, & k \geq 0, \\ \pi_{k2} = Kp\left[\frac{p\theta_w}{\bar{p}\mu_b}\sum\limits_{j=0}^{k-1} r^j\alpha^{k-1-j} + \frac{\theta_w\left(\theta_w+\bar{\theta}_w\mu_v\bar{r}\right)}{\mu_b}\sum\limits_{j=0}^{k-1}\beta^j\alpha^{k-1-j}\right], & k \geq 1, \end{cases} \tag{3.1}$$

where

$$K = \frac{\bar{p}\,\mu_b\bar{r}\bar{\alpha}\,\bar{\beta}}{p\bar{p}\mu_b\left(\theta_w + \bar{\theta}_w\bar{p}\mu_v\bar{r}\right)\bar{r}\bar{\alpha}\bar{\beta} + p^2\bar{\beta}\left(\theta_w + \bar{p}\bar{\theta}_w\bar{r}\mu_b\bar{\alpha}\right) + \bar{p}\theta_w\bar{r}\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right)(p + \bar{p}\mu_b\bar{\alpha})}. \tag{3.2}$$

Proof. Applying the method of matrix-geometric solution (see Neuts [18]), we have

$$\pi_k = (\pi_{k0}, \pi_{k1}, \pi_{k2}) = (\pi_{10}, \pi_{11}, \pi_{12})R^{k-1}, \quad k \geq 1, \tag{3.3}$$

$$(\pi_{00}, \pi_{01}, \pi_{10}, \pi_{11}, \pi_{12})B\,[R] = (\pi_{00}, \pi_{01}, \pi_{10}, \pi_{11}, \pi_{12}). \tag{3.4}$$

Expanding the equation (3.4) yields the following equations

$$\pi_{00} = \bar{\theta}_w \bar{p}\pi_{00} + \bar{\theta}_w \bar{p}\mu_v \pi_{10} + \bar{p}\mu_b \pi_{12}, \tag{3.5}$$

$$\pi_{01} = \theta_w \bar{p}\pi_{00} + \bar{p}\pi_{01} + \theta_w \bar{p}\mu_v \pi_{10}, \tag{3.6}$$

$$\pi_{10} = \bar{\theta}_w p\pi_{00} + \left[\bar{\theta}_w(1-\bar{p}\mu_v) - \frac{\theta_w r}{\bar{r}}\right]\pi_{10}, \tag{3.7}$$

$$\pi_{11} = \bar{\theta}_v p\pi_{01} + \bar{\theta}_v \bar{p}\pi_{11}, \tag{3.8}$$

$$\pi_{12} = \theta_w p\pi_{00} + \theta_v p\pi_{01} + \left[\theta_w(1-\bar{p}\mu_v) + \frac{\theta_w r}{\bar{r}}\right]\pi_{10} + (p+\theta_v\bar{p})\pi_{11} + (1-\bar{p}\mu_b)\pi_{12}. \tag{3.9}$$

Solving equations (3.5)–(3.9) in form of π_{00} yields:

$$\pi_{01} = \frac{\bar{p}\theta_w\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right)}{p\left(\theta_w + \bar{\theta}_w\bar{p}\mu_v\bar{r}\right)}\pi_{00}, \tag{3.10}$$

$$\pi_{10} = \frac{p\bar{\theta}_w\bar{r}}{\theta_w + \bar{\theta}_w\bar{p}\mu_v\bar{r}}\pi_{00}, \tag{3.11}$$

$$\pi_{11} = \frac{\bar{p}\theta_w\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right)\beta}{p\left(\theta_w + \bar{\theta}_w\bar{p}\mu_v\bar{r}\right)}\pi_{00}, \tag{3.12}$$

$$\pi_{12} = \frac{p\theta_w + \bar{p}\theta_w\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right)}{\bar{p}\mu_b\left(\theta_w + \bar{\theta}_w\bar{p}\mu_v\bar{r}\right)}\pi_{00}. \tag{3.13}$$

From (2.6), we can directly obtain that

$$R^k = \begin{bmatrix} r^k & 0 & \frac{\theta_w r}{\bar{\theta}_w \bar{p}\mu_b \bar{r}} \sum_{j=0}^{k-1} r^j \alpha^{k-1-j} \\ & \beta^k & \frac{p}{\bar{p}\mu_b} \sum_{j=0}^{k-1} \beta^j \alpha^{k-1-j} \\ & & \alpha^k \end{bmatrix}, \quad k = 1, 2, \dots. \tag{3.14}$$

Substituting $(\pi_{10},\ \pi_{11},\ \pi_{12})$ and R^{k-1} in (3.3), we can have

$$\pi_{k0} = \pi_{00} \frac{p\bar{\theta}_w \bar{r}}{\theta_w + \bar{\theta}_w \bar{p}\mu_v \bar{r}} r^{k-1}, \quad k = 1, 2, \dots, \tag{3.15}$$

$$\pi_{k1} = \pi_{00} \frac{\bar{p}\theta_w \left(\theta_w + \bar{\theta}_w \mu_v \bar{r}\right)}{p\left(\theta_w + \bar{\theta}_w \bar{p}\mu_v \bar{r}\right)} \beta^k, \quad k = 1, 2, \dots, \tag{3.16}$$

$$\pi_{k2} = \pi_{00} \left[\frac{p\theta_w}{\bar{p}\mu_b \left(\theta_w + \bar{\theta}_w \bar{p}\mu_v \bar{r}\right)} \sum_{j=0}^{k-1} r^j \alpha^{k-1-j} + \frac{\theta_w \left(\theta_w + \bar{\theta}_w \mu_v \bar{r}\right)}{\mu_b \left(\theta_w + \bar{\theta}_w \bar{p}\mu_v \bar{r}\right)} \sum_{j=0}^{k-1} \beta^j \alpha^{k-1-j} \right], \quad k = 1, 2, \dots \tag{3.17}$$

By utilizing normalization condition $\pi_{00} + \pi_{01} + \sum_{k=1}^{\infty} (\pi_{k0} + \pi_{k1} + \pi_{k2}) = 1$, we can determine the π_{00} after some manipulation:

$$\pi_{00} = \frac{p\bar{p}\mu_b \bar{r}\bar{\alpha}\bar{\beta}\left(\theta_w + \bar{\theta}_w \bar{p}\mu_v \bar{r}\right)}{p\bar{p}\mu_b \bar{r}\bar{\alpha}\bar{\beta}\left(\theta_w + \bar{\theta}_w \bar{p}\mu_v \bar{r}\right) + p^2 \bar{\beta}\left(\theta_w + \bar{p}\bar{\theta}_w \bar{r}\mu_b \bar{\alpha}\right) + \bar{p}\theta_w \bar{r}\left(\theta_w + \bar{\theta}_w \mu_v \bar{r}\right)\left(p + \bar{p}\mu_b \bar{\alpha}\right)}. \tag{3.18}$$

Hence, we can arrive at the results in Theorem 3.1. Then the proof is completed. □

Let $P(W)$, $P(V)$ and $P(B)$ be the probabilities that the server is in working vacation, vacation and regular busy period, respectively, then, according to the results in Theorem 3.1, we can easily obtain that

$$P(W) = \sum_{k=0}^{\infty} \pi_{k0} = Kp\left(p\bar{\theta}_w + \theta_w + \bar{p}\bar{\theta}_w \mu_v \bar{r}\right), \tag{3.19}$$

$$P(V) = \sum_{k=0}^{\infty} \pi_{k1} = \frac{K\bar{p}\theta_w \left(\theta_w + \bar{\theta}_w \mu_v \bar{r}\right)}{\bar{\beta}}, \tag{3.20}$$

$$P(B) = \sum_{k=1}^{\infty} \pi_{k2} = Kp\left[\frac{p\theta_w}{\bar{p}\mu_b \bar{r}\bar{\alpha}} + \frac{\theta_w \left(\theta_w + \bar{\theta}_w \mu_v \bar{r}\right)}{\mu_b \bar{\alpha}\bar{\beta}}\right]. \tag{3.21}$$

4. Stochastic decomposition structures

In this section, we focus on the analysis of the stochastic decomposition properties of the queueing system under consideration. At first, we consider the stochastic decomposition structure of system size Q in the steady state, before that, we give the following useful relationships:

$$\frac{z\bar{r}\,(1-\alpha z)}{1-rz} = \frac{z\bar{r}}{1-rz} - z\bar{r}\frac{\alpha}{r}\frac{rz-1+1}{1-rz} = \alpha\frac{\bar{r}}{r}z + \left(1-\frac{\alpha}{r}\right)\frac{z\bar{r}}{1-rz}, \tag{4.1}$$

similarly,

$$\frac{z\bar{\beta}\,(1-\alpha z)}{1-\beta z} = \frac{z\bar{\beta}}{1-\beta z} - z\bar{\beta}\frac{\alpha}{\beta}\frac{\beta z-1+1}{1-\beta z} = \alpha\frac{\bar{\beta}}{\beta}z + \left(1-\frac{\alpha}{\beta}\right)\frac{z\bar{\beta}}{1-\beta z}. \tag{4.2}$$

From (2.7), it is easily obtained that

$$\frac{\theta_w r}{\bar{\theta}_w} = \bar{p}\mu_v r^2 - (\bar{p}\mu_v + p\bar{\mu}_v)r + p\bar{\mu}_v = \bar{r}\,(p\bar{\mu}_v - \bar{p}\mu_v r)\,. \tag{4.3}$$

Further, from the above relationship, we can have

$$\theta_w + \bar{p}\bar{\theta}_w\mu_v\bar{r} = p\bar{\theta}_w\bar{\mu}_v\frac{\bar{r}}{r}. \tag{4.4}$$

Theorem 4.1. *If the conditions $\alpha < 1$ and $\mu_b > \mu_v$ are fulfilled, the stationary system size Q can be decomposed into the sum of two independent random variables: $Q = Q_0 + Q_d$, where Q_0 is the stationary system size in the classic Geom/Geom/1 queue and follows a geometric distribution with parameter $\bar{\alpha}$, and the additional system size Q_d is a mixture of four random variables*

$$Q_d = \delta_0 X_0 + \delta_1 X_1 + \delta_2 X_2 + \delta_3 X_3, \tag{4.5}$$

where $X_0 \equiv 0$, $X_1 \equiv 1$, X_2 and X_3 follow the geometric distribution with parameter $\bar{r}$ and $\bar{\beta}$, respectively, and

$$\delta_0 = K^*\left[p\bar{p}\,\mu_b\bar{r}\bar{\beta}\left(\theta_w + \bar{\theta}_w\bar{p}\mu_v\bar{r}\right) + \bar{p}^2\,\mu_b\theta_w\bar{r}\bar{\beta}\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right)\right],$$

$$\delta_1 = K^*\frac{p^3\,\bar{\mu}_b\bar{\theta}_w\bar{r}^2\bar{\beta}\mu_v}{r},$$

$$\delta_2 = \frac{K^* p^2\bar{\theta}_w\bar{\beta}\bar{r}\,(p + r\bar{p})\,(\mu_b - \mu_v)}{r},$$

$$\delta_3 = K^*\bar{p}\theta_w\bar{r}\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right)\left(\bar{p}\mu_b\beta + p\mu_b\right),$$

$$K^* = \left[p\bar{p}\mu_b\left(\theta_w + \bar{\theta}_w\bar{p}\mu_v\bar{r}\right)\bar{r}\bar{\alpha}\bar{\beta} + p^2\bar{\beta}\left(\theta_w + \bar{p}\bar{\theta}_w\bar{r}\mu_b\bar{\alpha}\right) + \bar{p}\theta_w\bar{r}\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right)(p + \bar{p}\mu_b\bar{\alpha})\right]^{-1}.$$

Proof. From equation (3.1), we can have the probability generating function (PGF) of the system size Q:

$$\begin{aligned}
Q(z) &= \sum_{k=0}^{\infty}\pi_{k0}z^k+\sum_{k=0}^{\infty}\pi_{k1}z^k+\sum_{k=1}^{\infty}\pi_{k2}z^k \\
&= K^*p\bar{p}\,\mu_b\bar{r}\bar{\alpha}\bar{\beta}\left(\theta_w+\bar{\theta}_w\bar{p}\mu_v\bar{r}\right)+K^*p^2\bar{p}\,\mu_b\bar{\theta}_w\bar{r}\bar{\alpha}\bar{\beta}\frac{\bar{r}z}{1-rz}+K^*\bar{p}^2\,\mu_b\theta_w\bar{r}\bar{\alpha}\bar{\beta}\left(\theta_w+\bar{\theta}_w\mu_v\bar{r}\right) \\
&\quad+K^*\bar{p}^2\,\mu_b\theta_w\bar{r}\bar{\alpha}\beta\left(\theta_w+\bar{\theta}_w\mu_v\bar{r}\right)\frac{\bar{\beta}z}{1-\beta z}+K^*p^2\theta_w\bar{\beta}\frac{\bar{r}z}{1-rz}\frac{\bar{\alpha}z}{1-\alpha z} \\
&\quad+K^*p\bar{p}\theta_w\,\bar{r}\left(\theta_w+\bar{\theta}_w\mu_v r\right)\frac{\bar{\beta}z}{1-\beta z}\frac{\bar{\alpha}z}{1-\alpha z},
\end{aligned}\tag{4.6}$$

where

$$K^*=\left[p\bar{p}\mu_b\left(\theta_w+\bar{\theta}_w\bar{p}\mu_v\bar{r}\right)\bar{r}\bar{\alpha}\bar{\beta}+p^2\bar{\beta}\left(\theta_w+\bar{p}\bar{\theta}_w\bar{r}\mu_b\bar{\alpha}\right)+\bar{p}\theta_w\bar{r}\left(\theta_w+\bar{\theta}_w\mu_v\bar{r}\right)\left(p+\bar{p}\mu_b\bar{\alpha}\right)\right]^{-1}.\tag{4.7}$$

Further, we can rewrite $Q(z)$ as follows:

$$Q(z)=\frac{\bar{\alpha}z}{1-\alpha z}Q_d(z),\tag{4.8}$$

where

$$\begin{aligned}
Q_d(z) &= K^*p\bar{p}\,\mu_b\bar{r}\bar{\beta}\left(\theta_w+\bar{\theta}_w\bar{p}\mu_v\bar{r}\right)(1-\alpha z)+K^*p^2\bar{p}\,\mu_b\bar{\theta}_w\bar{r}\bar{\beta}\frac{\bar{r}z\,(1-\alpha z)}{1-rz}+K^*\bar{p}^2\,\mu_b\theta_w\bar{r}\bar{\beta}\left(\theta_w+\bar{\theta}_w\mu_v\bar{r}\right)(1-\alpha z) \\
&\quad+K^*\bar{p}^2\,\mu_b\theta_w\bar{r}\beta\left(\theta_w+\bar{\theta}_w\mu_v\bar{r}\right)\frac{\bar{\beta}z\,(1-\alpha z)}{1-\beta z}+K^*p^2\theta_w\bar{\beta}\frac{\bar{r}z}{1-rz} \\
&\quad+K^*p\bar{p}\theta_w\,\bar{r}\left(\theta_w+\bar{\theta}_w\mu_v r\right)\frac{\bar{\beta}z}{1-\beta z}.
\end{aligned}\tag{4.9}$$

By utilizing (4.1), (4.2) and (4.4), we can rewrite $Q_d(z)$ as follows:

$$\begin{aligned}
Q_d(z) &= K^*p\bar{p}\,\mu_b\bar{r}\bar{\beta}\left(\theta_w+\bar{\theta}_w\bar{p}\mu_v\bar{r}\right)(1-\alpha z)+K^*p^2\bar{p}\,\mu_b\bar{\theta}_w\bar{r}\bar{\beta}\left[\alpha\frac{\bar{r}}{r}z+\left(1-\frac{\alpha}{r}\right)\frac{z\bar{r}}{1-rz}\right] \\
&\quad+K^*\bar{p}^2\,\mu_b\theta_w\bar{r}\bar{\beta}\left(\theta_w+\bar{\theta}_w\mu_v\bar{r}\right)(1-\alpha z)+K^*\bar{p}^2\,\mu_b\theta_w\bar{r}\beta\left(\theta_w+\bar{\theta}_w\mu_v\bar{r}\right)\left[\alpha\frac{\bar{\beta}}{\beta}z+\left(1-\frac{\alpha}{\beta}\right)\frac{z\bar{\beta}}{1-\beta z}\right] \\
&\quad+K^*p^2\theta_w\bar{\beta}\frac{z\bar{r}}{1-rz}+K^*p\bar{p}\theta_w\,\bar{r}\left(\theta_w+\bar{\theta}_w\mu_v r\right)\frac{z\bar{\beta}}{1-\beta z} \\
&= K^*\left[p\bar{p}\,\mu_b\bar{r}\bar{\beta}\left(\theta_w+\bar{\theta}_w\bar{p}\mu_v\bar{r}\right)+\bar{p}^2\,\mu_b\theta_w\bar{r}\bar{\beta}\left(\theta_w+\bar{\theta}_w\mu_v\bar{r}\right)\right] \\
&\quad+K^*\Big[-\alpha p\bar{p}\,\mu_b\bar{r}\bar{\beta}\left(\theta_w+\bar{\theta}_w\bar{p}\mu_v\bar{r}\right)-\alpha\bar{p}^2\,\mu_b\theta_w\bar{r}\bar{\beta}\left(\theta_w+\bar{\theta}_w\mu_v\bar{r}\right) \\
&\quad+p^2\bar{p}\,\mu_b\bar{\theta}_w\bar{r}\bar{\beta}\alpha\frac{\bar{r}}{r}+\bar{p}^2\,\mu_b\theta_w\bar{r}\beta\left(\theta_w+\bar{\theta}_w\mu_v\bar{r}\right)\alpha\frac{\bar{\beta}}{\beta}\Big]z \\
&\quad+K^*\left[p^2\bar{p}\,\mu_b\bar{\theta}_w\bar{r}\bar{\beta}\left(1-\frac{\alpha}{r}\right)+p^2\theta_w\bar{\beta}\right]\frac{z\bar{r}}{1-rz} \\
&\quad+K^*\left[\bar{p}^2\,\mu_b\theta_w\bar{r}\beta\left(\theta_w+\bar{\theta}_w\mu_v\bar{r}\right)\left(1-\frac{\alpha}{\beta}\right)+\theta_w\,\bar{r}\left(\theta_w+\bar{\theta}_w\mu_v r\right)\right]\frac{z\bar{\beta}}{1-\beta z}.
\end{aligned}\tag{4.10}$$

We now make some simplification for the expression of $Q_d(z)$, first of all, we denote the constant coefficient of right-hand-side of (4.10) by δ_0, that is,

$$\delta_0 = K^* \left[p\bar{p}\,\mu_b\bar{r}\bar{\beta}\left(\theta_w + \bar{\theta}_w\bar{p}\mu_v\bar{r}\right) + \bar{p}^2\,\mu_b\theta_w\bar{r}\bar{\beta}\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right)\right]. \tag{4.11}$$

Secondly, using the relationship $\bar{p}\mu_b\alpha = p\bar{\mu}_b$ and (4.4), the coefficient of z in the right-hand side of (4.10), denoted by δ_1, can be written as

$$\begin{aligned}
\delta_1 &= K^* \Big[-\alpha p\bar{p}\,\mu_b\bar{r}\bar{\beta}\left(\theta_w + \bar{\theta}_w\bar{p}\mu_v\bar{r}\right) - \alpha\bar{p}^2\,\mu_b\theta_w\bar{r}\bar{\beta}\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right) + p^2\bar{p}\,\mu_b\bar{\theta}_w\bar{r}\bar{\beta}\alpha\frac{\bar{r}}{r} \\
&\quad + \bar{p}^2\,\mu_b\theta_w\bar{r}\beta\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right)\alpha\frac{\bar{\beta}}{\beta}\Big] \\
&= K^* \left[\frac{-p^3\bar{\mu}_b\bar{r}^2\bar{\beta}\bar{\theta}_w\bar{\mu}_v}{r} - p\bar{p}\bar{\mu}_b\theta_w\bar{r}\bar{\beta}\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right) + p^2\bar{p}\,\mu_b\bar{\theta}_w\bar{r}\bar{\beta}\alpha\frac{\bar{r}}{r} + p\bar{p}\,\bar{\mu}_b\theta_w\bar{r}\bar{\beta}\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right)\right] \\
&= K^*\frac{p^3\,\bar{\mu}_b\bar{\theta}_w\bar{r}^2\bar{\beta}\mu_v}{r}.
\end{aligned} \tag{4.12}$$

Meanwhile, using the relationship $\bar{p}\mu_b\alpha = p\bar{\mu}_b$ and (4.3), the coefficient of $\frac{z\bar{r}}{1-rz}$ in the right-hand side of (2.14), denoted by δ_2, can be written as

$$\begin{aligned}
\delta_2 &= K^* \left[p^2\bar{p}\mu_b\bar{\theta}_w\bar{\beta}\bar{r}\left(1 - \frac{\alpha}{r}\right) + p^2\theta_w\bar{\beta}\right] \\
&= K^* p^2\bar{\theta}_w\frac{\bar{r}}{r}\bar{\beta}\left(r\bar{p}\mu_b - p\bar{\mu}_b + \frac{\theta_w r}{\bar{\theta}_w\bar{r}}\right) \\
&= \frac{K^* p^2\bar{\theta}_w\bar{\beta}\bar{r}\left(p + r\bar{p}\right)\left(\mu_b - \mu_v\right)}{r}.
\end{aligned} \tag{4.13}$$

Finally, the coefficient of $\frac{z\bar{\beta}}{1-\beta z}$ in the right-hand side of (2.14) is denoted by δ_3, that is

$$\delta_3 = K^* \left[\bar{p}^2\,\mu_b\theta_w\bar{r}\beta\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right)\left(1 - \frac{\alpha}{\beta}\right) + p\bar{p}\theta_w\bar{r}\left(\theta_w + \bar{\theta}_w\mu_v r\right)\right] \quad = K^*\bar{p}\theta_w\bar{r}\left(\theta_w + \bar{\theta}_w\mu_v\bar{r}\right)\left(\bar{p}\mu_b\beta + p\mu_b\right).$$

Then $Q_d(z)$ becomes to

$$Q_d(z) = \delta_0 + \delta_1 z + \delta_2\frac{z\bar{r}}{1 - rz} + \delta_3\frac{z\bar{\beta}}{1 - \beta z}. \tag{4.14}$$

We can verify that $\delta_0 + \delta_1 z + \delta_2 + \delta_3 = 1$, which indicate that $Q_d(z)$ is a PGF. Based on (4.14), we can directly arrive at the results in Theorem 4.1. □

Based on above stochastic decomposition properties, we can easily get the means

$$E(Q_d) = \delta_1 + \frac{\delta_2}{\bar{r}} + \frac{\delta_3}{\bar{\beta}}, E(Q) = \frac{\alpha}{1-\alpha} + \delta_1 + \frac{\delta_2}{\bar{r}} + \frac{\delta_3}{\bar{\beta}}. \tag{4.15}$$

Next, we are interesting in investigating the stochastic decomposition of the sojourn time W of an arbitrary customer in the steady state. To this end, we denote $W(s)$ be the PGF of W and utilize the classical relationship (see Kobayashi and Konheim [8])

$$Q(z) = W(\bar{p} + ps). \tag{4.16}$$

Theorem 4.2. *If the conditions $\alpha < 1$ and $\mu_b > \mu_v$ are fulfilled, the sojourn time W of a arbitrary customer can be decomposed into sum of two independent random variables, that is, $W = W_0 + W_d$, in which W_0 is the sojourn time in the corresponding classic Geom/Geom/1 queue, and W_d is the additional delay which is a mixture of three random variables:*

$$W_d = \tfrac{\delta_1}{p} Y_0 + \tfrac{\delta_2(1-\bar{p}\sigma)}{p} Y_1 + \tfrac{\delta_3(1-\bar{p}\tau)}{p} Y_2,$$

where $Y_0 \equiv 0$, Y_1 and Y_2 follow the geometric distribution with parameter $\bar{\sigma}$ and $\bar{\tau}$, respectively, and$\sigma = \frac{r}{p+r\bar{p}}$, $\tau = \bar{\theta}_v$.

Proof. By (4.8) and (4.14), we can have

$$Q(z) = \frac{\bar{\alpha}}{1-\alpha z}\left(\delta_0 + \delta_1 z + \delta_2 \frac{z\bar{r}}{1-rz} + \delta_3 \frac{z\bar{\beta}}{1-\beta z}\right). \tag{4.17}$$

Taking $s = \bar{p} + pz$ leads to

$$z = \frac{(s-\bar{p})}{p}. \tag{4.18}$$

Thus we have

$$\left.\frac{\bar{r}}{1-rz}\right|_{z=p^{-1}(s-\bar{p})} = \frac{\bar{\sigma}}{1-\sigma s}, \qquad \sigma = \frac{r}{p+r\bar{p}}, \tag{4.19}$$

$$\left.\frac{\bar{\alpha}}{1-\alpha z}\right|_{z=p^{-1}(s-\bar{p})} = \frac{\bar{\lambda}}{1-\lambda s}, \qquad \lambda = \frac{\bar{\mu}_b}{\bar{p}}, \tag{4.20}$$

$$\left.\frac{\bar{\beta}}{1-\beta z}\right|_{z=p^{-1}(s-\bar{p})} = \frac{\bar{\tau}}{1-\tau s}, \qquad \tau = \bar{\theta}_v. \tag{4.21}$$

Substituting (4.19)–(4.21) in (4.17) yields

$$\begin{aligned} W(s) &= \frac{\bar{\lambda}}{1-\lambda s}\left(\delta_0 + \delta_1 \frac{s-\bar{p}}{p} + \delta_2 \frac{s-\bar{p}}{p}\frac{\bar{\sigma}}{1-\sigma s} + \delta_3 \frac{s-\bar{p}}{p}\frac{\bar{\tau}}{1-\tau s}\right) \\ &= \frac{\bar{\lambda}}{1-\lambda s}\left(\delta_0 + \delta_1 \frac{s-\bar{p}}{p} + \delta_2 \frac{s-\bar{p}\sigma s+\bar{p}\sigma s-\bar{p}}{p}\frac{\bar{\sigma}}{1-\sigma s} + \delta_3 \frac{s-\bar{p}\tau s+\bar{p}\tau s-\bar{p}}{p}\frac{\bar{\tau}}{1-\tau s}\right) \\ &= \frac{\bar{\lambda}}{1-\lambda s}\left(\delta_0 - \delta_1 \frac{\bar{p}}{p} - \delta_2 \frac{\bar{p}\bar{\sigma}}{p} - \delta_3 \frac{\bar{p}\bar{\tau}}{p} + \delta_1 \frac{s}{p} + \delta_2 \frac{1-\bar{p}\sigma}{p}\frac{s\bar{\sigma}}{1-\sigma s} + \delta_3 \frac{1-\bar{p}\tau}{p}\frac{s\bar{\tau}}{1-\tau s}\right). \end{aligned} \tag{4.22}$$

We can verify that

$$\delta_0 - \delta_1 \frac{\bar{p}}{p} - \delta_2 \frac{\bar{p}\bar{\sigma}}{p} - \delta_3 \frac{\bar{p}\bar{\tau}}{p} = 0, \tag{4.23}$$

Then, (4.22) can be rewritten as

$$W(s) = W_0\,(s)\,W_d\,(s)\,, \tag{4.24}$$

where

$$W_0\,(s) = \frac{\bar{\lambda}}{1-\lambda s}, \tag{4.25}$$

that is PGF of sojourn time of classical Geom/Geom/1 queue and

$$W_d\,(s) = \delta_1 \frac{s}{p} + \delta_2 \frac{(1-\bar{p}\sigma)}{p}\frac{s\bar{\sigma}}{1-\sigma s} + \delta_3 \frac{(1-\bar{p}\tau)}{p}\frac{s\bar{\tau}}{1-\tau s}. \tag{4.26}$$

From the structure of the $W_d\,(s)$, we can directly arrive at the results in Theorem 4.2. □

Based on Theorem 4.2, it is readily obtained that

$$E(W_d) = \delta_2 \frac{(1-\bar{p}\sigma)}{p}\frac{\sigma}{\bar{\sigma}} + \delta_3 \frac{(1-\bar{p}\tau)}{p}\frac{\tau}{\bar{\tau}}, \tag{4.27}$$

$$E(W) = \frac{1}{\mu_b \bar{\alpha}} + \delta_2 \frac{(1-\bar{p}\sigma)}{p}\frac{\sigma}{\bar{\sigma}} + \delta_3 \frac{(1-\bar{p}\tau)}{p}\frac{\tau}{\bar{\tau}}. \tag{4.28}$$

5. The analysis of regular busy period and busy cycle

In this section, we mainly focus on the analysis of regular busy period and busy cycle. To avoid confusion, the regular busy period B considered here is the continuous duration in which the server works at the service rate of μ_b and a busy cycle C is defined as the period between two consecutive instants at which a regular busy period commences. In our queue model, a regular busy period can start at working vacation completion epoch or a vacation completion epoch. Thus, a busy cycle C can comprise a regular busy period B, the subsequent working vacation V_w and the vacation period V_v (if exists).

Since there is just a working vacation V_w in a busy cycle C, and the expected length of working vacation period $E(V_w) = \theta_w^{-1}$. Denote the expected length of a busy cycle by $E(C)$, and use the limiting theorem of the alternating renewal process and (3.19), we obtain

$$P(W) = \frac{E\,(V_w)}{E\,(C)} = Kp\left(p\bar{\theta}_w + \theta_w + \bar{p}\bar{\theta}_w \mu_v \bar{r}\right), \tag{5.1}$$

which leads to

$$E(C) = \frac{1}{\theta_w} + \frac{\bar{p}\left(\theta_w + \bar{\theta}_w \mu_v \bar{r}\right)}{p\bar{\beta}\left(\theta_w + \bar{p}\bar{\theta}_w \mu_v \bar{r} + p\bar{\theta}_w\right)} + \frac{\bar{p}\left(\theta_w + \bar{\theta}_w \mu_v \bar{r}\right)\bar{r} + p\bar{\beta}}{\bar{p}\mu_b\bar{\alpha}\bar{\beta}\bar{r}\left(\theta_w + \bar{p}\bar{\theta}_w \mu_v \bar{r} + p\bar{\theta}_w\right)}. \tag{5.2}$$

Similarly, by (3.20), we know

$$P(V) = \frac{E(V_v)}{E(C)} = \frac{K\bar{p}\theta_w\left(\theta_w + \bar{\theta}_w \mu_v \bar{r}\right)}{\bar{\beta}}, \tag{5.3}$$

and the expected length of vacation period in a busy cycle C is given by

$$E(V_v) = \frac{\bar{p}\left(\theta_w + \bar{\theta}_w \mu_v \bar{r}\right)}{p\bar{\beta}\left(\theta_w + \bar{p}\bar{\theta}_w \mu_v \bar{r} + p\bar{\theta}_w\right)}. \tag{5.4}$$

By (3.21), we have

$$P(B) = \frac{E(B)}{E(C)} = Kp\left[\frac{p\theta_w}{\bar{p}\mu_b \bar{r}\bar{\alpha}} + \frac{\theta_w\left(\theta_w + \bar{\theta}_w \mu_v \bar{r}\right)}{\mu_b \bar{\alpha}\bar{\beta}}\right]. \tag{5.5}$$

Hence, we get the expected length of regular busy period $E(B)$ in a busy cycle

$$E(B) = \frac{\bar{p}\left(\theta_w + \bar{p}\bar{\theta}_w \mu_v \bar{r}\right)\bar{r} + p\bar{\beta}}{\bar{p}\mu_b \bar{\alpha}\bar{\beta}\bar{r}\left(\theta_w + \bar{p}\bar{\theta}_w \mu_v \bar{r} + p\bar{\theta}_w\right)}. \tag{5.6}$$

Clearly, it is easily seen that

$$E(C) = E(B) + E(V_w) + E(V_v). \tag{5.7}$$

6. Relationship to the continuous-time counterpart

As we all know, one of the advantages of analyzing the discrete-time queues is that we can derive the continuous-time counterpart in a limiting case. Therefore, in this section, we are denoted to find the relationship between our discrete-time system and its continuous counterpart M/M/1/SWV+MV queue that was studied by Ye and Liu [27]. To this end, we first assume that the M/M/1/SWV+MV queue has the following parameter assumption: Customers arrive at the system according to a Poisson process with rate λ . The service times in a regular service period and working vacation are exponential distribution with parameter μ'_b and μ'_v, respectively. The durations of the working vacations and vacations are exponential distributions with parameters θ'_w, θ'_v, respectively. Suppose that time axis is slotted into sufficient small intervals of equal length Δ, then the M/M/1/SWV+MV queue can be approximated as follows.

$$p = \lambda\Delta, \quad \mu_b = \mu'_b\Delta, \quad \mu_v = \mu'_v\Delta, \quad \theta_w = \theta'_w\Delta, \quad \theta_v = \theta'_v\Delta. \tag{6.1}$$

By taking advantages of the Lvesque integral, we can know when $\Delta \to 0$,

$$r \to \frac{1}{2\mu'_v}\left(\lambda + \theta'_w + \mu'_v - \sqrt{\left(\lambda + \theta'_w + \mu'_v\right)^2 - 4\lambda\mu'_v}\right), \tag{6.2}$$

$$\alpha \to \frac{\lambda}{\mu'_b}, \beta \to \frac{\lambda}{\lambda + \theta'_v}. \tag{6.3}$$

Substituting above expression in (3.1), we can know that when $\Delta \to 0$,

$$\begin{cases} \pi_{k0} \to Kr^k, & k \geq 0, \\ \pi_{k1} \to K\frac{\theta_w}{\lambda}\beta^k, & k \geq 0, \\ \pi_{k2} \to K\left(\frac{\theta_w r}{\mu_b(1-r)}\sum\limits_{j=0}^{k-1} r^j\alpha^{k-1-j} + \frac{\theta_w}{\mu_b}\sum\limits_{j=0}^{k-1}\beta^j\alpha^{k-1-j}\right), & k \geq 1, \end{cases} \tag{6.4}$$

where

$$K=(1-r)(1-\beta)(1-\alpha)\left[(1-\beta)(1-\alpha)+\frac{\theta_w'}{\lambda}(1-r)(1-\alpha)+\frac{\theta_w'}{\mu_b'}(1-r)+\frac{\theta_w' r(1-\beta)}{\mu_b'(1-r)}\right]^{-1}. \tag{6.5}$$

We can find that the approximated probabilities obtained by (6.4) are consistent with that in Ye and Liu [27].

7. Special cases

Case 1: If $\theta_v=1$, it indicates that there is no vacation period, then the vacation period reduces to the idle period. So our model becomes the standard Geom/Geom/1 queue with single working vacation. To verify that, we find that when $\theta_v=1$, then $\beta=0$, and steady state probabilities becomes

$$\begin{cases}\pi_{00}=Kp\left(\theta_w+\bar\theta_w\bar p\mu_v\bar r\right), & \\ \pi_{01}=K\bar p\theta_w\left(\theta_w+\bar\theta_w\mu_v\bar r\right), & \\ \pi_{k0}=Kp^2\bar\theta_w\bar r r^{k-1}, & k\geq 1,\\ \pi_{k2}=Kp\left[\frac{p\theta_w}{\bar p\mu_b}\sum\limits_{j=0}^{k-1}r^j\alpha^{k-1-j}+\frac{\theta_w\left(\theta_w+\bar\theta_w\mu_v\bar r\right)}{\mu_b}\alpha^{k-1}\right], & k\geq 1,\end{cases} \tag{7.1}$$

where

$$K=\frac{\bar p\,\mu_b\bar r\bar\alpha}{p\bar p\mu_b\left(\theta_w+\bar\theta_w\bar p\mu_v\bar r\right)\bar r\bar\alpha+p^2\left(\theta_w+\bar p\bar\theta_w\bar r\mu_b\bar\alpha\right)+\bar p\theta_w\bar r\left(\theta_w+\bar\theta_w\mu_v\bar r\right)(p+\bar p\mu_b\bar\alpha)}. \tag{7.2}$$

which is consistent with the result in Li and Tian [10].

Case 2: If $\theta_v=1$ and $\mu_v=0$, then the vacation period reduces to the idle period, further, since the service rate during the working vacation degenerate to zero, the working vacation becomes the classical vacation, thus our model becomes Geom/Geom/1 queue with single vacation, and the steady state probabilities is

$$\begin{cases}\pi_{00}=Kp\left(\theta_w+\bar\theta_w\bar p\mu_v\bar r\right), & \\ \pi_{01}=K\bar p\theta_w\left(\theta_w+\bar\theta_w\mu_v\bar r\right), & \\ \pi_{k0}=Kp^2\bar\theta_w\bar r r^{k-1}, & k\geq 1,\\ \pi_{k2}=Kp\left[\frac{p\theta_w}{\bar p\mu_b}\sum\limits_{j=0}^{k-1}r^j\alpha^{k-1-j}+\frac{\theta_w\left(\theta_w+\bar\theta_w\mu_v\bar r\right)}{\mu_b}\alpha^{k-1}\right], & k\geq 1,\end{cases} \tag{7.3}$$

here,

$$r=\frac{\bar\theta_w p}{1-\bar\theta_w\bar p}, \tag{7.4}$$

and

$$K=\frac{\bar p\,\mu_b\bar r\bar\alpha}{p\bar p\mu_b\left(\theta_w+\bar\theta_w\bar p\mu_v\bar r\right)\bar r\bar\alpha+p^2\left(\theta_w+\bar p\bar\theta_w\bar r\mu_b\bar\alpha\right)+\bar p\theta_w\bar r\left(\theta_w+\bar\theta_w\mu_v\bar r\right)(p+\bar p\mu_b\bar\alpha)}.$$

Case 3: If $\mu_v=0$ and $\theta_w=\theta_v=\theta\,(0<\theta<1)$, then there is no service during the working vacation period and the working vacation and vacation follow the identical distribution, therefore, our model degenerates to Geom/Geom/1 queue with multiple vacation. Since $\mu_v=0$ and $\theta_w=\theta_v=\theta$, we can find that

$r=\beta=\bar{\theta}p\left(1-\bar{\theta}\bar{p}\right)^{-1}$, and the stationary probabilities is

$$\begin{cases}\pi_{00}=Kp\theta, & \\ \pi_{k0}=Kp^2\bar{\theta}\bar{r}r^{k-1}, & k\geq 1,\\ \pi_{k1}=K\bar{p}\theta^2 r^k, & k\geq 0,\\ \pi_{k2}=Kp\left(\frac{p\theta}{\bar{p}\mu_b}+\frac{\theta^2}{\mu_b}\right)\sum_{j=0}^{k-1}r^j\alpha^{k-1-j}, & k\geq 1,\end{cases} \tag{7.5}$$

where

$$K=\frac{\bar{p}\,\mu_b\bar{\alpha}\,\bar{r}^2}{p\bar{p}\mu_b\theta\bar{\alpha}\bar{r}^2+p^2\bar{r}\left(\theta+\bar{p}\bar{\theta}\bar{r}\mu_b\bar{\alpha}\right)+\bar{p}\theta\bar{r}\left(\theta+\bar{\theta}\mu_v\bar{r}\right)\left(p+\bar{p}\mu_b\bar{\alpha}\right)}. \tag{7.6}$$

Note here that states $(k,0)$ and $(k,1)$, for $k\geq 0$, both represent the system is in vacation period. In fact, $(k,0)$ means that the system stays in the first vacation after the regular busy period. So the probability that there are k customers in the system and the server stays in vacation period is $\pi_{k0}+\pi_{k1}$. If we let π_{kv} represent the probability that there are k customers in the system given that the server is in vacation period and π_{kb} represent the probability that there are k customers in the system given that the server is in normal service period. Then we can obtain the stationary probabilities as follows.

$$\begin{cases}\pi_{0v}=K\left(p\theta+\bar{p}\theta^2\right), & \\ \pi_{kv}=K\left(p^2\bar{\theta}\bar{r}+\bar{p}\theta^2 r\right)r^{k-1}, & k\geq 1,\\ \pi_{kb}=Kp\left(\frac{p\theta}{\bar{p}\mu_b}+\frac{\theta^2}{\mu_b}\right)\sum_{j=0}^{k-1}r^j\alpha^{k-1-j}, & k\geq 1,\end{cases} \tag{7.7}$$

where K is given by (7.6).

Case 4: If $\mu_v=0$ and $\theta_w=\theta_v=1$, then there is no working vacation or vacation in the system. Then our queue model becomes the classical Geom/Geom/1 queue without vacation. To verify that, we can easily observe that $r=\beta=0$ when $\mu_v=0$ and $\theta_w=\theta_v=1$, if we let $\pi_k\,(k=0,1,2,\ldots)$ be probability that there are k customers in the system, then stationary probabilities reduces to

$$\begin{cases}\pi_0=\bar{p}\left(1-\alpha\right), & \\ \pi_k=\frac{p(1-\alpha)}{\mu_b}\alpha^{k-1}, & k\geq 1.\end{cases} \tag{7.8}$$

Clearly, (7.8) is the stationary distribution of system in the classical Geom/Geom/1 queue.

From Cases 1–4, we can know that the queueing system we study is an extension of classical vacation queueing system and working vacation queueing system.

8. Numerical examples

In this section, based on the results we obtained previously, we perform several concrete numerical examples to illustrate the sensitivity of performance measures to changes in the parameters of systems.

Firstly, under the condition that the parameters $p=0.6$, $\theta_w=0.3$, $\theta_v=0.4$ are fixed, in Figures 2 and 3, we compare the behaviors of mean system size $E(Q)$ and mean waiting time $E(W)$ with the change of μ_v for different values of μ_b. Obviously, we can observe that μ_v and μ_b have the similar effect on the $E(Q)$ and $E(W)$, that is, both $E(Q)$ and $E(W)$ decrease along with the increase of μ_v or μ_b. That is because that the increase of μ_v or μ_b indicates the increase of average service rate, then the decrease of $E(Q)$ and $E(W)$ can be expected. Secondly, in Figures 4–6, given $p=0.6$, $\mu_b=1.6$, $\mu_v=0.3$, we compare the effects of θ_w and θ_v

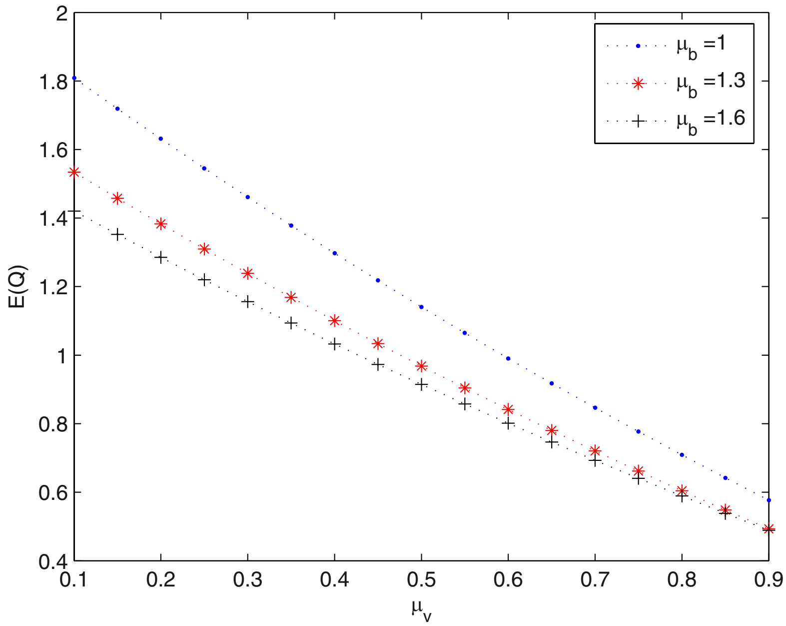

FIGURE 2. $E(Q)$ *versus* μ_v. (Color online.)

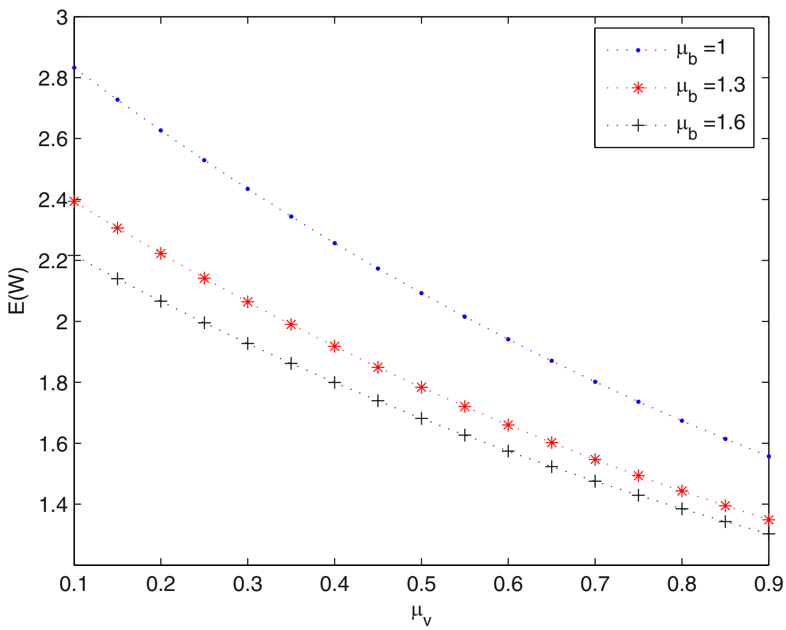

FIGURE 3. $E(W)$ *versus* μ_v. (Color online.)

on the probability that the server is in working vacation $P(W)$, the probability that the server is in vacation $P(V)$ and the probability that the server is in regular busy period $P(C)$, respectively. We first observe that an increase in θ_w results in the decrease of $P(W)$, which agrees with the intuitive expectations, actually, note that the expected length of working vacation period $E(V_w) = \theta_w^{-1}$, then as θ_w increases, the length of working

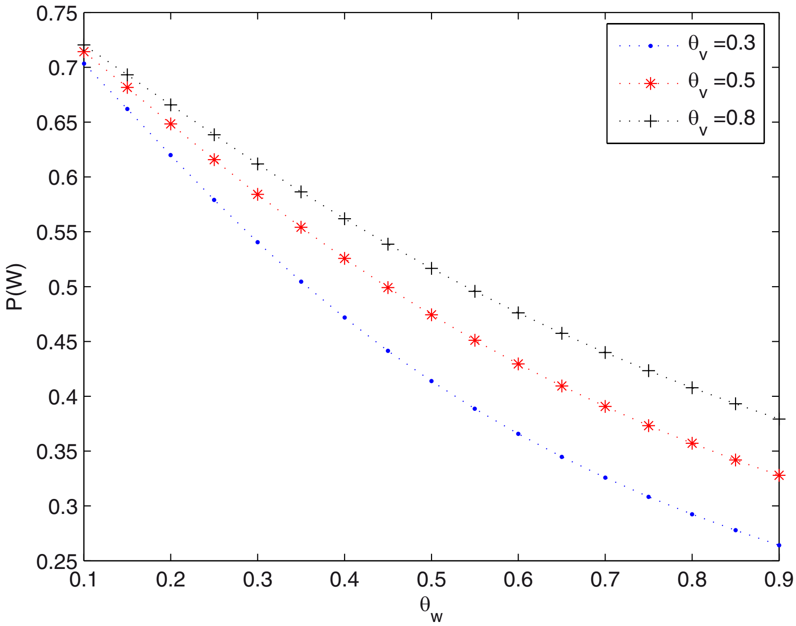

FIGURE 4. $P(W)$ *versus* θ_w. (Color online.)

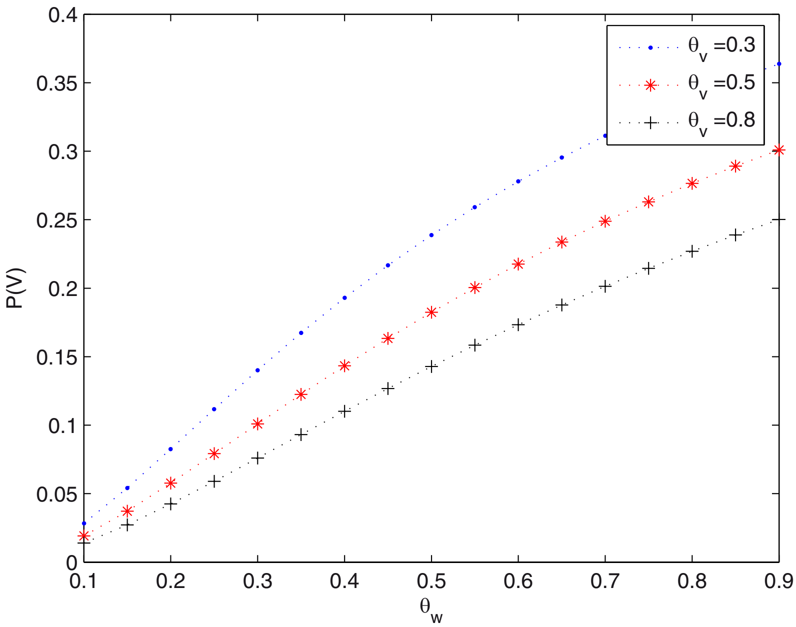

FIGURE 5. $P(V)$ *versus* θ_w. (Color online.)

vacation decrease, then the decrease of $P(W)$ can be expected. Contrary to Figure 4, from Figure 5 and 6, we observe that both $P(V)$ and $P(W)$ increase as θ_w increases. Further, we also find that the larger θ_v is, the larger $P(W)$ becomes, however, the larger θ_v leads to the smaller $P(V)$ and $P(B)$, respectively. Thirdly, in Figures 7–9, given that $p = 0.4$, $\mu_b = 0.9$, $\mu_v = 0.5$, we plot the trends of expected length of vacation period

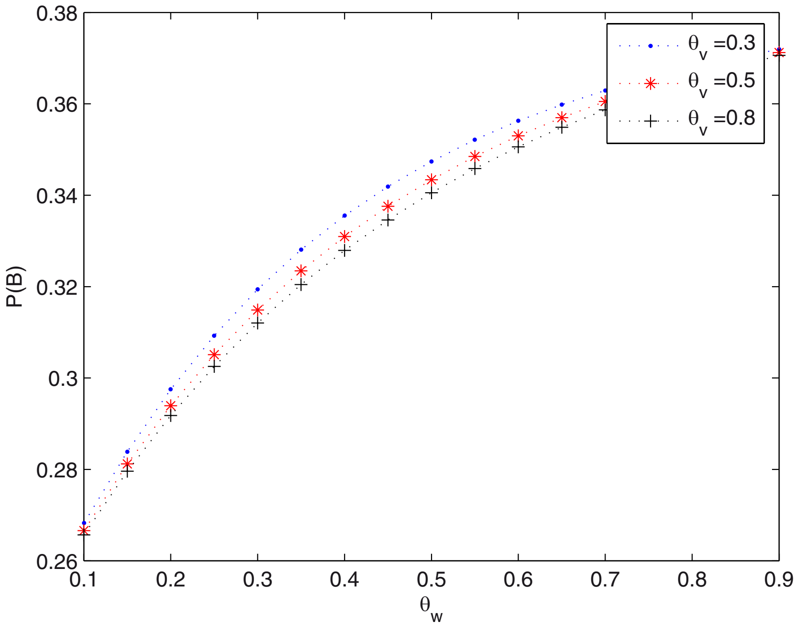

FIGURE 6. $E\,(V_v)$ *versus* θ_w. (Color online.)

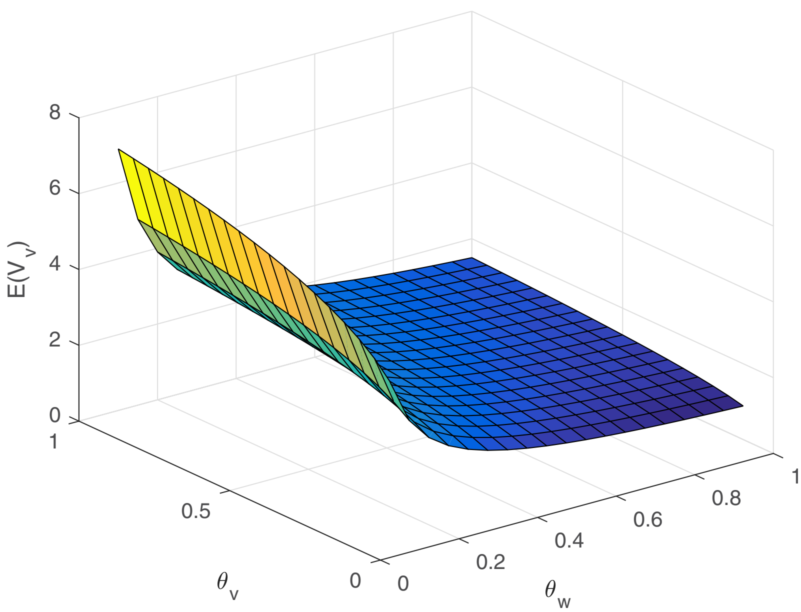

FIGURE 7. $P(B)$ *versus* θ_w. (Color online.)

$E(V_v)$, the expected length of regular busy period $E(B)$ and the expected length of busy cycle $E(C)$ with the change of θ_w and θ_v, respectively. Evidently, from Figure 7 and 8, it is observed that $E(V_v)$ and $E(B)$ have the similar change trend with the change of θ_w and θ_v, that is, both decreases as θ_w increases and increases as θ_v increases. In contrary, from Figure 9, we find that $E(C)$ increases as θ_w increases and decreases as θ_v increases. Further, we find that, $E(V_v)$ and $E(B)$ are more sensitive to θ_w than θ_v, however, $E(C)$ is more sensitive to θ_v

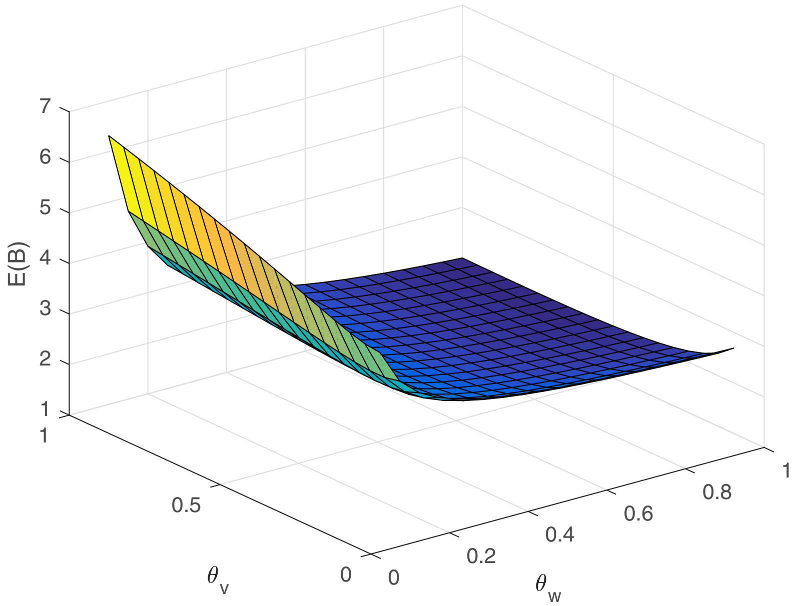

FIGURE 8. $E(B)$ *versus* θ_w and θ_v. (Color online.)

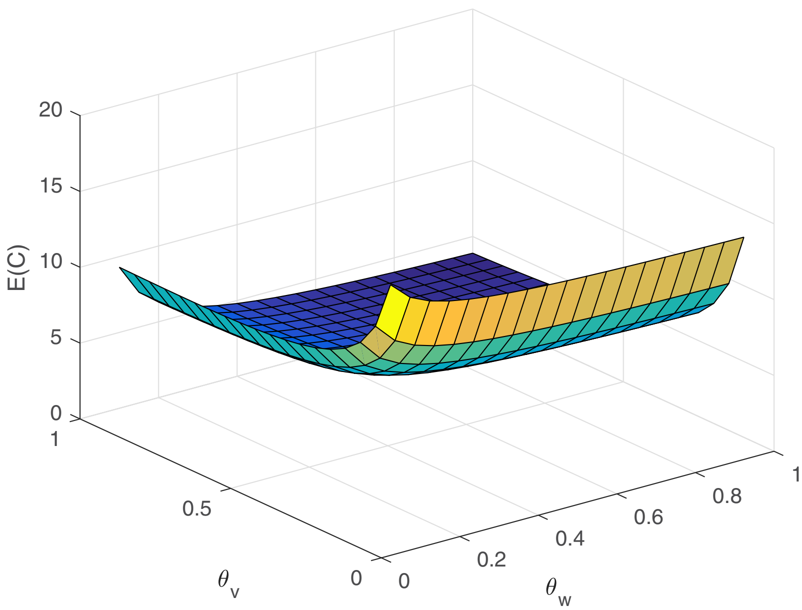

FIGURE 9. $E(C)$ *versus* θ_w and θ_v. (Color online.)

than θ_w. At last, Given that $p = 0.4$, $\mu_v = 0.5$, $\theta_w = 0.3$ and $\theta_v = 0.6$, Figure 10 presents a comparison of mean system size in three different vacation queue, that is, the Geom/Geom/1 queue with multiple vacations, the Geom/Geom/1 queue with single working vacation and the Geom/Geom/1 queue with single working vacation and multiple vacations that we study in this paper. An intuitive result we can obtain is that the system size

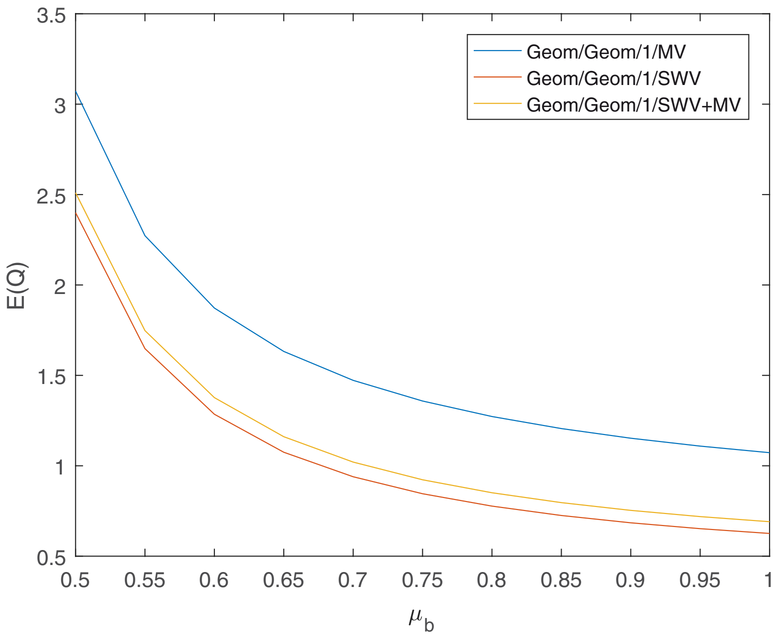

FIGURE 10. $E(Q)$*versus* μ_b. (Color online.)

of Geom/Geom/1 queue with single working vacation and multiple vacations is between the system size of Geom/Geom/1 queue with multiple vacations and the Geom/Geom/1 queue with single working vacation.

9. Conclusion

In this paper, we generalize the work in Ye and Liu [27] to the discrete time Geom/Geom/1 queue. We have done works in several aspects:

(1) Using matrix geometric solution method, we derive the explicit close-form expression for the stationary system size.
(2) The stochastic decomposition structures of the stationary system size and the sojourn time of an arbitrary customer.
(3) The regular busy period and busy cycle are analyzed by limiting theorem of the alternating renewal process.
(4) The relationship between the Geom/Geom/1/SWV+MV queue and the continuous time counterpart M/M/1/SWV+MV queue is analyzed.
(5) Some special cases are presented.
(6) The effects of various parameters on the performance measures are illustrated numerically and graphically.

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