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Minimizing the passengers' traveling time in the stop location problem

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In this paper we consider the location of stops along the edges of an already existing public transportation network. The positive effect of new stops is given by the better access of the passengers to the public transport network, while the passengers' traveling time increases due to the additional stopping activities of the trains, which is a negative effect for the passengers. The problem has been treated in the literature where the most common model is to cover all demand points with a minimal number of new stops. In this paper, we follow this line and seek for a set of new stops covering all demand points but instead of minimizing the number of new stops we minimize the additional passengers' traveling time due to the new stops. For computing this additional traveling time we do not only take the stopping times of the vehicles but also acceleration and deceleration of the vehicles into account. We show that the problem is NP-hard, but we are able to derive a finite candidate set and two tractable IP formulations. For linear networks we show that the problem is polynomially solvable. We also discuss the differences to the common models from literature showing that minimizing the number of new stops does not necessarily lead to a solution with minimal additional traveling times for the passengers. We finally provide a case study showing that our new model decreases the traveling times for the passengers while still achieving the minimal number of new stops.

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1. Introduction

The acceptance of public transportation depends on various components, among them convenience, punctuality, and reliability. In this paper, we address the question of convenience for the passengers. In particular, we investigate the problem of establishing additional stops (or stations) in a given public transportation network. The goal is, on the one hand, to improve the accessibility to the transportation network, but on the other hand not to increase the traveling time of passengers too much.

Due to their great potential for improving public transportation systems, several versions of the *stop location problem* (SL) (also called *station location problem*) have been considered in the literature. In order to find 'good' locations for new stops, several objective functions are possible. One of the most frequently discussed goals is to minimize the number of stops such that each demand point is *covered*, that is, it is within a tolerable distance from at least one stop. The maximal distance r that a passenger is willing to tolerate is called *covering radius*. For bus stops a covering radius of 400 m is common. In rail transportation, the covering radius is larger, often 2 km are used.

In the literature, *discrete* SLs have been introduced in Gleason (1975) and considered in Murray *et al* (1998); Murray (2001); Laporte *et al* (2002); Murray (2003); Wu and Murray (2005), see also references therein. In these papers, a finite set of potential stops is given. The goal is to choose a minimal number of stops from this set such that each demand point is covered. In contrast to this discrete setting, Hamacher *et al* (2001) are the first to allow a *continuous* set of possible locations for the stops, for instance, all points on the current bus routes or railway tracks. This is motivated by a real-world application within a project with the largest German rail company (Deutsche Bahn). The authors consider the trade-off between the positive and negative effects of stops. The negative effect of longer passengers' traveling times due to additional stops is compared with the positive effect of shorter access times. Based on this application, the continuous SL has also been treated theoretically in the literature. The goal is to locate a minimal number of stops along railway lines to cover a set of given demand points in the plane. Variants of this problem have been studied in Kranakis *et al* (2003); Schöbel (2005); Schöbel (2006); Schöbel *et al* (2009). The problem has been solved for the case of two intersecting lines, see Mammana *et al* (2004). Algorithmic approaches for solving the underlying covering problem have been studied in Mecke and Wagner (2004); Ruf and Schöbel (2004). Complexity and approximation issues have been presented in Mecke *et al* (2006).

A different objective function is to minimize the sum of distances from the passengers to the public transportation system, that is, the sum of the distances between the demand points and their closest stops, see Murray and Wu (2003);

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Poetranto *et al* (2009). Recently, covering a set of OD-pairs with a given number of stops has been studied, see, for example, Körner *et al* (2014) and references therein.

Research done so far mostly deals with minimizing the number of new stops. This can be seen as a rough approximation of the passengers' traveling time, namely adding a fixed time penalty (often assumed to be 2 min) for each stop, see Schöbel *et al* (2009). Since trains have a long acceleration and deceleration phase such an approximation is unrealistic in practice, in particular in metropolitan and regional transportation networks. In this paper we do not minimize the number of new stops but consider SLs minimizing the passengers' traveling time while taking the realistic vehicles' driving times including acceleration and deceleration into account. Note that we implicitly assume that there is no interference with other transportation modes that limits the speed of the vehicles, but that the trains can accelerate to their planned cruising speed without any restriction.

The remainder of the paper is structured as follows. We introduce the realistic driving time function and the SL based on this function in Section 2. In Section 3 we compare our new model with the model from Schöbel *et al* (2009). We derive a finite dominating set in Section 4. Approaches for solving our new model are given in Section 5. Its complexity is analyzed and two different integer programming formulations are given. Moreover, we show that the problem along one single line is polynomially solvable. In Section 6 we present a numerical study with numerical results on the quality and solvability of our new model.

2. Models for continuous stop location

In this section we first repeat the continuous stop location model common in the literature: Cover all demand points with a minimal number of new stops. We then introduce our new objective function in which we minimize the traveling time of the passengers instead of the number of new stops.

Basic definitions: Locating stops along the edges of a network

The continuous SL has been treated in the literature. We repeat its basic definitions.

Let $G=(V, E)$ be a given railway network with existing stops V and direct connections E between these stops. For each edge $e=(i, j)\in E$ an edge length $d_e \geqslant 0$ and a number of passengers $w_e \geqslant 0$ traveling along edge e is given.

A *point* in G is given as $s=(e, x)\in e$, that is, it is defined by the edge $e=(i, j)$ and its distance $d(i, s)=x$ to the start vertex of e. The distance from x to the end vertex of e hence is $d(s, j)=d_e-x$, $0\leqslant x\leqslant d_e$. Note that $i=(e, 0)$ and $j=(e, d_e)$. The set of points of G is denoted as $\mathcal{S}$. The set of points between two points $s_1=(e, x_1)$ and $s_2=(e, x_2)$ on the same edge e is denoted as $[s_1, s_2]=\{(e, x)\colon x_1\leqslant x\leqslant x_2\}$.

A new stop s in the network may be any point $s=(e,x)\in\mathcal{S}$. In the continuous SL we want to identify a set of points in $\mathcal{S}$ as new stops, see Figure 1 as an example of a network with six existing and four new stops.

The constraints: Covering all demand points

As in the common models in the literature we require that all (potential) passengers live close enough to at least one stop. To this end, we assume that a finite set $\mathcal{P}\subseteq\mathbb{R}^2$ of demand points is given, that the railway network $G=(V, E)$ is embedded in the plane and that the access times from the demand points to the railway network can be measured by a distance function d: $\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ being derived from a norm $\|\cdot\|$, that is,

$$d(x,y)=\|y-x\|,$$

where for $e=(i, j)$ we have $d_e=d(i, j)$.

Definition 2.1 *Let a covering radius $r\geqslant 0$ be given. For a set $S\subseteq\mathcal{S}$ and a set of demand points $\mathcal{P}$ we define: A demand point $p\in\mathcal{P}$ is covered by S if*

$$d(p,s)\leqslant r \text{ for some } s\in S.$$

The cover of S is given as

$$\text{cover}(S)=\{p\in\mathcal{P}: d(p,s)\leqslant r \text{ for some } s\in S\}.$$

If $\text{cover}(S)=\mathcal{P}$ we say that $\mathcal{P}$ *is covered* by S.

The goal is to find a set of stops $S\subseteq\mathcal{S}$ such that all demand points are covered, that is, with $\text{cover}(S\cup V)=\mathcal{P}$.

Since we assumed that all elements of V are stops, demand points p, which are covered by vertices $v\in V$, need not be considered. We hence may assume that

$$\mathcal{P}\subseteq\mathbb{R}^2\setminus\text{cover}(V). \tag{1}$$

The SL in the literature: Minimizing the number of new stops

In the literature, the SL has been considered: One looks for a set of stops $S^*\subseteq\mathcal{S}$ with minimal size covering all demand points:

(SL) Let $G=(V, E)$ be a graph and a finite set of points $\mathcal{P}\subseteq\mathbb{R}^2$ be given. Find a subset $S^*\subseteq\mathcal{S}$, such that $\text{cover}(S^*)=\mathcal{P}$ and $|S^*|$ is minimized.

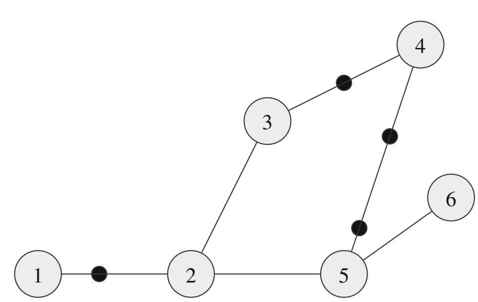

Figure 1 Locating 4 new stops on a network $G=(V, E)$.

Our new objective function: Minimizing the passengers' traveling times

In the SLs considered in the literature so far, the traveling times for passengers due to new stops can be estimated by adding a penalty time_{pen} for every stop to be located. This is an exact estimate if the distance between two stops is larger than the distance needed for acceleration and deceleration, and if time_{pen} gives the complete loss of vehicles' driving time resulting from the additional stop, that is, the loss resulting from waiting t min at the stop to allow passengers to board and alight and the loss resulting from decelerating and accelerating instead of going by full speed. As an example, time_{pen} is estimated as 2min for German regional trains (see Hamacher *et al*, 2001). However, since trains accelerate slowly, this estimate is not realistic if the distance between two stops is rather short.

In order to consider acceleration and deceleration in SLs we first introduce a function describing the driving time of a train between two consecutive stops. This function depends on the distance d between those two consecutive stops. It is a simple consequence from Newton's laws of motion and has been used, for example, in Kolesar *et al* (1975); Vuchic (1981); Drezner *et al* (2009).

Lemma 2.2 *Let a maximum cruising speed $v_0>0$, an acceleration of $a_0>0$ and a deceleration of $b_0>0$ of a vehicle be given. Then the vehicles' driving time for a distance d between two consecutive stops is given as*

$$T(d)=\begin{cases}\sqrt{\dfrac{2(a_0+b_0)}{a_0b_0}d} & \text{if } d\leqslant d^{\max}_{v_0,a_0,b_0}\\[2mm] \dfrac{d}{v_0}+\dfrac{v_0}{2a_0}+\dfrac{v_0}{2b_0} & \text{if } d\geqslant d^{\max}_{v_0,a_0,b_0}\end{cases}$$

$$\text{where } d^{\max}_{v_0,a_0,b_0}=\frac{v_0^2}{2a_0}+\frac{v_0^2}{2b_0}$$

In Kolesar *et al* (1975) the driving time function for the case $a_0=b_0$ is used and its practical relevance is analyzed for fire engines in New York City.

Note that $d^{\max}_{v_0,a_0,b_0}$ is the point where the driving time function turns from a square root behavior to a linear behavior. The shape and exact values of the function can be easily calculated; its main properties can be verified straightforwardly:

Lemma 2.3 *$T(d)$ is continuous, differentiable, concave, subadditive and monotonically increasing. Furthermore, for any d we have*

$$\sqrt{\frac{(2(a_0+b_0)}{a_0b_0)d}}\leqslant\frac{d}{v_0}+\frac{v_0}{2a_0}+\frac{v_0}{2b_0}.$$

A proof of these properties can be found, for example, in Drezner *et al* (2009).

We can now derive the passengers' traveling times, which depend on the new stop set $S\subseteq\mathcal{S}$ to be located. In order to do so, we need to refine the railway network according to the new stop set S. This is formalized next.

Given a finite set $S\subseteq\mathcal{S}$ of points of G, every set

$$S_e=S\cap e=\{s_1,\ \dots\ ,s_p\}\subseteq e$$

of points on $e=(i,j)$ can be naturally ordered along the edge e such that $d(s_1,i)\leqslant\dots\leqslant d(s_p,i)$, that is, 'from left to right'. Let $\leqslant_e$ denote this ordering. Adding the $p=|S_e|$ points of S_e as new stops to the edge $e=(i,j)$ splits the edge e into $|S_e|+1$ subedges

$$E'_e(S_e)=\{(i,s_1),(s_1,s_2),\ \dots\ ,(s_p,j)\}.$$

Defining

$$E'(S)=\bigcup_{e\in E}E'_e(S_e) \tag{2}$$

we receive a new network $(V\cup S, E'(S))$ which is a subdivision of the given network $G=(V,E)$ (see Figure 2). Note that every edge $e'\in E'(S)$ can be represented as $e'=((e,x_i),(e,x_j))$, that is, it belongs to one unique edge $e\in E$. For an edge $e'=((e,x_i),(e,x_j))\in E'(S)$ we define:

- its length as $d_{e'}=|x_j-x_i|$,
- the number of passengers traveling along edge e' as $w_{e'}=w_e$.

Given a set S of points on the graph G, we can finally define the *passengers' traveling time function* as

$$\begin{aligned}g(S)&:=\sum_{e\in E}w_e\left(|S_e|t+\sum_{e'\in E'_e(S_e)}T(d_{e'})\right)\\&=\sum_{e\in E}w_e|S_e|t+\sum_{e'\in E'(S)}w_{e'}T(d_{e'}).\end{aligned} \tag{3}$$

Note that this function depends on the parameters $v_0>0$, $a_0>0$ and $b_0>0$, which model the vehicles' behavior in the vehicles' driving time function T.

In this formula, we sum over the traveling times on all edges. The first term, $|S_e|\,t$, refers to the time that is given by stopping t min at every additional stop on edge $e\in E$. Note that the stopping times at the already existing stops $i\in V$ are independent of the location of the new stops and hence neglected in the formula. The second term, $\sum_{e'\in E'_e(S_e)}T(d_{e'})$, accounts for the vehicles' driving time on edge e including acceleration and deceleration at every new stop in S_e. Both values are multiplied by the number of passengers w_e on edge e to obtain the

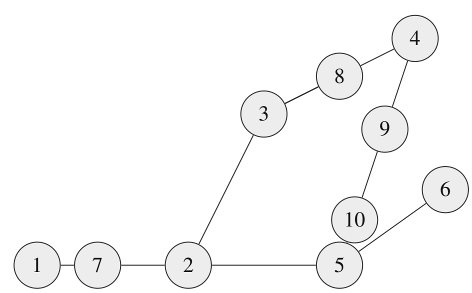

Figure 2 The new network $(V\cup S, E'(S))$ with 4 new stops.

passengers' traveling time along this edge. The *additional* traveling time for the passengers is given as

$$f(S) = g(S) - g(\varnothing).$$

In the case that no new stop is built along an edge $e \in E$ we have that $S_e = \varnothing$ and $E'_e(S_e) = \{e\}$ and hence obtain

$$|S_e|t + \sum_{e' \in E'_e(S_e)} T(d_{e'}) = T(d_e).$$

The SL minimizing passengers' traveling time

We summarize our new model: cover all demand points with a set of stops S such that the passengers' traveling time function $g(S)$ is minimal:

(**SL***) Let $G = (V, E)$ be a graph with edge weights w_e for all $e \in E$, $\mathcal{P} \subseteq \mathbb{R}^2$ be a finite set of points, and $t \geqslant 0$. Moreover, let $v_0 > 0$, $a_0 > 0$ and $b_0 > 0$ be the parameters for the vehicles' driving time. Find a subset $S^* \subseteq \mathcal{S}$, such that $\text{cover}(S^*) = \mathcal{P}$ and $g(S^*)$ is minimized.

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3. Comparing (SL) and (SL*)

(SL) minimizes the number of new stops while (SL*) considers the traveling times for the passengers. We start by showing that these may have different optimal solutions: Our example shows a case in which the passengers' traveling time can be reduced by building two stops instead of only one.

Example 3.1 *In Figure 3 two demand points p_1 and p_2 have to be covered by stops on $e = (v_1, v_2)$. As indicated in the figure, $d(p_1, v_1)$ and $d(p_2, v_2)$ is only a little bit larger than the covering radius r, and both demand points can be covered from the midpoint of the edge e. For this example let $w_e = 1$.*

In order to minimize the number of stops it is sufficient to build only one stop, namely the midpoint of e, s_2. It means, $S = \{s_2\}$ is an optimal solution to (SL). We compare the solution provided by S with that of $\tilde{S} = \{s_1, s_3\}$, where s_1 and s_3 are the leftmost and rightmost points on e from which we can cover p_1 and p_2, respectively. Given some $\varepsilon < d^{max}_{v_0,a_0,b_0}$ the points s_1, s_2, and s_3 can be constructed such that $d(v_1, s_1) = d(v_2, s_3) = \varepsilon$ and such that $d(v_1, s_2), d(s_2, v_2) \geqslant d^{\max}_{v_0,a_0,b_0}$.

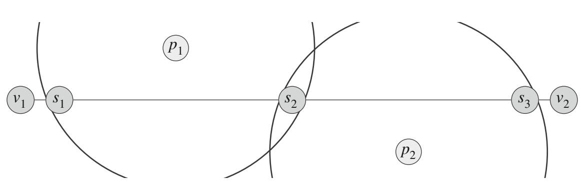

Figure 3 Example in which an optimal solution to (SL*) has more stops than an optimal solution to (SL).

Then the passengers' traveling times can be computed as

$$g(S) = t + T(d(v_1, s_2)) + T(d(s_2, v_2)) = t + \tfrac{d_e}{v_0} + \tfrac{v_0}{a_0} + \tfrac{v_0}{b_0}$$

$$g(\tilde{S}) = 2t + T(d(v_1, s_1)) + T(d(s_1, s_3)) + T(d(s_3, v_2))$$

$$= 2t + 2\sqrt{\tfrac{2(a_0 + b_0)}{a_0 b_0}\varepsilon} + \tfrac{d_e - 2\varepsilon}{v_0} + \tfrac{v_0}{2a_0} + \tfrac{v_0}{2b_0}$$

and by letting ε tend to 0, we see that the relation between a_0, b_0, v_0 and t determines whether $g(\tilde{S}) < g(S)$. We obtain that $g(\tilde{S}) < g(S)$ if $(v_0/2a_0) + (v_0/2b_0) > t$, that is, if breaking and accelerating takes longer than halting.

Even more, examples of the same pattern can be constructed, which show that the number of stops in an optimal solution to (SL) can differ by an arbitrarily large number from the number of stops in an optimal solution to (SL).*

It is trivial that in (SL) more stops increase the value of the objective function. The previous example shows that this is not the case for (SL*), the objective function g is better for the two stops s_1 and s_3 as for the single stop s_2. However, adding an *additional* stop to a set of stops S always worsens the objective function even in (SL*):

Lemma 3.2 *Let $S^1 \subseteq S^2$ be two sets of points. Then $g(S^1) \leqslant g(S^2)$.*

Proof We show that adding a single new stop increases the traveling time, that is, that $g(S^1) \leqslant g(S^2)$ for $S^2 = S^1 \cup \{s\}$. If $s \in S^1$ there is nothing to show. Hence, let $s \notin S^1$, and let $s = (e, x)$ for some edge $e = (i, j) \in E$. Clearly, $|S^1_e| = |S^1 \cap e| \leqslant |S^2 \cap e| = |S^2_e|$. Moreover, let $s_a = (e, x_a) \leqslant_e s \leqslant_e s_b = (e, x_b)$ for $s_a, s_b \in S^1_e \cup \{i, j\}$ being the neighbors of s in S^1_e. Then we have

$$T(x_b - x_a) \leqslant T(x_b - x) + T(x - x_a)$$

due to the subadditivity of T (see Lemma 2.3), and since $w_e \geqslant 0$ the result follows. □

In the next lemma we show that the two models are equivalent if we require a minimal distance of $d^{\max}_{v_0,a_0,b_0}$ between any two new stops and between any new stop and an existing stop $v \in V$. It means that the distance between any two stops is long enough such that a train always reaches the maximal cruising speed.

Lemma 3.3 *(SL*) with $w_e = 1$ for all $e \in E$ and (SL) are equivalent if both models have the additional constraints*

$$d(s, s') \geqslant d^{\max}_{v_0,a_0,b_0} \quad \text{for all } s, s' \in S$$

$$d(s, v) \geqslant d^{\max}_{v_0,a_0,b_0} \quad \text{for all } s \in S, v \in V.$$

Proof We compute the objective function g using the assumptions of the lemma and obtain

$$\begin{aligned} g(S) &= t\sum_{e\in E}|S_e| + \sum_{e\in E}\sum_{e'\in E'_e(S_e)} T(d_{e'}) \\ &= t|S| + \sum_{e\in E}\sum_{e'\in E'_e(S_e)} \tfrac{d_{e'}}{v_0}+\tfrac{v_0}{2a_0}+\tfrac{v_0}{2b_0} \\ &= t|S| + \sum_{e\in E}\tfrac{d_e}{v_0} + \sum_{e\in E}(|S_e|+1)\left(\tfrac{v_0}{2a_0}+\tfrac{v_0}{2b_0}\right) \\ &= \left(t+\tfrac{v_0}{2a_0}+\tfrac{v_0}{2b_0}\right)|S| + constant, \end{aligned}$$

that is, minimizing $g(S)$ is equivalent to minimizing the number $|S|$ of new stops. □

In Theorem 4.7 in Section 4.2 we will provide a special case in which the additional constraints of the previous lemma are always satisfied, that is, in which (SL) and (SL*) are equivalent.

Note that a numerical comparison between the solutions provided by (SL) and those provided by (SL*) will be presented within a case study in Section 6.

4. Feasibility and a finite candidate set for (SL*)

In this section we analyze the problem (SL*). We discuss its feasibility and provide a finite candidate set for (SL*). Let us start with the feasibility of (SL*).

4.1. Feasibility

(SL*) need not be feasible, but if it is it admits a finite solution whose objective value is bounded.

Lemma 4.1

- *(SL*) has a feasible solution if and only if $cover(\mathcal{S}) = \mathcal{P}$.*
- *If (SL*) has a feasible solution, then it also has a finite solution and*

$$g(S^*) \leqslant \max_{e\in E} w_e\left(|\mathcal{P}|t + (|E|+|\mathcal{P}|)\left(\max_{e\in E}\frac{d_e}{v_0}+\frac{v_0}{2a_0}+\frac{v_0}{2b_0}\right)\right)$$

Proof The first part of the lemma is obvious. For the second part, let (SL*) be feasible. Then there exists a point $s_p \in \mathcal{S}$ such that $d(p, s_p) \leqslant r$ for every demand point $p \in \mathcal{P}$. Choose $S := \{s_p : p \in \mathcal{P}\}$ as a feasible solution, ie $|S| \leqslant |\mathcal{P}|$. Each stop $s \in S$ adds at most one new edge to $E'(S)$, hence $|E'(S)| \leqslant |E| + |\mathcal{P}|$. Let $e' = (i, j) \in E'(S)$ be a new edge with $i = (\bar{e}, x_i), j = (\bar{e}, x_j)$ for some $\bar{e} \in E$. Then we estimate $d_{e'} \leqslant d_{\bar{e}} \leqslant \max_{e\in E} d_e$, and since T is monotone we obtain

$$T(d_{e'}) \leqslant \max_{e\in E} T(d_e) \leqslant \max_{e\in E}\frac{d_e}{v_0}+\frac{v_0}{2a_0}+\frac{v_0}{2b_0},$$

where the second inequality holds due to Lemma 2.3. Hence,

$$\begin{aligned} g(S^*) \leqslant g(S) &\leqslant \max_{e\in E} w_e \sum_{e\in E}\left(|S_e|t + \sum_{e'\in E'(S_e)} T(d_{e'})\right) \\ &\leqslant \max_{e\in E} w_e\left(|\mathcal{P}|t + (|E|+|\mathcal{P}|)\left(\max_{e\in E}\frac{d_e}{v_0}+\frac{v_0}{2a_0}+\frac{v_0}{2b_0}\right)\right) \end{aligned}$$

and the result follows. □

4.2. A finite dominating set for (SL)*

In the following we show that (SL*) can be reduced to a discrete problem by identifying a finite dominating set, that is, a finite set of candidates $\mathcal{S}_{cand} \subseteq \mathcal{S}$, for which we know that it contains an optimal solution S^* if the problem is feasible. Such a finite dominating set will enable us to derive an IP formulation in Section 5.2. It turns out that we can use the same finite dominating set which has been used as candidate set for solving (SL) (see Schöbel *et al*, 2009). Throughout this section, let us assume that (SL*) is feasible.

For an edge $e = (i, j) \in E$ we define

$$T^e(p) = \{s \in e : d(p, s) \leqslant r\}$$

as the set of all points on the edge $e \subseteq \mathcal{S}$ that can be used to cover demand point p.

Since $T^e(p) = e \cap \{x \in \mathbb{R}^2 : d(p, x) \leqslant r\}$ is the intersection of two convex sets, and contained in e, it turns out to be a line segment itself. This observation is due to Schöbel *et al* (2009).

Lemma 4.2 (*Schöbel et al, 2009) For each demand point $p \in \mathbb{R}^2$ the set $T^e(p)$ is either empty or an interval contained in edge e.*

Let f_p^e, l_p^e denote the endpoints of the interval $T^e(p)$ We write $[f_p^e, l_p^e] = T^e(p)$. For each edge $e = (i, j)$ we define

$$\mathcal{S}^e_{cand} := \bigcup_{p\in\mathcal{P}}\{f_p^e, l_p^e\} \cup \{i, j\},$$

which can be ordered along the edge e with respect to $\leqslant_e$. Let the resulting set be given as $\mathcal{S}^e_{cand} = \{s_0, s_1, \ldots, s_{N_e}\}$. In the following we show that

$$\mathcal{S}_{cand} = \bigcup_{e\in E}\mathcal{S}^e_{cand}$$

is a finite dominating set for (SL*).

From Schöbel *et al* (2009) we know that moving a point $s \in \mathcal{S}$ until it reaches an element of $\mathcal{S}_{cand}$ does not decrease cover($\{s\}$):

Lemma 4.3 (Schöbel *et al*, 2009) *Let $s \in e$ for an edge e of E, and let $s_j, s_{j+1} \in \mathcal{S}_{cand}$ be two consecutive elements of the finite dominating set with $s_j <_e s <_e s_{j+1}$. Then*

$$cover(\{s\}) \subseteq cover(\{s_j\}) \cap cover(\{s_{j+1}\}),$$

in particular, the cover of s does not decrease when moving s to s_j or to s_{j+1}.

Now we are able to prove that $\mathcal{S}_{cand} = \bigcup_{e\in E} \mathcal{S}^e_{cand}$ is, indeed, a finite dominating set for (SL*).

Theorem 4.4 *Either (SL*) is infeasible, or there exists an optimal solution $S^* \subseteq \mathcal{S}_{cand} \setminus V$.*

Proof Let $S^* \subseteq \bigcup_{e\in E, p\in\mathcal{P}} T^e(p)$ be a feasible solution. Assume that $S^* \subseteq \mathcal{S}_{cand}$ does not hold. Our goal is to replace each $\tilde{s} \in S^* \setminus \mathcal{S}_{cand}$ by a point in $\mathcal{S}_{cand}$ without loosing feasibility and without worsening the objective function. To this end, take some $\tilde{s} \in S^* \setminus \mathcal{S}_{cand}$. If $\tilde{s} \in V$ there is nothing to show. Otherwise $\tilde{s} = (e, x) \in E$ with $0 < x < d_e$. Now find the following points on edge e:

- $s_j = (e, x_j), s_{j+1} = (e, x_{j+1}) \in \mathcal{S}_{cand}$ with $s_j <_e \tilde{s} <_e s_{j+1}$ for two consecutive elements of $\mathcal{S}^e_{cand}$ (which exist on e), and
- moreover, find a point $s_{left} \in e$ such that $s_{left} \in (S^* \cup V) \cap e$, $s_{left} <_e \tilde{s}$ and no other point $s' \in (S^* \cup V) \cap e$ exists with $s_{left} <_e s' <_e \tilde{s}$.

  Analogously, find $s_{right} \in e$ such that $s_{right} \in (S^* \cup V) \cap e$, $\tilde{s} <_e \tilde{s}_{right}$ and no other point $s' \in (S^* \cup V) \cap e$ exists with $\tilde{s} <_e s' <_e s_{right}$.

We investigate the objective function if we move $\tilde{s}$. For all $s = (e, x)$ with $s_{left} <_e s <_e s_{right}$ the objective function $h(x) := g(S \setminus \tilde{s} \cup \{(e, x)\})$ is given as

$$h(x) = constant + w_e\big(T(x - x_{left}) + T(x_{right} - x)\big)$$

where the constant part is independent of the choice of x in $s = (e, x)$, since x only influences the subedges (s_{left}, s) and (s, s_{right}) and leaves all other parts of the objective function untouched.
We hence have the following two properties:

(P1) Since the composition of a concave and a linear function is concave, and the sum of two concave functions is also concave, we obtain that $h(x)$ is concave in x on the segment between s_{left} and s_{right}.

(P2) From Lemma 4.3 we furthermore know that $\text{cover}(\{s\}) \supseteq \text{cover}(\{\tilde{s}\})$ for all $s = (e, x)$ between s_j and s_{j+1}, so we can shift $\tilde{s}$ between s_i and s_{i+1} without loosing feasibility.
Hence, due to the concavity (see (P1)) of h, the minimization problem

$$\min\big\{h(x) = w_e\big(T(x - x_{left}) + T(x_{right} - x)\big) : \max\{x_{left}, x_j\} \leqslant x \leqslant \min\{x_{right}, x_{j+1}\}\big\}$$

has an optimal solution

$$x^* \in \big\{\max\{x_{left}, x_j\}, \min\{x_{right}, x_{j+1}\}\big\}$$

which is feasible for (SL*), see (P2).

- In case that $x^* = x_j$ or $x^* = x_{j+1}$ we may replace $\tilde{s}$ by $s = (e, x^*) \in \mathcal{S}_{cand}$ and hence obtain a feasible solution with the same objective value.
- In case that $x^* = x_{left}$ or $x^* = x_{right}$ we may delete $\tilde{s}$ since the new solution is still feasible and improve the same objective value.

In both cases, we have reduced the number of points in $S^* \setminus \mathcal{S}_{cand}$. Proceeding with the remaining points of S^*, which do not belong to $\mathcal{S}_{cand}$, finally yields a feasible solution that is completely contained in $\mathcal{S}_{cand}$ and has the same (or a better) objective value as S^*.
In the proof we added the existing stops V to $\mathcal{S}_{cand}$. This has only been done for technical reasons. Since we know that no optimal solution contains a stop from V (since we assumed in (1) that $\mathcal{P} \subseteq \mathbb{R}^2 \setminus \text{cover}(V)$) we can also delete V from $\mathcal{S}_{cand}$. This finally finishes the proof. □

Remark 4.5 Note that the proof shows more than stated in the theorem, namely: For every feasible solution $S' \subseteq \mathcal{S}$ to (SL*) there exists a solution $S \subseteq \mathcal{S}_{cand}$, which is also feasible for (SL*) and satisfies $g(S) \leqslant g(S')$.

We hence have shown that (SL*) is equivalent to the following *discrete* problem:

(SL*-discrete) Let $G = (V, E)$ be a graph with edge weights w_e for all $e \in E$, $\mathcal{P} \subseteq \mathbb{R}^2$ be a finite set of points, and $t \geqslant 0$. Moreover, let $v_0 > 0$, $a_0 > 0$ and $b_0 > 0$ be the parameters for the vehicles' driving time. Find a subset $S^* \subseteq \mathcal{S}_{cand} \setminus V$, such that $\text{cover}(S^*) = \mathcal{P}$ and $g(S^*)$ is minimized.

We remark that there can be at most two candidates for each demand point on every edge, hence the total number of candidates $|\mathcal{S}_{cand}|$ is bounded by

$$|\mathcal{S}_{cand}| \leqslant 2|E||\mathcal{P}| + |V|.$$

Given the finite candidate set $\mathcal{S}_{cand}$ we define the candidate edge set

$$\mathcal{E}_{cand} = \{c = (s, s') : s, s' \in \mathcal{S}_{cand} \text{ and } s, s' \in e \text{ for some edge } e \in E \text{ and } s \leqslant_e s'\}, \tag{4}$$

and d_c as the length of $c \in \mathcal{E}_{cand}$. Note that

$$|\mathcal{E}_{cand}| = \sum_{e\in E} \binom{|\mathcal{S}^e_{cand}|}{2}.$$

From Theorem 4.4 we obtain the following corollary, which addresses the edges of the refined railway network $E'(S^*)$ with new stops S^*, see (2).

Corollary 4.6 *If (SL*) is feasible there exists an optimal solution S^* with $S^* \subseteq \mathcal{S}_{cand} \setminus V$ and $E'(S^*) \subseteq \mathcal{E}_{cand}$.*

As another consequence of Theorem 4.4 we will present a case in which (SL) and (SL*) are equivalent (as already promised in Section 3).

Theorem 4.7 *Let $d_c \geqslant d^{\max}_{v_0,a_0,b_0}$ for all $c \in \mathcal{E}_{cand}$, and let $w_e = 1$ for all $e \in E$. Then (SL) and (SL*) are equivalent.*

Proof Let S^* be an optimal solution to (SL*). Due to Theorem 4.4 we can assume without loss of generality that $S^* \subseteq \mathcal{S}_{cand} \setminus V$. From the assumption that $d_c \geqslant d^{\max}_{v_0,a_0,b_0}$ for all $c \in \mathcal{E}_{cand}$ we know that

$$d(s, s') \geqslant d^{\max}_{v_0,a_0,b_0} \quad \text{for all } s, s' \in S^*$$

$$d(s, v) \geqslant d^{\max}_{v_0,a_0,b_0} \quad \text{for all } s \in S^* \text{ and for all } v \in V.$$

Since $w_e = 1$ for all $e \in E$ we can use Lemma 3.3 and conclude that (SL) and (SL*) are equivalent. □

Theorem 4.4 has several other important consequences that will be exploited next: First, using the finite candidate set we can clarify the complexity status of (SL*) (see Section 5.1). Second, we can use the finite candidate set to derive IP formulations in Section 5.2. Finally, it also helps to solve the problem in polynomial time in a special case in Section 5.3.

5. Solving (SL*)

Using the finite candidate set $\mathcal{S}_{cand} \setminus V$ we can theoretically enumerate all potential solutions $S \subseteq \mathcal{S}_{cand} \setminus V$. However, this leads to a number of $\mathcal{O}(2^{2|E|\,|\mathcal{P}|})$ different solutions to be tested. In the following we hence discuss the complexity, IP formulations and a polynomially solvable case of (SL*-discrete).

5.1. Complexity of (SL)*

Having the finite candidate set we can now clarify the complexity status of (SL*).

Theorem 5.1 *(SL*) is NP-hard.*

Proof To see that (SL*) is NP-hard, we reduce the decision version of (SL), which is NP-complete according to Schöbel *et al* (2009), to the decision version of (SL*). To this end, we need another result of Schöbel *et al* (2009) (similar to Remark 4.5), namely that for every feasible solution $S' \subseteq \mathcal{S}$ for (SL) there exists a solution $S \subseteq \mathcal{S}_{cand}$, which is also feasible for (SL) and satisfies $|S| \leqslant |S'|$. Let an instance of (SL) be given. We define the following instance of (SL*):

- We leave the network G and the demand points $p \in \mathcal{P}$ as they are.
- We choose $w_e = 1$ for all $e \in E$.
- For choosing the parameters v_0, a_0 and b_0 we proceed as follows. We use the finite candidate set $\mathcal{S}_{cand}$ for (SL) and determine $\underline{m} := \min\{d(s, s') : s, s' \in \mathcal{S}_{cand}, \text{ and } s, s' \in e \text{ for some } e \in E\}$ as the closest distance between two candidate locations on the same edge. We then choose v_0, a_0, b_0 such that $d^{\max}_{v_0,a_0,b_0} \leqslant \underline{m}$. (This is possible since $d^{max}_{v_0,a_0,b_0} \to 0$ if $a_0, b_0 \to \infty$).

We now claim that a solution to (SL) with $|S| \leqslant K$ exists if and only if a solution S^* to (SL*) exists with

$$g(S^*) \leqslant Kt + \sum_{e \in E} \frac{d_e}{v_0} + (|E| + K)\left(\frac{v_0}{2a_0} + \frac{v_0}{2b_0}\right).$$

To see this, we reformulate the objective g as follows: If $d_{e'} \geqslant d^{\max}_{a_0,b_0,v_0}$ for all $e' \in E'(S)$ and $w_e = 1$ for all $e \in E$ the objective function of (SL*) becomes

$$\begin{aligned} g(S) &= |S|t + \sum_{e' \in E'(S)} \left(\frac{d_{e'}}{v_0} + \frac{v_0}{2a_0} + \frac{v_0}{2b_0}\right) \\ &= |S|t + \frac{1}{v_0} \sum_{e' \in E'(S)} d_{e'} + |E'(S)| \left(\frac{v_0}{2a_0} + \frac{v_0}{2b_0}\right) \\ &= |S|t + \frac{1}{v_0} \sum_{e \in E} d_e + |E'(S)| \left(\frac{v_0}{2a_0} + \frac{v_0}{2b_0}\right) \qquad (5) \end{aligned}$$

"$\Rightarrow$" Let S be a solution to (SL) with $|S| \leqslant K$. Then there exists a solution S^* to (SL) with $S^* \subseteq \mathcal{S}_{cand}$ according to the result of (Schöbel *et al*, 2009) mentioned above. S^* is the required solution to (SL*): Clearly, S^* is feasible for (SL*). Furthermore, $d_{e'} \geqslant \underline{m} \geqslant d^{\max}_{a_0,b_0,v_0}$ for all $e' \in E'(S^*)$ (since $S^* \subseteq \mathcal{S}_{cand}$), and $|E'(S^*)| \leqslant |E| + K$. Hence

$$g(S^*) \leqslant |K|t + \sum_{e \in E} \frac{d_e}{v_0} + (|E| + K)\left(\frac{v_0}{2a_0} + \frac{v_0}{2b_0}\right).$$

"$\Leftarrow$" Let S^* be a solution to (SL*) with

$$g(S^*) \leqslant |K|t + \sum_{e \in E} \frac{d_e}{v_0} + (|E| + K)\left(\frac{v_0}{2a_0} + \frac{v_0}{2b_0}\right).$$

From Remark 4.5 we know that there exists $S \subseteq \mathcal{S}_{cand}$ with $g(S^*) = g(S)$. We show that S is the required solution: Clearly, S is feasible. Since $d_{e'} \geqslant \underline{m} \geqslant d^{\max}_{a_0,b_0,v_0}$ for all $e' \in E'(S^*)$ and $|E'(S)| = |E| + |S|$ we have

$$\begin{aligned} |K|t + \sum_{e \in E} \frac{d_e}{v_0} + (|E| + K)\left(\frac{v_0}{2a_0} + \frac{v_0}{2b_0}\right) &\geqslant g(S^*) \\ = g(S) = |S|t + \sum_{e \in E} \frac{d_e}{v_0} + (|E| + |S|)\left(\frac{v_0}{2a_0} + \frac{v_0}{2b_0}\right) \end{aligned}$$

which is equivalent to $|S| \leqslant K$. □

As an immediate consequence we obtain:

Corollary 5.2 *(SL*-discrete) is NP-hard.*

Proof This follows directly from Theorems 5.1 and 4.4. □

5.2. Integer programming formulations for (SL)*

In this section we develop two different IP-formulations for (SL*) and discuss their performance. Both IP-formulations are based on the candidate set developed in Section 4.2. We first define the *covering matrix* $A^{cov} = (a_{p,s})_{p\in\mathcal{P}, s\in\mathcal{S}_{cand}}$ by

$$a_{p,s} = \begin{cases} 1 & \text{if } p \in \text{cover}(\{s\}) \\ 0 & \text{otherwise.} \end{cases}$$

Then (SL) can be formulated as a covering problem (see, eg, Murray, 2001), namely

$$(\text{IP}-\text{SL}) \qquad \min \sum_{s\in\mathcal{S}_{cand}} x_s \tag{6}$$

$$s.t. \quad \sum_{s\in\mathcal{S}_{cand}} a_{p,s}x_s \geqslant 1 \qquad \forall p \in \mathcal{P} \tag{7}$$

$$x \in \{0,1\}^{|\mathcal{S}_{cand}|}$$

where the variables x_s have the following meaning:

$$x_s = \begin{cases} 1 & \text{if stop } s \in \mathcal{S}_{cand} \text{ is built.} \\ 0 & \text{otherwise.} \end{cases}$$

We now develop two IP formulations for (SL*).

To this end, let $\geqslant_e$ denote the canonical ordering along edge $e\in E$, $\mathcal{E}_{cand}$ be defined as in (4), and let the length d_c and $T(d_c)$ be pre-calculated for all $c \in \mathcal{E}_{cand}$. Our first—straightforward—IP formulation of the discrete version of (SL*) is given by a classical covering formulation in which we additionally require that all edges between consecutive stops must be built.

$$(\text{IP1}-\text{SL*}) \qquad \min \sum_{e\in E} w_e \left(t \sum_{\substack{s \in \mathcal{S}_{cand} \\ s\in e}} x_s + \sum_{\substack{c \in \mathcal{E}_{cand} \\ c \subset e}} T(d_c)y_c \right) \tag{8}$$

$$s.t. \quad \sum_{s\in\mathcal{S}_{cand}} a_{ps}x_s \geqslant 1 \qquad \forall p \in \mathcal{P} \tag{9}$$

$$x_{s_i} + x_{s_j} - \sum_{\substack{s\in[s_i,s_j]\cap\mathcal{S}_{cand} \\ s\notin\{s_i,s_j\}}} x_s \leqslant y_c + 1$$

$$\forall c = (s_i, s_j) \in \mathcal{E}_{cand} \tag{10}$$

$$x_s = 1 \qquad \forall s \in V \tag{11}$$

$$x \in \{0,1\}^{|\mathcal{S}_{cand}|} \tag{12}$$

$$y \in \{0,1\}^{|\mathcal{E}_{cand}|} \tag{13}$$

The variables x_s have the same meaning as before while the variables y_c for $c \in \mathcal{E}_{cand}$ have the following interpretation:

$$y_c = y_{s_i,s_j} = \begin{cases} 1 & \text{if edge } c = (s_i, s_j) \in \mathcal{E}_{cand} \text{ is built.} \\ 0 & \text{otherwise.} \end{cases}$$

Constraint (9) ensures that every demand point is covered as in (7). Constraints of type (10) ensure that a candidate edge is considered in the objective function if and only if it is built, that is, if and only if its two endpoints are stops and no candidate between the two endpoints is also a stop. Recall that all $v\in V$ are considered as stops. This is ensured by constraint (11). Finally, the objective function (8) gives the passengers' traveling time:

Lemma 5.3 *(IP1-SL*) is a correct IP formulation for (SL*).*

Proof It is obvious that a solution is only feasible if all demand points are covered. This is ensured by constraint (9) of the IP formulation. Furthermore we need to show that in any optimal solution a candidate edge c is built if and only if its two endpoints are chosen as stops and there is no other stop on c, that is, that for $c = (s_i, s_j) \in \mathcal{E}_{cand}$ we have:

$$\begin{matrix} x_{s_i} = 1 \\ x_{s_j} = 1 \\ x_{s_k} = 0 \quad \forall i<k<j \end{matrix} \quad \Leftrightarrow \quad y_c = 1$$

"$\Rightarrow$" Suppose $x_{s_i} = 1$, $x_{s_j} = 1$ and $x_s = 0$ for all $s \in [s_i, s_j] \cap \mathcal{S}_{cand} \setminus \{s_i, s_j\}$. Then $2 = x_{s_i} + x_{s_j} - \sum_{\substack{s\in[s_i,s_j]\cap\mathcal{S}_{cand} \\ s\notin\{s_i,s_j\}}} x_s \leqslant y_c + 1$ (since (10) is satisfied), that is, we conclude that $y_c = 1$.
"$\Leftarrow$" Suppose we do not have $x_{s_i} = 1$, $x_{s_j} = 1$ and $x_{s_k} = 0$ for all $s_k \in [s_i, s_j]$ and $s_k \notin \{s_i, s_j\}$. Then $x_{s_i} + x_{s_j} - \sum_{\substack{s\in[s_i,s_j]\cap\mathcal{S}_{cand} \\ s\notin\{s_i,s_j\}}} x_s \leqslant 1$, that is, $y_c = 0$ satisfies (10), hence, due to optimality $y_c \neq 1$. □

The second IP-formulation is based on an interpretation of the problem as a flow problem. It involves less constraints. In our numerical results we show that it is clearly superior to (IP1-SL*). The definition of the variables x and y of the IP-formulation remains the same as in (IP1-SL*). The IP-formulation (IP2-SL*) is given as

$$(\text{IP2}-\text{SL*})$$

$$\min \sum_{e\in E} w_e \left(t \sum_{\substack{s \in \mathcal{S}_{cand} \\ s\in e}} x_s + \sum_{\substack{c \in \mathcal{E}_{cand} \\ c \subset e}} T(d_c)y_c \right) \tag{14}$$

$$s.t. \quad \sum_{s\in\mathcal{S}_{cand}} a_{p,s}x_s \geqslant 1 \qquad \forall p \in \mathcal{P} \tag{15}$$

$$\sum_{\substack{s<_e s_j \\ s \in \mathcal{S}^e_{cand}}} y_{s,s_j} = x_{s_j} \qquad \forall e \in E, \qquad \forall s_j \in \mathcal{S}^e_{cand} \tag{16}$$

$$\sum_{\substack{s_i<_e s \\ s \in \mathcal{S}^e_{cand}}} y_{s_i,s} = x_{s_i} \qquad \forall e \in E, \qquad \forall s_i \in \mathcal{S}^e_{cand} \tag{17}$$

$$x_s = 1 \qquad \forall s \in V \tag{18}$$

$$x \in \{0, 1\}^{|\mathcal{S}_{cand}|} \tag{19}$$

$$y \in \{0, 1\}^{|\mathcal{E}_{cand}|} \tag{20}$$

Constraints (15) define the covering condition (see (7) in (IP-SL)). The flow understanding of the problem is expressed in constraints (16)–(17), which ensure that exactly one candidate edge is built on either side of a stop s with $x_s = 1$, and no candidate edge is built, which ends at a stop s with $x_s = 0$. Finally, as in (IP1-SL*), constraint (18) ensures that all $v \in V$ are considered as stops.

Lemma 5.4 *(IP2-SL*) is a correct IP-formulation for (SL*).*

Proof Let (x, y) be a feasible solution to (IP2-SL*). Assuming x to be fixed we may rewrite constraints (16) and (17) for every edge $e \in E$ to

$$\sum_{\substack{s' <_e s \\ s' \in \mathcal{S}^e_{cand}}} y_{s', s} = \sum_{\substack{s <_e s' \\ s' \in \mathcal{S}^e_{cand}}} y_{s, s'} \text{ for all } s \text{ with } x_s = 1, \text{and} \tag{21}$$

$$\sum_{\substack{u <_e s' \\ s' \in \mathcal{S}^e_{cand}}} y_{u, s'} = 1 \tag{22}$$

$$\sum_{\substack{s' <_e v \\ s' \in \mathcal{S}^e_{cand}}} y_{s', v} = 1, \tag{23}$$

that is, the y-variables describe a path passing exactly through the set of stops s with $x_s = 1$. We now show that a candidate edge $c = (s_i, s_j)$ is built if and only if its endpoints s_i, s_j are built and no stop in between s_i and s_j is built.

"$\Leftarrow$": If s_i and s_j are built and no stop between s_i and s_j are built then

$$\sum_{\substack{s_i <_e s' \\ s' \in \mathcal{S}^e_{cand}}} y_{s_i, s'} = 1$$

hence there exists a unique s with $y_{s_i, s} = 1$. If $s_i <_e s' <_e s_j$ then from (16) it follows $x_{s'} = 1$ which is a contradiction to the assumption that no stop between s_i and s_j is built. If $s_j <_e s'$ then from (21) it follows that there exists a path from s_i to v. But since $x_{s_j} = 1$ and (21) there also exists a path (independent to the $s_i - v$-path due to (21)) from s_j to v. This is a contradiction to (23). Hence, $s' = s_j$ and $c = (s_i, s_j)$ is built.

"$\Rightarrow$": We now assume a candidate edge $c = (s_i, s_j)$ is built and show that, hence, the stops s_i and s_j are built and no stop between s_i and s_j is built. Since c is built, from (16) we know that s_i is built and from (17) we know that s_j is built. From (21) it follows that s_j and v are connected by a path. Assume there was any $\tilde{s}$ such that $s_i <_e \tilde{s} <_e s_j$ and $x_{\tilde{s}} = 1$. From (21) we then know that there is also a path from $\tilde{s}$ to v (independent to the $s_i - v$-path due to (21)). Hence, we have a contradiction to (23) as the $s_i - v$-path and the $\tilde{s} - v$-path can not exist at the same time.

As before, (15) ensures that all demand points are covered. □

Note that any feasible solution to (IP2-SL*) consists of a set of paths in the network $(\mathcal{S}_{cand}, \mathcal{E}_{cand})$, namely exactly one path from u to v for any edge $e = (u, v)$ in the original network $G = (V, E)$. Hence, (IP2-SL*) is similar to the *shortest covering path problem* as defined in Current *et al* (1994). The differences are that the we seek for a set of shortest paths—one on each edge—such that the covering constraints are satisfied while in Current *et al* (1994) one single shortest path is sought. Another difference is that the demand points to be covered are contained in the plane and are not nodes of the network (as in Current *et al*, 1994); that is, in our version we combine the discrete nature of choosing a potential solution with the geometric aspect of covering.

We now compare the sizes of the two different IP-formulations. The number of variables is the same for both formulations, namely $|\mathcal{S}_{cand}| + |\mathcal{E}_{cand}|$, but the number of constraints varies quite substantially. In the first formulation (IP1-SL*) we have $|\mathcal{P}| + |\mathcal{E}_{cand}|$ constraints, which is of order $\mathcal{O}(|\mathcal{S}_{cand}|^2)$ while in (IP2-SL*) we have $|\mathcal{P}| + 2|\mathcal{S}_{cand}| + 2|V|$ constraints, that is, of linear order $\mathcal{O}(|\mathcal{S}_{cand}|)$ only.

5.3. The special case (Line-SL): Covering all demand points from a line*

In this section we consider a special case in which (SL*) is polynomially solvable. Namely, we consider the problem (SL*) on a line, that is, $V = \{u, v\}$ and $E = \{e\}$. To refer to the line structure of the underlying network G we call this problem (Line-SL*).

(Line-SL*) Let $G = (\{u, v\}, \{e\})$ be a (single-edge) graph, $\mathcal{P} \subseteq \mathbb{R}^2$ be a finite set of points and $t \geqslant 0$. Moreover, let $v_0 > 0$, $a_0 > 0$ and $b_0 > 0$ be the parameters for the vehicles' driving time. Find a subset $S^* \in \mathcal{S}$, such that cover $(S^*) = \mathcal{P}$ and $g(S^*)$ is minimized.

Note that G only consists of one edge and hence there is only one weight w_e appearing in the objective function which in consequence has no effect on the solution and hence can be neglected.

As before, we will use the finite dominating set $\mathcal{S}_{cand}$ instead of the set of all points $\mathcal{S}$. Since only one edge is considered we can assume $\mathcal{S}_{cand}$ to be ordered with respect to $\leqslant_e$. This order has a nice property (see Hamacher *et al*, 2001; Schöbel, 2006), namely, in the covering matrix A^{cov} the ones appear consecutively in every row. This property is called *consecutive-ones property* (C1P). It ensures that (SL) can be solved in polynomial time by linear programming. In Hamacher *et al* (2001) and Schöbel (2006) a more efficient shortest path algorithm on a special graph is designed for solving (SL) on a line. In the following we transfer these results to our problem (Line-SL*), that is, to the case of minimizing the passengers' traveling times.

Let us define

$$l_p := \min\{j : s_j \in \mathcal{S}_{cand}, a_{p,s_j} = 1\} \text{ for } p \in \mathcal{P},$$

$$r_p := \max\{j : s_j \in \mathcal{S}_{cand}, a_{p,s_j} = 1\} \text{ for } p \in \mathcal{P}.$$

Recall that a matrix A^{cov} with (C1P) is called strictly monotone if $l_{p_1} < \cdots < l_{p_m}$ and $r_{p_1} < \cdots < r_{p_m}$ hold.

Lemma 5.5 (Schöbel, 2006) *Let A^{cov} define the covering matrix for an instance of (Line-SL*), then there exists an equivalent (Line-SL*) problem with covering matrix $\overline{A}^{cov}$, such that $\overline{A}^{cov}$ is a strictly monotone matrix.*

Thus, we can assume A^{cov} to be given as a strictly monotone matrix. Implicitly, this gives an ordering of $\mathcal{P}$ according to the ordering of the rows in the strictly monotone matrix. From the ordering also the minimum and the maximum are well defined. We will assume that $\mathcal{P} = \{p_1, \ldots, p_m\}$ is given in this ordering. Let us now define

$$\bar{l}_s := \min\{i : p_i \in \mathcal{P}, a_{p_i,s} = 1\} \text{ for } s \in \mathcal{S}_{cand},$$

$$\bar{r}_s := \max\{i : p_i \in \mathcal{P}, a_{p_i,s} = 1\} \text{ for } s \in \mathcal{S}_{cand}.$$

Now, we are able to construct a graph $\mathcal{N} = (\mathcal{V}, \mathcal{A})$ on which we will later solve a shortest path problem. Let

$$\mathcal{V} := \mathcal{S}_{cand}.$$

Note that $u, v \in \mathcal{S}_{cand}$. Furthermore, let

$$\mathcal{A} := \{(s_j, s_k) : s_j < s_k \text{ and } \bar{l}_{s_k} \leqslant \bar{r}_{s_j} + 1\}$$
$$\cup \{(u, s_j) : \bar{l}_{s_j} = 1\} \cup \{(j, v) : \bar{r}_{s_j} = m\}.$$

Finally, we define costs for each edge $a = (s_j, s_k) \in \mathcal{A}$ as follows

$$cost_{s_j,s_k} = \begin{cases} T\big(d(s_j, s_k)\big) & \text{if } s_k = v \\ t + T\big(d(s_j, s_k)\big) & \text{else.} \end{cases}$$

$\mathcal{N}$ is a directed cycle-free graph. Let P be any $u-v-$path in $\mathcal{N}$. Then $P \subseteq \mathcal{S}_{cand}$ is uniquely described by its nodes, and its costs are given by $\text{cost}(P) = \sum_{a \in P} \text{cost}_a$. Based on this notation the following theorem holds.

Theorem 5.6 (Schöbel, 2006) *Let $P \subseteq \mathcal{V}$. Then cover $(P) = \mathcal{P}$ if and only if P is an $u-v-$path in $\mathcal{N}$.*

With these results we can now provide a polynomial algorithm for solving (Line-SL*).

Theorem 5.7 *(Line-SL*) is solvable in polynomial time.*

Proof The problem (Line-SL*) is solved as a shortest $u-v-$path problem on the respective graph $\mathcal{N}$. Let $P = \{u, s_{j_1}, \ldots, s_{j_p}, v\}$ be an optimal solution of the shortest $u-v-$path problem in $\mathcal{N}$. From Theorem 5.6 we obtain that P fulfills the covering constraint and thus is a feasible solution to (Line-SL*). We have to show that the solution is also optimal for (Line-SL*). Suppose no, that is, there exists a solution $P^* = \{u, s^*_{j_1}, \ldots, s^*_{j_q}, v\}$ for (Line-SL*) with

$$g(P) > g(P^*).$$

Again using Theorem 5.6 we may interpret this solution as $u-v-$path in $\mathcal{N}$. This means

$$\begin{aligned}
Cost(P) &= cost_{u,s_{j_1}} + \sum_{i=1}^{p-1} cost_{s_{j_i},s_{j_{i+1}}} + cost_{s_{j_p},v} \\
&= pt + T(d(u, s_{j_1})) + \sum_{i=1}^{p-1} T(d(s_{j_i}, s_{j_{i+1}})) \\
&\quad + T(d(s_{j_p}, v)) \\
&> qt + T(d(u, s^*_{j_1})) + \sum_{i=1}^{q-1} T(d(s^*_{j_i}, s^*_{j_{i+1}})) \\
&\quad + T(d(s^*_{j_q}, v)) \\
&= cost_{u,s^*_{j_1}} + \sum_{i=1}^{q-1} cost_{s^*_{j_i},s^*_{j_{i+1}}} + cost_{s^*_{j_q},v} \\
&= Cost(P^*)
\end{aligned}$$

Hence, P^* defines a shorter $u-v-$path in $\mathcal{N}$ which is a contradiction to the optimality of P. The complexity applying Dijkstra's shortest path algorithm results in $\mathcal{O}(|\mathcal{P}| log |\mathcal{P}|)$, where $|S_{cand}|$ and $|\mathcal{P}|$ are of the same order since we consider a linear graph G. □

We conclude that simplifications of the general problem turn out to be solvable in polynomial time, while (SL*) is NP-hard due to Theorem 5.1.

6. Experiments

Environment. All our experiments were conducted on a PC with 24 six-core Intel Xenon X5650 Processor running at 2.67 GHz with 12 MB cache and a main memory of 94 GB. The IPs were solved using Xpress Optimizer v27.01.02. The running time limit of the solver was set to 300 s.

Benchmark sets. The southern part of the existing railway network of Lower Saxony, Germany, is used as the existing network $G = (V, E)$. From the same area the 34 largest cities are considered as demand points if they are not already close enough to an existing stop (as assumed in (1)). This is the setting for the first benchmark set (LoSa = Lower Saxony).

In our second benchmark set we removed existing stops which have only two adjacent edges in order to obtain another set (LoSaRe = Lower Saxony Reduced) with longer tracks and more uncovered demand points. This set has higher complexity.

The values for the vehicles' driving times are chosen according to realistic properties, which are an acceleration and deceleration of $0.6\,m/s^2$, a cruising speed of $160\,km/h$ and a stopping time t of $30\,s$. For a set of different radii ($r \in \{1500, 2000, 2500, \ldots, 15000\}$ (in meters)), we constructed instances containing all demand points which can be covered by r, that is,

$$\{p \in \mathcal{P} \setminus \text{cover}(V) : \text{there exists } s \in \mathcal{S} : d(p, s) \leqslant r\}.$$

This means that $\mathcal{P}$ increases with the radius.

Setup. The quality and computation times when solving the IP formulations for (SL) and for (SL*) are compared. To this end, for each of the benchmark sets and every radius $r \in \{1500, 2000, 2500, \ldots, 15000\}$ both models have been solved by Xpress Optimizer. Then for each run the quality of the solution is measured by evaluating the passengers' traveling time and the number of stops.

If it is not explicitly stated, the IP-formulation (IP2-SL*) is used to solve (SL*) since it turned out to be more efficient, see Section 6.1.

Results. Table 1 summarizes our results calculating the average values of the additional traveling time for all instances. With additional traveling time for a solution S we mean the passengers traveling time $g(S)$ reduced by the traveling time which is not avoidable, that is, the traveling time $g(\varnothing)$ obtained when no additional stop is established:

$$f(S) = g(S) - g(\varnothing).$$

Note that minimizing the traveling time $g(S)$ and minimizing the additional traveling time $f(S)$ are equivalent problems, since $g(\varnothing)$ is a constant.

6.1. Computing times for solving (SL) and (SL)*

The generation of instances for different radii also allows to study the computation time for the IP-formulation of (SL) and for the two IP formulations (IP1-SL*) and (IP2-SL*) for (SL*). Note that all the IP formulations are based on the finite candidate set $\mathcal{S}_{cand}$.

In Figure 4 (lower part) we depict that the number of such candidates increases (linearly) with increasing radius. However, even a linear increase in $|\mathcal{S}_{cand}|$ leads to an exponential increase of the number of possible solutions $S \subset \mathcal{S}_{cand}$. We hence expect an exponential increase of the running time. Such an exponential increase is shown in the upper part of Figure 4, but only for solving (SL*) with the first IP formulation (IP1-SL*).

Table 1 Average values of the additional traveling time for the solutions of (SL) and (SL*)

	(SL)	*(SL*)*
Instance (LoSa):		
Passengers' traveling time $f(S)$	243.6	230.5
Number of stops $\lvert S\rvert$	35.9	35.9
Instance (LoSaRe):		
Passengers' traveling time $f(S)$	1019.6	998.6
Number of stops $\lvert S\rvert$	17.6	17.6

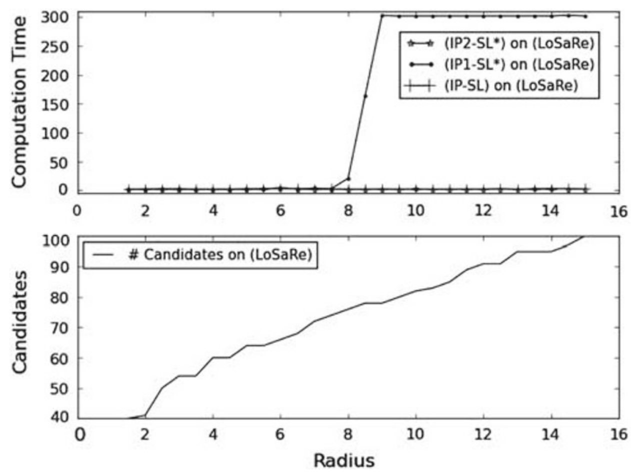

Figure 4 Comparing computing time and number of candidates.

(Since the maximal running time is set to 300 s we have a flat part at the end of the graph.)

The figure also clearly shows that the IP-formulations for (SL) and our second IP formulation (IP2-SL*) for (SL*) are good enough to be still able to solve the problem of bigger sizes in the same time (about 2 s). We conclude that (IP2-SL*) is much better than (IP1-SL*).

This justifies using the solutions of (IP2-SL*) for comparing the quality of the solutions in the following tests.

6.2. Comparison of objective function values of (SL) and (SL)*

(SL) minimizes the number of new stops and (SL*) minimizes the passengers' traveling time. We tested on our two benchmark sets (LoSa) and (LoSaRe) how big these differences are. The results are depicted in Figure 5.

We see that the solutions of (SL) and (SL*) in terms of the number of stops do not vary at all while in terms of the resulting additional passengers' traveling times (SL*) performs better than (SL).

- On the benchmark set (LoSa) the average additional passengers' traveling time can be reduced by 13 *s*, which amounts to an improvement of about 5.6% by using (SL*) instead of (SL). The maximal improvement on this instance reduces the additional passengers' traveling time by about 75 *s*, which are about 16.6%.
- On the instance (LoSaRe) we obtain an average time saving of 21 *s*, which results in an improvement of 2.1%. The maximal improvement achieves to save 78 *s*, which means an improvement of 5.1%. We conclude that on real-world instances (SL*) is not worse in the number of new stops but improves the additional passengers' traveling time.

Note that in terms of the absolute time saving the model (SL*) performs better on the network with longer edges, first because

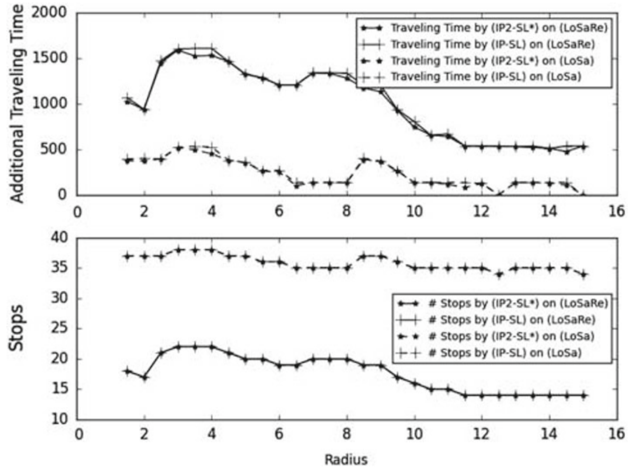

Figure 5 Comparing passengers' traveling time and number of new stops.

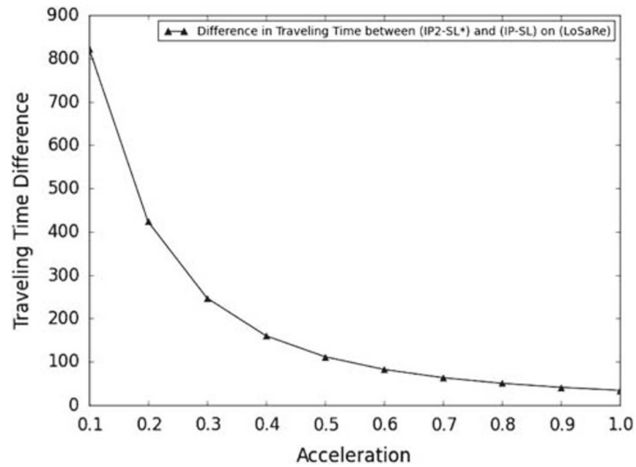

Figure 6 Difference of traveling time values for (IP2-SL*) and (IP-SL) with different acceleration and deceleration values on (LoSaRe).

in (LoSaRe) more demand points have to be covered (less are uncovered), so more new stops are needed. Second, longer edges grant more freedom to do a bad allocation of the stops in terms of the traveling time (since the network structure remains the same for (LoSa) and (LoSaRe)).

We summarize that the relative improvement is higher on (LoSa) while the absolute improvement is larger on (LoSaRe).

We also looked at some other properties of our solutions.

Dependence on the covering radius. There is another interesting observation to be mentioned when looking at Figure 5: We generated a new instance for every radius r (as described in the benchmark set), which consists of all demand points that can potentially be covered by some new stop. The effect of the radius on the number of new stops and on the traveling time cannot be clearly predicted since we have two conflicting effects: On the one hand, the number of demand points to be covered increases, we hence may need more stops. On the other hand, every stop covers more demand points if r is increased. In our experiments we see that the latter effect is the dominating one. Although there are instances in which the number of stops and the passengers' traveling time increases with increasing radius, the tendency is that increasing the radius leads to fewer new stops and less traveling time.

Dependence on acceleration and deceleration. We computed optimal solutions for (SL) and (SL*) on (LoSaRe) for different values for the acceleration and deceleration. For simplicity, acceleration and deceleration are assumed to be equal. As expected, with increasing values of acceleration and deceleration the traveling time values decrease. The traveling time function T behaves in a_0 and b_0 like x^{-1}, which is of a similar shape as the traveling time values depicted in Figure 6. In order to justify Theorem 4.7 we compared the solutions of (SL) and (SL*) in terms of traveling times as follows: Let $a_0 = b_0$ be fixed, then denote by S_{a_0} an optimal solution to (SL) and by $S^*_{a_0}$ an optimal solution to (SL*) for the specific acceleration and deceleration a_0. We evaluate the difference of travel times between these solutions, that is,

$$h(a_0) := g(S_{a_0}) - g(S^*_{a_0}).$$

Figure 6 shows this function h. We recognize that the gap between the two traveling time values decreases for increasing values for acceleration and deceleration. It means that the smaller the acceleration and deceleration, the higher the difference between the traveling time values for (SL) and (SL*), that is, the bigger the error when using (SL) instead of (SL*). Hence, for smaller acceleration and deceleration values such as in train transportation it is more desirable to use the (SL*) model with an accurate computation of the traveling time.

6.3. Summary

From our numerical experiments, we hence conclude

1. (IP2-SL*) is clearly superior to (IP1-SL*).
2. In the experiments, (SL*) leads to the same (minimal) number of stops as (SL) but improves the additional traveling time.
3. The difference between (SL*) and (SL) is larger for small acceleration and deceleration. It means that in bus transportation one can use the easier model (SL) and expect a good traveling time while it becomes more important to use (SL*) in train transportation.
4. Increasing the covering radius leads to fewer new stops and less traveling time.

7. Conclusion and further research

In this paper we minimized the passengers' traveling time function in SLs. We derived a finite dominating set and two IP formulations and showed the applicability of the model on two different benchmark sets. It turns out that the solutions of (SL*) slightly outperform the solutions of (SL) with similar computation time values and with the same (minimal) number of new stops. Even more we have seen two different IP-formulations for (SL*) and their performance. Judging from the experiments, the second of those (IP2-SL*) has a high potential to even solve complex instances.

Further research on this topic goes into two directions. First, we assumed that all vertices of the existing network are built as stops. However, it may be better to close or move some of these. In order to model this appropriately, an integration with line planning is necessary.

Second, the traveling time for the passengers could be even more realistic if OD-pairs are considered. Minimizing an OD-pair based traveling time leads to a different model and thus analysis. It would also allow to take congestion into account: if more passengers are boarding or alighting this leads to larger stopping times of the trains at the stops. Integrating routing of passengers into planning problems has only been done rarely, for example, in Körner *et al* (2014); Perea *et al* (2014), whereas first approaches exist for line planning (see, eg, (Schmidt, 2014; Schmidt and Schöbel, 2015a), timetabling (Borndörfer *et al*, 2015; Schmidt and Schöbel, 2015b), or delay management (Dollevoet *et al*, 2012; Schmidt, 2013). It is a challenging task to develop similar integrated approaches also for stop location.

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