Time Consistent Behavioral Portfolio Policy for Dynamic Mean-Variance Formulation

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Abstract

When one considers an optimal portfolio policy under a mean-risk formulation, it is essential to correctly model investors' risk aversion which may be time variant, or even state-dependent. In this paper, we propose a behavioral risk aversion model, in which risk aversion is a piecewise linear function of the current excess wealth level with a reference point at the discounted investment target (either surplus or shortage), to reflect a behavioral pattern with both house money and break-even effects. Due to the time inconsistency of the resulting multi-period mean-variance model with adaptive risk aversion, we investigate the time consistent behavioral portfolio policy by solving a nested mean-variance game formulation. We derive a semi-analytical time consistent behavioral portfolio policy which takes a piecewise linear feedback form of the current excess wealth level with respect to the discounted investment target. Finally, we extend the above results to time consistent behavioral portfolio selection for dynamic mean-variance formulation with a cone constraint.

Key Words: Investment analysis, state-dependent risk aversion, dynamic mean-variance formulation, time consistency, behavioral portfolio policy.

1 Introduction

According to the classical investment doctrine in Markowitz (1952), an investor of a mean-variance type needs to strike a balance between maximizing the expected value of the terminal wealth, $\mathbb{E}[X_1|X_0]$, and minimizing the investment risk measured by the variance of the terminal wealth, $\mathrm{Var}(X_1|X_0)$, by solving the following mean-variance formulation,

$$(MV(\gamma)): \qquad \min \quad \mathrm{Var}(X_1|X_0) - \gamma\mathbb{E}[X_1|X_0],$$

where X_0 is the initial wealth level, X_1 is the terminal wealth at the end of the (first) time period and $\gamma \geq 0$ is the trade-off parameter between the two conflicting objectives. We call γ the risk

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aversion parameter, which represents the risk aversion attitude of the investor. The larger the value of γ, the less the risk aversion of the investor. Mathematically, $(MV(\gamma))$ is equivalent to the following formulation,

$$(MV(\omega)): \qquad \max \quad \mathbb{E}[X_1|X_0] - \omega \mathrm{Var}(X_1|X_0),$$

with the risk aversion parameter $\omega = 1/\gamma$.

In a dynamic investment environment, the risk aversion attitude of a mean-variance investor may change from time to time, or could be even state-dependent (i.e., dependent on the investor's current wealth level X_t realized at time t). Björk et al. (2014) and Wu (2013) proposed, respectively, in continuous-time and discrete-time settings, that the risk aversion parameter ω takes the following simple form of the current wealth level X_t,

$$\omega(X_t) = \frac{\omega}{X_t}, \quad (\omega \geq 0).$$

The intuition behind this setting is clear: The larger the current wealth x_t, the smaller the degree of risk aversion. Due to the positiveness of the wealth process X_t in the continuous-time setting, $\omega(X_t)$ proposed by Björk et al. (2014) is always nonnegative and is a decreasing function of the current wealth level. Applying the same model to a discrete-time multi-period setting as proposed in Wu (2013), however, could encounter some problem, as there is no guarantee for the positiveness of the wealth process in a discrete-time setting. When the wealth level is negative, $\omega(X_t)$ becomes negative, which leads to an irrationality of the investor to maximize both the expected value and the variance of the terminal wealth, resulting in an infinite position on the riskiest asset (See Theorem 7 in Wu, 2013).

In a continuous time setting, Hu et al. (2012) also introduced the risk aversion parameter γ as a linear function of the current wealth level X_t,

$$\gamma(X_t) = \mu_1 X_t + \mu_2, \quad (\mu_1 \geq 0).$$

When the wealth level is less than $-\mu_2/\mu_1$, $\gamma(X_t)$ becomes negative. It also leads to an irrationality of the investor, which contradicts the original interests of the investor of a mean-variance type.

In this paper, we propose a behavioral risk aversion model as follows,

$$\gamma_t(X_t) = \begin{cases} \gamma_t^+(X_t - \rho_t^{-1}W), & \text{if } X_t \geq \rho_t^{-1}W, \\ -\gamma_t^-(X_t - \rho_t^{-1}W), & \text{if } X_t < \rho_t^{-1}W, \end{cases} \tag{1}$$

where W is the investment target set by the investor at time 0, ρ_t^{-1} is the risk-free discount factor from the current time t to the terminal time T, and $\gamma_t^+ \geq 0$ and $\gamma_t^- \geq 0$ are t-dependent risk aversion coefficients for the ranges of X_t on the right and left sides of $\rho_t^{-1}W$, respectively. Basically, we consider a piecewise linear state-dependent risk aversion function in our behavioral risk aversion model.

This proposed behavioral risk aversion model is pretty flexible in incorporating the behavioral pattern of a mean-variance investor. If the current wealth level is the same as the discounted investment target, the investor becomes fully risk averse and thus invests only in the risk-free asset. If the current wealth level is larger than the discounted investment target, the investor may consider the surplus over the discounted target level as house money and the larger the surplus the less the risk aversion. If the current wealth level is less than the discounted target level, the investor may intend to break-even and the larger the shortage under the discounted target, the stronger the desire to break-even (the less the risk aversion). The magnitude of γ_t^+ (or γ_t^-) represents the risk aversion reduction with respect to one unit increase of the surplus (or the shortage). Apparently, different mean-variance investors may have different choices of γ_t^+ and γ_t^-. For example, an investor, who is eager for breaking-even when facing shortage and feels less sensitive with the levels of surplus, may set $\gamma_t^- > \gamma_t^+$. Although we use the same terms of "house money" and "break-even" as in behavioral finance, their meanings are slightly different. In behavioral finance, the house money effect describes the behavior that people take greater risk following prior gains, while the break-even effect describes the behavior that people take greater risk following prior losses (see, for examples, Staw, 1976, Thaler and Johnson, 1990, Weber and Zuchel, 2005). While no risk attitude is assumed for investors in their study, all investors are assumed to be risk-averse under the dynamic mean-variance framework discussed in this paper. Nevertheless, the key concepts behind the house money and break-even effects are the same for both our study and the literature in behavior finance: investors become less risk averse when experiencing either larger gains or larger losses.

The main challenge of solving multi-period mean-variance portfolio selection problem with the proposed behavioral risk aversion parameter is the time inconsistency of the problem. To see this, let us consider the simple problem with constant risk aversion parameter. At time 0, the investor faces the following global mean-variance portfolio selection problem over the entire investment time horizon,

$$(MV_0(\gamma)): \qquad \min \quad \mathrm{Var}(X_T|X_0) - \gamma\mathbb{E}[X_T|X_0],$$

whose pre-committed optimal mean-variance policy is derived by Li and Ng (2000) and given as follows,

$$\mathbf{u}_j^* = -\mathbb{E}^{-1}[\mathbf{P}_j\mathbf{P}_j']\mathbb{E}[\mathbf{P}_j]s_j(X_j - \lambda_0\rho_j^{-1}), \quad j = 0, 1, \cdots, T-1,$$

where $\mathbf{P}_j$ is the vector of excess return rates of risky assets, X_j is the wealth level at time j and

$$\lambda_0 = \rho_0 X_0 + \frac{\gamma}{2}\frac{1}{\prod_{k=0}^{T-1}(1 - \mathbb{E}[\mathbf{P}_k']\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}_k']\mathbb{E}[\mathbf{P}_k])}.$$

However, for $t > 0$, the investor may reconsider the mean-variance portfolio selection problem for a truncated time horizon from t to T,

$$(MV_t(\gamma)): \qquad \min \quad \mathrm{Var}(X_T|X_t) - \gamma\mathbb{E}[X_T|X_t],$$

whose local optimal mean-variance policy is given by

$$\bar{\mathbf{u}}_j = -\mathbb{E}^{-1}[\mathbf{P}_j\mathbf{P}_j']\mathbb{E}[\mathbf{P}_j]s_j(X_j - \lambda_t\rho_j^{-1}), \quad j = t, t+1, \cdots, T-1,$$

where

$$\lambda_t = X_t\rho_t + \frac{\gamma}{2}\frac{1}{\prod_{k=t}^{T-1}(1-\mathbb{E}[\mathbf{P}_k']\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}_k']\mathbb{E}[\mathbf{P}_k])}.$$

Since $\lambda_0 \neq \lambda_t,\quad t = 1, 2, \cdots, T-1$, this leads to $\mathbf{u}_j^* \neq \bar{\mathbf{u}}_j,\quad j = t, t+1, \cdots, T-1$, i.e., the local optimal policy is different from the pre-committed optimal policy. This interesting phenomenon is called time inconsistency (see Basak and Chabakauri, 2010, Wang and Forsyth, 2011, Cui et al., 2012). In the language of dynamic programming, the *Bellman's principle of optimality* is not applicable to this model formulation, as the global and local objectives are not consistent (See Artzner et al., 2007, Cui et al., 2012). In the fields of dynamic risk measures and dynamic risk management, time consistency is considered to be a basic requirement (see Rosazza Gianin, 2006, Boda and Filar, 2006, Artzner et al., 2007 and Jobert and Rogers, 2008).

In fact, there exists a unique trade-off $\gamma(X_t)$ which depends on the wealth X_t, termed as the trade-off induced by the pre-committed optimal policy, such that the optimal mean-variance policy of the truncated-time horizon problem, $(MV_t(\gamma(X_t)))$, is the same as the pre-committed optimal policy, i.e., the pre-committed optimal policy of $(MV_0(\gamma(X_0)))$ could become a time consistent policy of $(MV_t(\gamma(X_t)))$ when all the trade-offs are set as the induced trade-offs (see Cui et al., 2012). Furthermore, Cui et al. (2012) showed that the trade-off induced by the pre-committed optimal policy is a linear function in terms of the current wealth level X_t, the initial wealth level X_0 and the initial risk aversion parameter $\gamma_0(X_0)$. Thus, the induced trade-off may become negative over a finite time investment horizon, which implies that investors may take irrational actions. This actually reveals that a linear trade-off is a hidden reason behind time inconsistency. Thus, to better the performance of the dynamic mean-variance formulation, the setting of trade-off parameter should go beyond the class of linear functions.

Strotz (1956) suggested two possible actions to overcome time inconsistency: (1) "He may try to pre commit his future activities either irrevocably or by contriving a penalty for his future self if he should misbehave", which is termed as the *strategy of precommitment*; and (2) "He may resign himself to the fact of inter temporal conflict and decide that his 'optimal' plan at any date is a will-o'-the-wisp which cannot be attained, and learn to select the present action which will be best in the light of future disobedience", which is termed the *strategy of consistent planning*. Strategy of consistent planning is also called *time consistent policy* in the literature. For a dynamic mean-variance model, Basak and Chabakauri (2010) reformulated it as an intrapersonal game model where the investor optimally chooses the policy at any time t, on the premise that he has already decided his time consistent policies in the future. More specifically, in a framework of time consistency, the investor faces the following nested portfolio selection problem,

$$\begin{aligned}(NMV_0(\gamma)): \qquad &\min_{u_t} \quad \mathrm{Var}(X_T|X_t) - \gamma\mathbb{E}[X_T|X_t],\\ &\text{s.t.} \quad u_j \text{ solves } (NMV_j(\gamma)), \quad t \leq j \leq T,\end{aligned}$$

with the terminal period problem given as

$$(NMV_{T-1}(\gamma)): \qquad \min_{u_{T-1}} \quad \mathrm{Var}(X_T|X_{T-1}) - \gamma\mathbb{E}[X_T|X_{T-1}].$$

The time consistent policy is then the equilibrium solution of the above nested problem, which can be derived by a backward induction. Björk et al. (2014), Hu et al. (2012) and Wu (2013) extended the results in Basak and Chabakauri (2010) by considering different state-dependent risk aversion mentioned before in this section. For a general class of continuous-time mean-field linear-quadratic control problems, please refer to Yong (2015). In the original setting of dynamic mean-variance portfolio selection with constant risk aversion, the time inconsistency is caused by the appearance of the variance of the terminal wealth in the objective function, which does not satisfy the smoothing property. Our model in this study that adopts a more realistic time-varying and wealth dependent risk aversion further complicates the extent of time inconsistency, which forces us to consider time consistent policies in this paper.

In this paper, we focus on studying time consistent behavioral portfolio policies under the proposed behavioral risk aversion model. The remaining parts of this paper are organized as follows: In Section 2, we provide the basic market setting and formulate the nested mean-variance portfolio selection problem. We derive in Section 3 the semi-analytical time consistent behavioral portfolio policy, which takes a piecewise linear feedback form of the surplus or the shortage with respect to the discounted wealth target. In Section 4, we extend our main results to cone constrained markets. After we offer in Section 5 numerical analysis to show the trading patterns of investors with different risk aversion coefficients, we conclude the paper in Section 6.

2 Market Setting and Problem Formulation

We consider an arbitrage-free capital market of T-time periods, which consists of one risk-free asset with deterministic rate of return and n risky assets with random rates of return. An investor with an initial wealth X_0 joins the market at time 0 and allocates wealth among the risk-free asset and n risky assets at time 0 and the beginning of each of the following $(T-1)$ consecutive periods. The deterministic rate of return of the risk-free asset at time period t is denoted by s_t (> 1) and the random rates of return of the risky assets at time period t are denoted by the vector $\mathbf{e}_t = [e_t^1, \cdots, e_t^n]'$, where e_t^i is the random rate of return of asset i at time period t. It is assumed that $\mathbf{e}_t$, $t = 0, 1, \cdots, T-1$, are statistically independent, absolutely integrable continuous random vectors, whose finite first and second moments, $\mathbb{E}[\mathbf{e}_t]$ and $\mathbb{E}[\mathbf{e}_t\mathbf{e}_t']$, are known for every t and whose covariance matrixes $\mathrm{Cov}(\mathbf{e}_t) = \mathbb{E}[\mathbf{e}_t\mathbf{e}_t'] - \mathbb{E}[\mathbf{e}_t]\mathbb{E}[\mathbf{e}_t']$, $t = 0, 1, \cdots, T-1$, are positive definite[1]. All of the random vectors are defined in a filtered probability space $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}, P)$, where $\mathcal{F}_t = \sigma(\mathbf{e}_0, \mathbf{e}_1, \cdots, e_{t-1})$ and $\mathcal{F}_0$ is the trivial σ-algebra over Ω. Let A' denote the transpose operation of matrix or vector A.

Let X_t be the wealth of the investor at the beginning of period t, and u_t^i, $i = 1, 2, \cdots, n$, be the amount invested in the i-th risky asset at period t. Then, $X_t - \sum_{i=1}^n u_t^i$ is the amount invested in

[1] Our main results can be readily extended to situations where random vectors $\mathbf{e}_t$, $t = 0, 1, \cdots, T-1$, are correlated. This extension can be achieved based on the concept of the so-called opportunity-neutral measure introduced by Černý and Kallsen (2009).

the risk-free asset at period t. Thus, the wealth at the beginning of period $t+1$ is given as

$$X_{t+1} = s_t\Big(X_t - \sum_{i=1}^{n} u_t^i\Big) + \mathbf{e}_t'\mathbf{u}_t = s_t X_t + \mathbf{P}_t'\mathbf{u}_t,$$

where

$$\mathbf{P}_t = [P_t^1, P_t^2, \cdots, P_t^n]' = [(e_t^1 - s_t), (e_t^2 - s_t), \cdots, (e_t^n - s_t)]'$$

is the vector of the excess rates of return and $\mathbf{u}_t = [u_t^1, u_t^2, \cdots, u_t^n]'$ is the portfolio policy. We confine all admissible investment policies to be $\mathcal{F}_t$-measurable Markov control, whose realizations are in $\mathbb{R}^n$. Then, $\mathbf{P}_t$ and $\mathbf{u}_t$ are independent, the controlled wealth process $\{X_t\}$ is an adapted Markovian process and $\mathcal{F}_t = \sigma(X_t)$.

An investor of a mean-variance type considers the following portfolio decision problem at the beginning of period t,

$$\begin{aligned} (MV_t(\gamma_t(X_t))) \qquad & \min \ \ \mathrm{Var}_t(X_T) - \gamma_t(X_t)\mathbb{E}_t[X_T], \\ & \text{s.t.} \ \ X_{j+1} = s_j X_j + \mathbf{P}_j'\mathbf{u}_j, \quad j = t, t+1, \cdots, T-1, \end{aligned} \tag{2}$$

where $\mathrm{Var}_t(X_T) = \mathrm{Var}(X_T|X_t)$, $\mathbb{E}_t[X_T] = \mathbb{E}[X_T|X_t]$, $\rho_t^{-1} = \prod_{j=t}^{T-1} s_j^{-1}$ is the risk-free discount factor with $\rho_T^{-1} = 1$ and $\gamma_t(X_t)$ is given by

$$\gamma_t(X_t) = \begin{cases} \ \ \gamma_t^+(X_t - \rho_t^{-1}W), & \text{if } X_t \geq \rho_t^{-1}W, \\ -\gamma_t^-(X_t - \rho_t^{-1}W), & \text{if } X_t < \rho_t^{-1}W. \end{cases}$$

As we have already pointed out in the introduction section, the risk aversion parameter $\gamma_t(X_t)$ proposed in this paper keeps nonnegative in the discrete-time setting and captures the features of both house-money and break-even effects.

Due to the time inconsistency of $(MV_t(\gamma_t(X_t)))$, we aim to derive the time consistent behavioral portfolio policy. More specifically, similar to the approach in Basak and Chabakauri (2010), we formulate the multi-period mean-variance model into an interpersonal game model in which the investor optimally chooses the policy at any time t, on the premise that he has already decided his time consistent policy in the future. Then the time consistent behavioral portfolio policy (or time consistent policy in short) is the optimizer of the following nested mean-variance problem (NMV),

$$\begin{aligned} (NMV_t(\gamma_t(X_t))) \qquad & \min_{\mathbf{u}_t} \ \ \mathrm{Var}_t(X_T) - \gamma_t(X_t)\mathbb{E}_t[X_T], \\ & \text{s.t.} \ \ X_{t+1} = s_t X_t + \mathbf{P}_t'\mathbf{u}_t, \\ & \qquad X_{j+1} = s_j X_j + \mathbf{P}_j'\mathbf{u}_j^{TC}, \quad j = t+1, \cdots, T-1, \\ & \qquad \mathbf{u}_j^{TC} \text{ solves } (MV_j(\gamma_j(X_j))), \quad j = t+1, \cdots, T-1, \end{aligned} \tag{3}$$

with terminal period problem given as

$$\begin{aligned} (NMV_{T-1}(\gamma_{T-1}(X_{T-1}))) \qquad & \min_{\mathbf{u}_{T-1}} \ \ \mathrm{Var}_{T-1}(X_T) - \gamma_{T-1}(X_{T-1})\mathbb{E}_{T-1}[X_T], \\ & \text{s.t.} \ \ X_t = s_{T-1}X_{T-1} + \mathbf{P}_{T-1}'\mathbf{u}_{T-1}, \end{aligned} \tag{4}$$

which can be solved by a backward induction. Since the stage-trade off $\gamma_t(X_t)$ reflects certain behavioral pattern of an investor in terms of his wealth level, we call the optimal policy to (NMV) time consistent behavioral portfolio policy.

3 Semi-analytical Time Consistent Policy

In this section, we derive the semi-analytical time consistent behavioral portfolio policy. Before presenting our main results, we define the following two deterministic continuous functions, $F_t^-(\mathbf{K})$ and $F_t^+(\mathbf{K})$, on $\mathbb{R}^n$ for $t = 0, 1, \cdots, T-1$,

$$\begin{aligned}
F_t^+(\mathbf{K}) =& \rho_{t+1}^2 \mathbf{K}'(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t])\mathbf{K} \\
&+ \mathbb{E}\left[(2\rho_{t+1}a_{t+1}^+ + b_{t+1}^+)(s_t + \mathbf{P}_t'\mathbf{K})^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}\geq 0\}}\right] \\
&+ \mathbb{E}\left[(2\rho_{t+1}a_{t+1}^- + b_{t+1}^-)(s_t + \mathbf{P}_t'\mathbf{K})^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}< 0\}}\right] \\
&- \left(\mathbb{E}\left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}) 1_{\{s_t+\mathbf{P}_t'\mathbf{K}\geq 0\}}\right] + \mathbb{E}\left[a_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}) 1_{\{s_t+\mathbf{P}_t'\mathbf{K}< 0\}}\right]\right)^2 \\
&- 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}) 1_{\{s_t+\mathbf{P}_t'\mathbf{K}\geq 0\}}\right](s_t + \mathbb{E}[\mathbf{P}_t']\mathbf{K}) \\
&- 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}) 1_{\{s_t+\mathbf{P}_t'\mathbf{K}< 0\}}\right](s_t + \mathbb{E}[\mathbf{P}_t']\mathbf{K}) \\
&- \gamma_t^+ \left(\mathbb{E}[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}) 1_{\{s_t+\mathbf{P}_t'\mathbf{K}\geq 0\}}] + \mathbb{E}[a_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}) 1_{\{s_t+\mathbf{P}_t'\mathbf{K}< 0\}}]\right) \\
&- \rho_{t+1}\gamma_t^+(s_t + \mathbb{E}[\mathbf{P}_t']\mathbf{K}), \\
F_t^-(\mathbf{K}) =& \rho_{t+1}^2 \mathbf{K}'(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t])\mathbf{K} \\
&+ \mathbb{E}\left[(2\rho_{t+1}a_{t+1}^+ + b_{t+1}^+)(s_t + \mathbf{P}_t'\mathbf{K})^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}\leq 0\}}\right] \\
&+ \mathbb{E}\left[(2\rho_{t+1}a_{t+1}^- + b_{t+1}^-)(s_t + \mathbf{P}_t'\mathbf{K})^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}> 0\}}\right] \\
&- \left(\mathbb{E}\left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}) 1_{\{s_t+\mathbf{P}_t'\mathbf{K}\leq 0\}}\right] + \mathbb{E}\left[a_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}) 1_{\{s_t+\mathbf{P}_t'\mathbf{K}> 0\}}\right]\right)^2 \\
&- 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}) 1_{\{s_t+\mathbf{P}_t'\mathbf{K}\leq 0\}}\right](s_t + \mathbb{E}[\mathbf{P}_t']\mathbf{K}) \\
&- 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}) 1_{\{s_t+\mathbf{P}_t'\mathbf{K}> 0\}}\right](s_t + \mathbb{E}[\mathbf{P}_t']\mathbf{K}) \\
&+ \gamma_t^- \left(\mathbb{E}[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}) 1_{\{s_t+\mathbf{P}_t'\mathbf{K}\leq 0\}}] + \mathbb{E}[a_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}) 1_{\{s_t+\mathbf{P}_t'\mathbf{K}> 0\}}]\right) \\
&+ \rho_{t+1}\gamma_t^-(s_t + \mathbb{E}[\mathbf{P}_t']\mathbf{K}),
\end{aligned}$$

with a_{t+1}^+, a_{t+1}^-, b_{t+1}^+ and b_{t+1}^- being deterministic parameters.
The following proposition ensures that the optimizers of $\min_{\mathbf{K}\in\mathbb{R}^n} F_t^+(\mathbf{K})$ and $\min_{\mathbf{K}\in\mathbb{R}^n} F_t^-(\mathbf{K})$ are finite.

Proposition 3.1 *Suppose that deterministic numbers a_{t+1}^+, a_{t+1}^-, b_{t+1}^+ and b_{t+1}^- satisfy*

$$b_{t+1}^+ - (a_{t+1}^+)^2 \geq 0, \quad b_{t+1}^- - (a_{t+1}^-)^2 \geq 0.$$

Then we have

$$\lim_{\|\mathbf{K}\|\to+\infty} F_t^+(\mathbf{K}) = +\infty, \qquad \lim_{\|\mathbf{K}\|\to+\infty} F_t^-(\mathbf{K}) = +\infty,$$

where $\|\mathbf{K}\|$ *denotes the Euclidean norm of vector* $\mathbf{K}$.

Proof. See Appendix A. $\square$

According to Proposition 3.1, we denote the finite optimizers of $\min_{\mathbf{K}\in\mathbb{R}^n} F_t^+(\mathbf{K})$ and $\min_{\mathbf{K}\in\mathbb{R}^n} F_t^-(\mathbf{K})$ as follows,

$$\mathbf{K}_t^+ = \operatorname*{argmin}_{\mathbf{K}\in\mathbb{R}^n} F_t^+(\mathbf{K}), \quad \mathbf{K}_t^- = \operatorname*{argmin}_{\mathbf{K}\in\mathbb{R}^n} F_t^-(\mathbf{K}),$$

and define the deterministic parameters a_t^+, a_t^-, b_t^+ and b_t^-, $t = 0, 1, \cdots, T-1$, by the following backward recursions, respectively,

$$\begin{aligned}
a_t^+ &= \rho_{t+1}\mathbb{E}[\mathbf{P}_t']\mathbf{K}_t^+ + \mathbb{E}\left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_t^+)1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+\geq 0\}}\right] \\
&\quad + \mathbb{E}\left[a_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}_t^+)1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+< 0\}}\right], && (5)\\
a_t^- &= \rho_{t+1}\mathbb{E}[\mathbf{P}_t']\mathbf{K}_t^- + \mathbb{E}\left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_t^-)1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^-\leq 0\}}\right] \\
&\quad + \mathbb{E}\left[a_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}_t^-)1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^-> 0\}}\right], && (6)\\
b_t^+ &= \rho_{t+1}^2(\mathbf{K}_t^+)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\mathbf{K}_t^+ + 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_t^+)\mathbf{P}_t'\mathbf{K}_t^+1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+\geq 0\}}\right] \\
&\quad + 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}_t^+)\mathbf{P}_t'\mathbf{K}_t^+1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+< 0\}}\right] \\
&\quad + \mathbb{E}\left[b_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_t^+)^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+\geq 0\}}\right] + \mathbb{E}\left[b_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}_t^+)^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+< 0\}}\right], && (7)\\
b_t^- &= \rho_{t+1}^2(\mathbf{K}_t^-)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\mathbf{K}_t^- + 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_t^-)\mathbf{P}_t'\mathbf{K}_t^-1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^-\leq 0\}}\right] \\
&\quad + 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}_t^-)\mathbf{P}_t'\mathbf{K}_t^-1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^-> 0\}}\right] \\
&\quad + \mathbb{E}\left[b_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_t^-)^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^-\leq 0\}}\right] + \mathbb{E}\left[b_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}_t^-)^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^-> 0\}}\right], && (8)
\end{aligned}$$

with terminal condition $a_T^+ = a_T^- = 0$ and $b_T^+ = b_T^- = 0$.

Remark 3.1 *In general, functions* $F_t^+(\mathbf{K})$ *and* $F_t^-(\mathbf{K})$ *are not convex functions with respect to* $\mathbf{K}$*. However, when* $a_{t+1}^+ \geq 0 \geq a_{t+1}^-$*, it is easy to show that* $F_t^+(\mathbf{K})$ *and* $F_t^-(\mathbf{K})$ *are d.c. functions (difference of convex functions) with respect to* $\mathbf{K}$ *(see Horst and Thoai, 1999). In such cases, we can use the existing global search methods for d.c. functions in the literature to derive the optimizers,* $\mathbf{K}_t^+$ *and* $\mathbf{K}_t^-$*.*

With the above notations, we show now that the time consistent behavioral portfolio policy takes a piecewise linear feedback form of the current excess wealth level (either surplus or shortage with respect to the discounted investment target) in the following theorem.

Theorem 3.1 *The time consistent behavioral portfolio policy of* $(NMV_t(\gamma_t(X_t)))$ *is given as follows for* $t = 0, \ldots, T-1$,

$$\mathbf{u}_t^{TC} = \mathbf{K}_t^+(X_t - \rho_t^{-1}W)1_{\{X_t\geq\rho_t^{-1}W\}} + \mathbf{K}_t^-(X_t - \rho_t^{-1}W)1_{\{X_t<\rho_t^{-1}W\}}, \tag{9}$$

in which the parameters a_t^+, a_t^-, b_t^+ and b_t^- defined in (5)-(8) satisfy

$$b_t^+ - (a_t^+)^2 \geq 0, \quad b_t^- - (a_t^-)^2 \geq 0.$$

Furthermore, the mean and the variance of the terminal wealth achieved by the time consistent behavioral portfolio policy are

$$\mathbb{E}_0[X_T]|_{\mathbf{u}^{TC}} = \rho_0 X_0 + a_0^+(X_0 - \rho_0^{-1}W)1_{\{X_0 \geq \rho_0^{-1}W\}} + a_0^-(X_0 - \rho_0^{-1}W)1_{\{X_0 < \rho_0^{-1}W\}}, \tag{10}$$

$$\mathrm{Var}_0(X_T)|_{\mathbf{u}^{TC}} = \left[(b_0^+ - (a_0^+)^2)1_{\{X_0 \geq \rho_0^{-1}W\}} + (b_0^- - (a_0^-)^2)1_{\{X_0 < \rho_0^{-1}W\}}\right](X_0 - \rho_0^{-1}W)^2. \tag{11}$$

Proof. See Appendix B. □

Remark 3.2 *Proposition 3.1 and Theorem 3.1 have revealed that the nested mean-variance problem $(NMV_t(\gamma_t(X_t)))$ is a well-posed problem in the sense of the existence of a finite subgame perfect Nash equilibrium policy.*

Remark 3.3 *In our behavioral risk aversion model, the functions $F_t^+(\mathbf{K})$ and $F_t^-(\mathbf{K})$ are no longer convex functions with respect to $\mathbf{K}$. However, the optimal investment funds $\mathbf{K}_t^+$ and $\mathbf{K}_t^-$ can be derived off-line via some global search methods, thus reducing the dynamic optimization problem into T static optimization problems.*

Remark 3.4 *In the proofs of Proposition 3.1 and Theorem 3.1, the assumption of $\gamma_t^+ \geq 0$ and $\gamma_t^- \geq 0$ is not used. Therefore, our main results remain valid for more general case with $\gamma_t^+ \in \mathbb{R}$ and $\gamma_t^- \in \mathbb{R}$.*

4 Extension to Cone Constrained Markets

In real financial markets, realizations of ($\mathcal{F}_t$-measurable) admissible policy are often confined in a subset of $\mathbb{R}^n$, instead of the whole space $\mathbb{R}^n$. In this section, we consider the situation that the realizations of admissible policies are required to be in a cone. Besides to represent portfolio restrictions, such cone-type constraints have been widely adopted to model regulatory restrictions, for example, restrictions for no-short selling or non-tradeable assets (see Cuoco, 1997, Li et al., 2001, Napp, 2003, Cui et al., 2014, Cui et al., 2015 and Shi et al., 2015 for detailed examples). We express the feasible set of the realizations of admissible polices as $\mathcal{A}_t = \{\mathbf{u}_t \in \mathbb{R}^n | A\mathbf{u}_t \geq 0,\ A \in \mathbb{R}^{m\times n}\}$. Now, mean-variance investors would face the following cone-constrained nested mean-variance problem,

$$\begin{aligned}
(CNMV_t(\gamma_t(X_t))) \qquad \min_{\mathbf{u}_t}\ & \mathrm{Var}_t(X_T) - \gamma_t(X_t)\mathbb{E}_t[X_T], \\
\text{s.t.}\ & X_{t+1} = s_t X_t + \mathbf{P}_t'\mathbf{u}_t, \\
& X_{j+1} = s_j X_j + \mathbf{P}_j'\mathbf{u}_j^{TC}, \quad j = t+1, \cdots, T-1, \\
& \mathbf{u}_t \in \mathcal{A}_t, \\
& \mathbf{u}_j^{TC} \text{ solves } (MV_j(\gamma_j(X_j))), \quad j = t+1, \cdots, T-1,
\end{aligned} \tag{12}$$

with the problem in the last stage given as

$$
\begin{aligned}
(CNMV_{T-1}(\gamma_{T-1}(X_{T-1}))) \quad \min_{\mathbf{u}_{T-1}} \quad & \mathrm{Var}_{T-1}(X_T) - \gamma_{T-1}(X_{T-1})\mathbb{E}_{T-1}[X_T], \\
\text{s.t.} \quad & X_t = s_{T-1}X_{T-1} + \mathbf{P}'_{T-1}\mathbf{u}_{T-1}, \\
& \mathbf{u}_{T-1} \in \mathcal{A}_T.
\end{aligned} \tag{13}
$$

Theorem 4.1 *The time consistent behavioral portfolio policy of $(CNMV_t(\gamma_t(X_t)))$ is given as follows for $t = 0, \ldots, T-1$,*

$$
\mathbf{u}_t^{TC} = \widetilde{\mathbf{K}}_t^{+}(X_t - \rho_t^{-1}W)1_{\{X_t \geq \rho_t^{-1}W\}} + \widetilde{\mathbf{K}}_t^{-}(X_t - \rho_t^{-1}W)1_{\{X_t < \rho_t^{-1}W\}}. \tag{14}
$$

The optimal investment funds $\widetilde{\mathbf{K}}_t^{+}$ and $\widetilde{\mathbf{K}}_t^{-}$ are given by,

$$
\widetilde{\mathbf{K}}_t^{+} = \operatorname*{argmin}_{\mathbf{K}\in\mathcal{A}_t} F_t^{+}(\mathbf{K}), \quad \widetilde{\mathbf{K}}_t^{-} = \operatorname*{argmin}_{\mathbf{K}\in-\mathcal{A}_t} F_t^{-}(\mathbf{K}),
$$

where $-\mathcal{A}_t = \{-\mathbf{u}_t | \mathbf{u}_t \in \mathcal{A}_t\}$ is the negative cone of $\mathcal{A}_t$, and the deterministic parameters in $F_t^{-}(\mathbf{K})$ and $F_t^{+}(\mathbf{K})$, i.e., a_t^{+}, a_t^{-}, b_t^{+} and b_t^{-}, are computed according to recursive functions (5)-(8) with $\mathbf{K}^{+}$ and $\mathbf{K}_t^{-}$ replaced by $\widetilde{\mathbf{K}}_t^{+}$ and $\widetilde{\mathbf{K}}_t^{-}$, respectively.

Proof. See Appendix C. □

In cone constrained markets, the time consistent behavioral portfolio policy remains a piecewise linear feedback form of the current wealth level with respect to the discounted investment target. The only difference from unconstrained markets is that we need to search the optimal investment funds in an $\mathcal{A}_t$ related cone instead of the entire space.

5 Sensitivity Analysis

In this section, we study a numerical example to analyze the property of the time consistent behavioral portfolio policy proposed in this paper, and compare the time consistent behavioral portfolio policy with the existing time consistent portfolio policy in the literature.

Consider a pension fund consisting of S&P 500 (SP), the index of Emerging Market (EM), Small Stock (MS) of the U.S market and a bank account with annual rate of return equal to 5% ($s_t = 1.05$). Based on the data provided in Elton et al. (2007), we list the expected values, variances and correlation coefficients of the annual rates of return of these three indices in Table 1.

We assume that the annual rates of return of the three risky indices follow a joint lognormal distribution. An investor with initial wealth $X_0 = 1$ is considering an investment opportunity of three years ($T = 3$). And his behavioral risk aversion $\gamma_t(X_t)$ given in equation (1) with $\gamma_0^{+} = \gamma_1^{+} = \gamma_2^{+} = \gamma^{+}$ and $\gamma_0^{-} = \gamma_1^{-} = \gamma_2^{-} = \gamma^{-}$.

Firstly, by simulating 20,000 sample paths for annual rates of return of the three risky indices and adopting a global search method, we can compute the optimal investment funds $\mathbf{K}_t^{+}$, $\mathbf{K}_t^{-}$ and the

Table 1: Data for the asset allocation example

	SP	EM	MS
Expected Return	14%	16%	17%
Standard Deviation	18.5%	30%	24%
Correlation coefficient			
SP	1	0.64	0.79
EM		1	0.75
MS			1

deterministic parameters a_t^+, a_t^-, b_t^+ and b_t^- backwards. We provide these results in Table 2 and Table 3. Please note that $F_t^+(\mathbf{K})$ and $F_t^-(\mathbf{K})$ are now d.c. functions with respect to $\mathbf{K}$ based on Remark 3.1.

Table 2: Optimal investment funds and parameters ($\gamma^+ = 1$)

γ^+	γ^-	$\mathbf{K}_2^+$	$\mathbf{K}_2^-$	a_2^+	a_2^-	b_2^+	b_2^-
1	0.5	[0.6347,-0.0764,0.7221]′	[-0.3174,0.0382,-0.3610]′	0.1349	-0.0675	0.0857	0.0214
1	1	[0.6347,-0.0764,0.7221]′	[-0.6347,0.0764,-0.7220]′	0.1349	-0.1349	0.0857	0.0857
1	1.5	[0.6347,-0.0764,0.7221]′	[-0.9520,0.1146,-1.0831]′	0.1349	-0.2024	0.0857	0.1927
1	2	[0.6347,-0.0764,0.7221]′	[-1.2694,0.1528,-1.4441]′	0.1349	-0.2698	0.0857	0.3427
1	2.5	[0.6347,-0.0764,0.7221]′	[-1.5867,0.1910,-1.8051]′	0.1349	-0.3373	0.0857	0.5354
γ^+	γ^-	$\mathbf{K}_1^+$	$\mathbf{K}_1^-$	a_1^+	a_1^-	b_1^+	b_1^-
1	0.5	[0.4292,-0.0503,0.4775]′	[-0.3274,0.0384,-0.3641]′	0.2492	-0.1388	0.1944	0.0528
1	1	[0.4292,-0.0503,0.4775]′	[-0.6968,0.0814,-0.7687]′	0.2492	-0.2759	0.1944	0.2030
1	1.5	[0.4292,-0.0503,0.4775]′	[-1.0429,0.1188,-1.1317]′	0.2492	-0.3996	0.1944	0.4223
1	2	[0.4292,-0.0503,0.4775]′	[-1.3655,0.1550,-1.4718]′	0.2492	-0.5166	0.1944	0.7102
1	2.5	[0.4292,-0.0503,0.4775]′	[-1.6767,0.1918,-1.8098]′	0.2492	-0.6329	0.1944	1.0802
γ^+	γ^-	$\mathbf{K}_0^+$	$\mathbf{K}_0^-$	a_0^+	a_0^-	b_0^+	b_0^-
1	0.5	[0.3309,-0.0492,0.3432]′	[-0.3505,0.0521,-0.3635]′	0.3505	-0.2129	0.3189	0.0948
1	1	[0.3309,-0.0492,0.3432]′	[-0.7788,0.1159,-0.7974]′	0.3505	-0.4171	0.3189	0.3477
1	1.5	[0.3309,-0.0492,0.3432]′	[-1.1312,0.1669,-1.1423]′	0.3505	-0.5810	0.3189	0.6674
1	2	[0.3309,-0.0492,0.3432]′	[-1.4274,0.2116,-1.4624]′	0.3505	-0.7289	0.3189	1.0643
1	2.5	[0.3309,-0.0492,0.3432]′	[-1.7402,0.2516,-1.7733]′	0.3505	-0.8783	0.3189	1.5860

We can find some interesting features from Table 2 and Table 3. First, the larger the absolute value of $\mathbf{K}_t^-$ (or $\mathbf{K}_t^+$), the larger the absolute value of a_t^- and b_t^- (or a_t^+ and b_t^+). According to Theorem 3.1 and its proof, parameters a_t^-, b_t^-, a_t^+ and b_t^+ are related to the conditional expected value and variance of the terminal wealth. The large positions in risky assets, i.e., the large absolute value of $\mathbf{K}_t^-$ and $\mathbf{K}_t^+$, may result in large conditional expected value and variance of the terminal wealth, i.e., the large absolute value of a_t^-, b_t^-, a_t^+ and b_t^+. Second, the larger the value of γ^- (or γ^+), the larger the absolute value of $\mathbf{K}_t^-$ (or $\mathbf{K}_t^+$). The reason behind is quite straightforward. The larger the value of γ^- (or γ^+), the less risk aversion of the investor at time t in the domain $X_t < \rho_t^{-1}W$ (or

Table 3: Optimal investment portfolios and parameters ($\gamma^- = 1$)

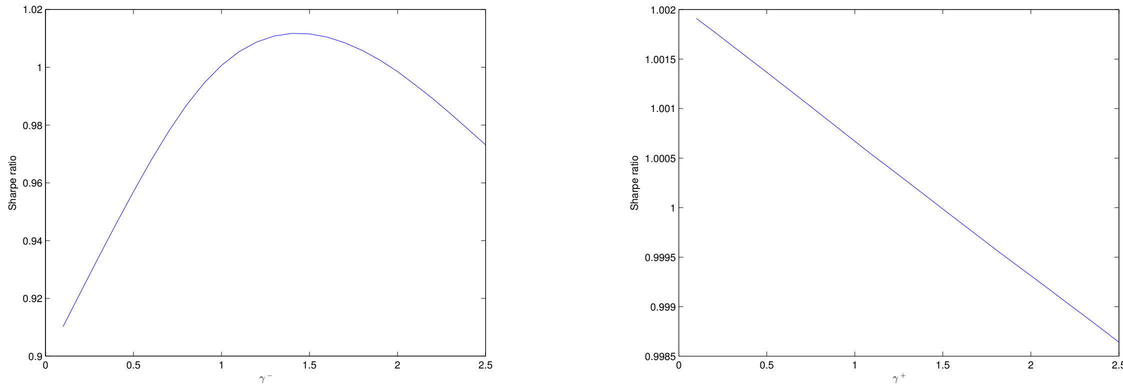
γ^+	γ^-	$\mathbf{K}_2^+$	$\mathbf{K}_2^-$	a_2^+	a_2^-	b_2^+	b_2^-
0.5	1	[0.3173,-0.0382,0.3610]′	[-0.6347,0.0764,-0.7220]′	0.0675	-0.1349	0.0214	0.0857
1	1	[0.6347,-0.0764,0.7221]′	[-0.6347,0.0764,-0.7220]′	0.1349	-0.1349	0.0857	0.0857
1.5	1	[0.9520,-0.1146,1.0831]′	[-0.6347,0.0764,-0.7220]′	0.2024	-0.1349	0.1927	0.0857
2	1	[1.2694,-0.1528,1.4441]′	[-0.6347,0.0764,-0.7220]′	0.2698	-0.1349	0.3427	0.0857
2.5	1	[1.5867,-0.1910,1.8051]′	[-0.6347,0.0764,-0.7220]′	0.3373	-0.1349	0.5354	0.0857
γ^+	γ^-	$\mathbf{K}_1^+$	$\mathbf{K}_1^-$	a_1^+	a_1^-	b_1^+	b_1^-
0.5	1	[0.2526,-0.0296,0.2810]′	[-0.6985,0.0816,-0.7722]′	0.1305	-0.2764	0.0508	0.2036
1	1	[0.4292,-0.0503,0.4775]′	[-0.6968,0.0814,-0.7687]′	0.2492	-0.2759	0.1944	0.2030
1.5	1	[0.5427,-0.0636,0.6037]′	[-0.6951,0.0811,-0.7653]′	0.3562	-0.2755	0.4159	0.2024
2	1	[0.6092,-0.0714,0.6777]′	[-0.6930,0.0809,-0.7615]′	0.4533	-0.2749	0.7029	0.2017
2.5	1	[0.6431,-0.0754,0.7153]′	[-0.6906,0.0806,-0.7578]′	0.5427	-0.2744	1.0467	0.2009
γ^+	γ^-	$\mathbf{K}_0^+$	$\mathbf{K}_0^-$	a_0^+	a_0^-	b_0^+	b_0^-
0.5	1	[0.2149,-0.0319,0.2228]′	[-0.7838,0.1166,-0.8057]′	0.1897	-0.4186	0.0868	0.3499
1	1	[0.3309,-0.0492,0.3432]′	[-0.7788,0.1159,-0.7974]′	0.3505	-0.4171	0.3189	0.3477
1.5	1	[0.3888,-0.0578,0.4032]′	[-0.7736,0.1155,-0.7893]′	0.4866	-0.4156	0.6575	0.3454
2	1	[0.4145,-0.0616,0.4299]′	[-0.7683,0.1148,-0.7815]′	0.6041	-0.4140	1.0761	0.3431
2.5	1	[0.4226,-0.0628,0.4382]′	[-0.7638,0.1143,-0.7739]′	0.7080	-0.4125	1.5587	0.3408

$X_t \geq \rho_t^{-1}W$). Thus, the investor intends to invest a riskier portfolio, which has larger proportion $\mathbf{K}_t^-$ (or $\mathbf{K}_t^+$) with respect to the shortage (or surplus) of the current wealth level. Third, the larger the value of γ^+, the less the absolute value of $\mathbf{K}_t^-$. In contrast, $\mathbf{K}_t^+$ is almost fixed when γ^- is increasing. Actually, the absolute value of $\mathbf{K}_t^+$ also decreases as γ^- increases. But the differences are too small to be identified in Table 2. The reason behind of this finding is more complicated. From the expression of $F_t^-(\mathbf{K})$ and the optimality condition of $\mathbf{K}_t^-$, we can see that the optimal investment fund $\mathbf{K}_t^-$ is influenced by γ^+ through parameters a_{t+1}^+, b_{t+1}^+ and a concomitant event $\{s_t + \mathbf{P}_t'\mathbf{K}_t^- \leq 0\}$. If the probability of the event is large, the parameter γ^+ has large influence on $\mathbf{K}_t^-$. Similar argument holds for γ^- and $\mathbf{K}_t^+$. Taking $\gamma^+ = \gamma^- = 1$ and $t = 1$ as an example, we have

$$Pr(s_1 + \mathbf{P}_1'\mathbf{K}_1^- \leq 0) = 0.0038, \quad Pr(s_1 + \mathbf{P}_1'\mathbf{K}_1^+ < 0) = 0.0000.$$

Thus, comparing to the impact of parameter γ^- on $\mathbf{K}_t^+$, the parameter γ^+ has higher impact on $\mathbf{K}_t^-$.

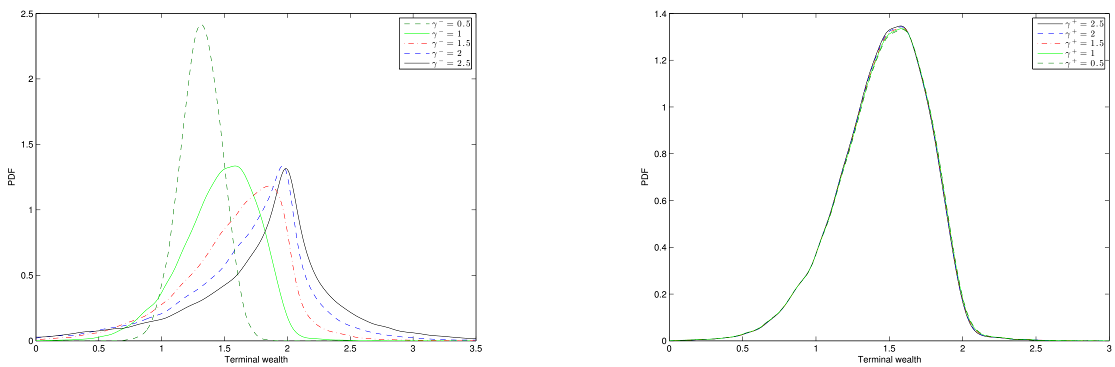
Secondly, we analyze the global investment performance of the time consistent behavioral portfolio policy proposed in this paper. We assume that the investor chooses a very natural investment target $W = 2$, which is twice of the value of X_0. Thus, the investor starts from the domain $X_0 < \rho_0^{-1}W$ and takes $\mathbf{K}_0^-(X_0 - \rho_0^{-1}W)$ positions on risky assets at time 0. Figures 1(a) and 1(b) show the Sharpe ratio with respect to γ^-(with $\gamma^+ = 1$) and γ^+(with $\gamma^- = 1$), respectively. Figures 2(a) and 2(b) show the probability density functions (PDFs) of terminal wealth levels under different γ^- and γ^+ settings.



(a) Sharpe ratio v.s. γ^-

(b) Sharpe ratio v.s. γ^+

Figure 1: Sharpe ratio under different parameter settings



(a) PDFs of different γ^-

(b) PDFs of different γ^+

Figure 2: PDFs of terminal wealth level under different parameter settings

We can see that different investors may achieve different global investment performances under their different time consistent behavioral portfolio policies. However, for the situations of $\gamma^- = 1$, all the investors' time consistent policies are quite similar (see column $\mathbf{K}_t^-$ in Table 3), which results in similar Sharpe ratios and PDFs of the terminal wealth levels. In other words, under our setting (i.e., starting from the domain $X_0 < \rho_0^{-1}W$), the negative risk aversion coefficient γ^- has a higher impact on the global investment performance of the model[2]. The reason can be explained by the following numerical results. For the case of $\gamma^+ = 1$ and $\gamma^- = 1$, it is easy to compute that

$$
\begin{aligned}
Pr(X_1 < \rho_1^{-1}W | X_0 < \rho_0^{-1}W) &= Pr(s_0 + \mathbf{P}_0'\mathbf{K}_0^- > 0) = 0.9947, \\
Pr(X_2 < \rho_2^{-1}W | X_1 < \rho_1^{-1}W) &= Pr(s_1 + \mathbf{P}_1'\mathbf{K}_1^- > 0) = 0.9962, \\
Pr(X_3 < \rho_3^{-1}W | X_2 < \rho_2^{-1}W) &= Pr(s_2 + \mathbf{P}_2'\mathbf{K}_2^- > 0) = 0.9977.
\end{aligned}
$$

[2]When starting from the domain $X_0 \geq \rho_0^{-1}W$, the positive risk aversion coefficient γ^+ would have a higher impact on the global investment performance.

We can see that the investor has very large probability staying in the domains of $X_t < \rho_t^{-1}W$, where γ^- is mainly in effect.

Thirdly, we compare the derived time consistent behavioral portfolio policy with the time consistent portfolio policy of Wu (2013). According to Wu (2013), the investor's portfolio selection problem at the beginning of period t is formulated as

$$(MV_t(\omega(X_t))) \quad \max \quad \mathbb{E}_t[X_T] - \frac{\omega}{X_t}\mathrm{Var}_t(X_T),$$

where X_t is known. By applying Theorem 7 in Wu (2013), we have the linear time consistent portfolio policy $\hat{\mathbf{u}}_t = \hat{\mathbf{K}}_t X_t$, where

$$\hat{\mathbf{K}}_t = \frac{\hat{a}_{t+1} + 2\omega\left((\hat{a}_{t+1})^2 - \hat{b}_{t+1}\right)s_t}{2\omega}(\xi_{t+1})^{-1}\mathbb{E}[\mathbf{P}_t],$$

and parameters $\hat{a}_{t+1}$, $\hat{b}_{t+1}$ and ξ_{t+1} satisfy the following backward equations,

$$\begin{aligned}
\xi_{t+1} &= \hat{b}_{t+1}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - (\hat{a}_{t+1})^2\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t'],\\
\hat{a}_t &= \hat{a}_{t+1}(s_t + \mathbb{E}[\mathbf{P}_t']\hat{\mathbf{K}}_t),\\
\hat{b}_t &= \hat{b}_{t+1}\mathbb{E}\left[(s_t + \mathbf{P}_t'\hat{\mathbf{K}}_t)^2\right],
\end{aligned}$$

with $\hat{a}_T = \hat{b}_T = 1$. To let the comparison make more sense, we choose the derived time consistent behavioral portfolio policy with $\gamma^+ = \gamma^- = 1$, $X_0 = 1$ and $W_0 = 0$ (starting from the domain $X_0 \geq \rho_0^{-1}W$), and the time consistent portfolio policy of Wu (2013) with $\omega = 1$, $X_0 = 1$. The optimal investment funds $\mathbf{K}_t^+$, $\mathbf{K}_t^-$, $\hat{\mathbf{K}}_t$ and the global investment performances of two policies are listed in Table 4.

Table 4: Comparison between two types of time consistent portfolio policies

Panel A: The optimal investment funds $\mathbf{K}_t^+$, $\mathbf{K}_t^-$, $\hat{\mathbf{K}}_t$

t	$\mathbf{K}_t^+$	$\mathbf{K}_t^-$	$\hat{\mathbf{K}}_t$
2	[0.6347,-0.0764,0.7221]′	[-0.6347,0.0764,-0.7220]′	[2.6941,-0.3073,2.8142]′
1	[0.4292,-0.0503,0.4775]′	[-0.6968,0.0814,-0.7687]′	[-1.1927,0.1360,-1.2459]′
0	[0.3309,-0.0492,0.3432]′	[-0.7788,0.1159,-0.7974]′	[-0.3862,0.0440,-0.4034]′

Panel B: The global investment performances

	$\mathbb{E}[X_T\|X_0]$	$\mathrm{Var}(X_T\|X_0)$	Sharpe ratio
$\mathbf{u}_t^{TC}$	1.5086	0.1937	0.7975
$\hat{\mathbf{u}}_t$	1.2551	1.4847	0.0800

In the first period, the derived time consistent behavioral portfolio policy and the time consistent portfolio policy of Wu (2013) have the risky positions of similar size, but in different directions. In the second and third periods, time consistent portfolio policy of Wu (2013) has much larger risky positions. In contrast, the derived time consistent behavioral portfolio policy has much better global investment performance than the time consistent portfolio policy of Wu (2013). Figure 3, which represents the PDFs of the wealth level at time 2 achieved by two policies, reveals the reason behind. We can see that there is a large probability for X_2 to be negative when applying the time consistent portfolio policy of Wu (2013). For the cases of $X_2 < 0$, the risk aversion model $\frac{\omega}{X_2}$ makes no sense. The bad global investment performance can be considered as a shortcoming of the irrational risk aversion model in discrete-time setting.

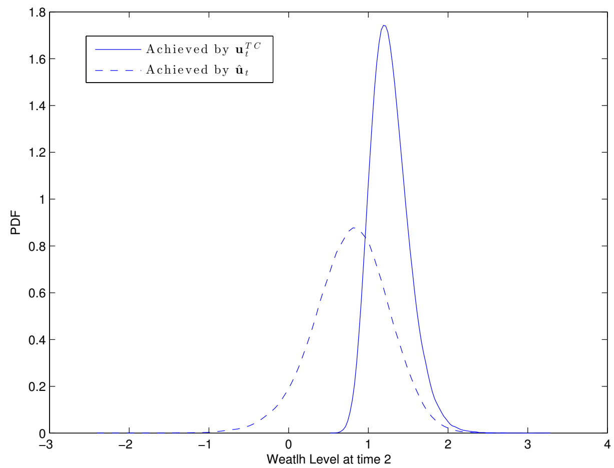


Figure 3: PDFs of the wealth level at time 2

At last, we analyze our data in a cone constrained market. We present our brief results under a no shorting constraint in Table 5. Due to the presence of the no-shorting constraint, the position on risky indices is forced to zero whenever the discounted investment target is larger than the current wealth level, i.e., $\mathbf{K}_t^- = \mathbf{0}$.

In summary, we mainly have three findings through the sensitivity analysis. First, when the risk aversion of the investor decreases in one of the two wealth domains, $X_t \geq \rho_t^{-1}W$ and $X_t < \rho_t^{-1}W$, the investor would like to increase the risky assets' positions in that domain and decrease the risky assets' positions in the other domain. Second, when the investor adopts the time consistent behavioral portfolio policy, it is most likely for him to remain in the domain where he starts. Thus, the risk aversion attitude of the investor in the initial domain has much larger influence on the global investment performance. Third, comparing to the irrational risk aversion model in discrete-time setting $\omega(X_t) = \frac{\omega}{X_t}$, the behavioral risk aversion model proposed in this paper may avoid irrational investment behaviors and achieve much better global investment performance.

6 Conclusions

When we implement a portfolio selection framework under a mean-risk formulation, it is crucial to assess the investor's subjective trade-off between maximizing the expected terminal wealth and minimizing the investment risk. It in turn requires a good understanding of the investor's risk aversion, which is in general an adaptive process of the wealth level. We propose in this paper a behavioral risk aversion model to describe the risk attitude of a mean-variance investor, which

Table 5: Optimal investment portfolios and parameters ($\gamma^+ \geq \gamma^-$)

γ^+	γ^-	$\mathbf{K}_2^+$	$\mathbf{K}_2^-$	a_2^+	a_2^-	b_2^+	b_2^-
1	0.5	[0.6204,0,0.6594]′	[0,0,0]′	0.1346	0	0.0855	0
1	1	[0.6204,0,0.6594]′	[0,0,0]′	0.1346	0	0.0855	0
1	1.5	[0.6204,0,0.6594]′	[0,0,0]′	0.1346	0	0.0855	0
1	2	[0.6204,0,0.6594]′	[0,0,0]′	0.1346	0	0.0855	0
0.5	1	[0.3091,0,0.3324]′	[0,0,0]′	0.0675	0	0.0215	0
1	1	[0.6204,0,0.6594]′	[0,0,0]′	0.1346	0	0.0855	0
1.5	1	[0.9308,0,0.9897]′	[0,0,0]′	0.2020	0	0.1924	0
2	1	[1.2430,0,1.3160]′	[0,0,0]′	0.2690	0	0.3415	0
γ^+	γ^-	$\mathbf{K}_1^+$	$\mathbf{K}_1^-$	a_1^+	a_1^-	b_1^+	b_1^-
1	0.5	[0.4189,0,0.4364]′	[0,0,0]′	0.2484	0	0.1936	0
1	1	[0.4189,0,0.4364]′	[0,0,0]′	0.2484	0	0.1936	0
1	1.5	[0.4189,0,0.4364]′	[0,0,0]′	0.2484	0	0.1936	0
1	2	[0.4189,0,0.4364]′	[0,0,0]′	0.2484	0	0.1936	0
0.5	1	[0.2450,0,0.2565]′	[0,0,0]′	0.1302	0	0.0507	0
1	1	[0.4189,0,0.4364]′	[0,0,0]′	0.2484	0	0.1936	0
1.5	1	[0.5284,0,0.5516]′	[0,0,0]′	0.3550	0	0.4143	0
2	1	[0.5970,0,0.6193]′	[0,0,0]′	0.4521	0	0.7006	0
γ^+	γ^-	$\mathbf{K}_0^+$	$\mathbf{K}_0^-$	a_0^+	a_0^-	b_0^+	b_0^-
1	0.5	[0.3171,0,0.3047]′	[0,0,0]′	0.3491	0	0.3170	0
1	1	[0.3171,0,0.3047]′	[0,0,0]′	0.3491	0	0.3170	0
1	1.5	[0.3171,0,0.3047]′	[0,0,0]′	0.3491	0	0.3170	0
1	2	[0.3171,0,0.3047]′	[0,0,0]′	0.3491	0	0.3170	0
0.5	1	[0.2071,0,0.1973]′	[0,0,0]′	0.1890	0	0.0864	0
1	1	[0.3171,0,0.3047]′	[0,0,0]′	0.3491	0	0.3170	0
1.5	1	[0.3747,0,0.3590]′	[0,0,0]′	0.4851	0	0.6547	0
2	1	[0.3999,0,0.3825]′	[0,0,0]′	0.6023	0	1.0720	0

takes the piecewise linear form of the surplus or the shortage with respect to some preset investment target. Our new risk aversion model is flexible enough to incorporate the features of "house money" and "breaking-even", thus enriching the modeling power to capture the essence of the investor's risk attitude.

As the resulting dynamic mean-variance model with adaptive risk aversion is time inconsistent, we focus on its time consistent policy by solving a nested mean-variance game formulation. Fortunately, we obtain the semi-analytical time consistent behavioral portfolio policy and reveal its piecewise linear form of the excess wealth level with respect to the discounted wealth target. Our numerical analysis sheds light on some prominent features of the time consistent behavioral portfolio policy established in our theoretical derivations.

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Appendix

Appendix A: The Proof of Proposition 3.1

Proof. Define $\xi = \|\mathbf{K}\|$, $\mathbf{L} = \mathbf{K}\xi^{-1}$ (which implies $\|\mathbf{L}\| = 1$) and $y_t = \mathbf{P}_t'\mathbf{L}$. Then, for any $\mathbf{L}$, we have $M \geq \mathrm{Var}(y_t) = \mathbf{L}'\mathrm{Cov}(\mathbf{P}_t)\mathbf{L} > 0$, where M is the largest eigenvalue of $\mathrm{Cov}(\mathbf{P}_t)$.

If $y_t 1_{\{y_t \geq 0\}}$ is zero, (i.e., $y_t \leq 0$ almost surely), we can construct an arbitrage portfolio by shorting $\mathbf{L}$ and holding $\mathbf{L}'\mathbf{1}$ risk-free asset. Similarly, if $y_t 1_{\{y_t < 0\}}$ is zero, (i.e., $y_t \geq 0$ almost surely), we also can construct an arbitrage portfolio by holding $\mathbf{L}$ and shorting $\mathbf{L}'\mathbf{1}$ risk-free asset. Thus, we conclude that $y_t 1_{\{y_t \geq 0\}}$ and $y_t 1_{\{y_t < 0\}}$ are nontrivial random variables with finite second moment.

Moreover, $\mathbf{P}_t$ is absolutely integrable, so do y_t, $y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}}$ and $y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}$. Then, for given $\mathbf{L}$, we have

$$F_t^+(\mathbf{K}) = \tilde{F}_t^+(\xi),$$

where

$$\begin{aligned}
\tilde{F}_t^+(\xi) =& \rho_{t+1}^2 \mathrm{Var}(y_t)\xi^2 + \mathbb{E}\left[(2\rho_{t+1}a_{t+1}^+ + b_{t+1}^+)(s_t + \xi y_t)^2 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right] \\
&+ \mathbb{E}\left[(2\rho_{t+1}a_{t+1}^- + b_{t+1}^-)(s_t + \xi y_t)^2 1_{\{y_t < \frac{-s_t}{\xi}\}}\right] \\
&- \left(\mathbb{E}\left[a_{t+1}^+(s_t + \xi y_t) 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right] + \mathbb{E}\left[a_{t+1}^-(s_t + \xi y_t) 1_{\{y_t < \frac{-s_t}{\xi}\}}\right]\right)^2 \\
&- 2\rho_{t+1}\left(\mathbb{E}\left[a_{t+1}^+(s_t + \xi y_t) 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right] + \mathbb{E}\left[a_{t+1}^-(s_t + \xi y_t) 1_{\{y_t < \frac{-s_t}{\xi}\}}\right]\right)(s_t + \mathbb{E}[y_t]\xi) \\
&- \gamma_t^+\left(\mathbb{E}\left[a_{t+1}^+(s_t + \xi y_t) 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right] + \mathbb{E}\left[a_{t+1}^-(s_t + \xi y_t) 1_{\{y_t < \frac{-s_t}{\xi}\}}\right]\right) \\
&- \rho_{t+1}\gamma_t^+(s_t + \mathbb{E}[y_t]\xi).
\end{aligned}$$

Furthermore, we have

$$
\begin{aligned}
\tilde{F}_t^+(\xi) \geq\, & \rho_{t+1}^2 \mathrm{Var}(y_t)\xi^2 + (a_{t+1}^+)^2 \mathbb{E}\left[y_t^2 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right]\xi^2 + (a_{t+1}^-)^2 \mathbb{E}\left[y_t^2 1_{\{y_t < \frac{-s_t}{\xi}\}}\right]\xi^2 \\
& - \left(a_{t+1}^+ \mathbb{E}\left[y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right]\xi + a_{t+1}^- \mathbb{E}\left[y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right]\xi\right)^2 \\
& + 2\rho_{t+1}\left(a_{t+1}^+ \mathbb{E}\left[y_t^2 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right]\xi^2 + a_{t+1}^- \mathbb{E}\left[y_t^2 1_{\{y_t < \frac{-s_t}{\xi}\}}\right]\xi^2\right) \\
& - 2\rho_{t+1}\left(a_{t+1}^+ \mathbb{E}\left[y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right] + a_{t+1}^- \mathbb{E}\left[y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right]\right)\mathbb{E}[y_t]\xi^2 + O(\xi) \\
=\, & \rho_{t+1}^2 \mathrm{Var}\left(y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right)\xi^2 + \rho_{t+1}^2 \mathrm{Var}\left(y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right)\xi^2 \\
& + 2\rho_{t+1}^2\left(\mathbb{E}\left[y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}} y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right] - \mathbb{E}\left[y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right]\mathbb{E}\left[y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right]\right)\xi^2 \\
& + (a_{t+1}^+)^2 \mathrm{Var}\left(y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right)\xi^2 + (a_{t+1}^-)^2 \mathrm{Var}\left(y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right)\xi^2 \\
& + 2a_{t+1}^+ a_{t+1}^-\left(\mathbb{E}\left[y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}} y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right] - \mathbb{E}\left[y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right]\mathbb{E}\left[y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right]\right)\xi^2 \\
& + 2\rho_{t+1}\left(a_{t+1}^+ \mathrm{Var}\left(y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right)\xi^2 + a_{t+1}^- \mathrm{Var}\left(y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right)\xi^2\right) \\
& + 2\rho_{t+1} a_{t+1}^+\left(\mathbb{E}\left[y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}} y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right] - \mathbb{E}\left[y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right]\mathbb{E}\left[y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right]\right)\xi^2 \\
& + 2\rho_{t+1} a_{t+1}^-\left(\mathbb{E}\left[y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}} y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right] - \mathbb{E}\left[y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}}\right]\mathbb{E}\left[y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right]\right)\xi^2 \\
& + O(\xi) \\
=\, & \left[\rho_{t+1} + a_{t+1}^+, \rho_{t+1} + a_{t+1}^-\right] \mathrm{Cov}\begin{bmatrix} y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}} \\ y_t 1_{\{y_t < \frac{-s_t}{\xi}\}} \end{bmatrix}\begin{bmatrix} \rho_{t+1} + a_{t+1}^+ \\ \rho_{t+1} + a_{t+1}^- \end{bmatrix}\xi^2 + O(\xi),
\end{aligned}
$$

where $O(\xi)$ is the infinity of the same order as ξ and the second equality holds due to the fact of $\mathbb{E}\left[y_t 1_{\{y_t \geq \frac{-s_t}{\xi}\}} y_t 1_{\{y_t < \frac{-s_t}{\xi}\}}\right] = 0$. Hence,

$$
\begin{aligned}
\lim_{\xi \to +\infty} \tilde{F}_t^+(\xi) = & \lim_{\xi \to +\infty} \left[\rho_{t+1} + a_{t+1}^+, \rho_{t+1} + a_{t+1}^-\right] \mathrm{Cov}\begin{bmatrix} y_t 1_{\{y_t \geq 0\}} \\ y_t 1_{\{y_t < 0\}} \end{bmatrix}\begin{bmatrix} \rho_{t+1} + a_{t+1}^+ \\ \rho_{t+1} + a_{t+1}^- \end{bmatrix}\xi^2 \\
& + O(\xi) = +\infty.
\end{aligned}
$$

Based on the discussion for all possible $\mathbf{L}$, we make our conclusion for $F_t^+(\mathbf{K})$. Similarly we can prove the result of $F_t^-(\mathbf{K})$. $\square$

Appendix B: The Proof of Theorem 3.1

Proof. Let $Y_t = X_t - \rho_t^{-1} W$. Then,

$$\begin{aligned} Y_{t+1} &= X_{t+1} - \rho_{t+1}^{-1} W \\ &= s_t X_t + \mathbf{P}_t' \mathbf{u}_t - \rho_{t+1}^{-1} W \\ &= s_t (X_t - \rho_t^{-1} W) + \mathbf{P}_t' \mathbf{u}_t \\ &= s_t Y_t + \mathbf{P}_t' \mathbf{u}_t, \end{aligned}$$

and $\gamma_t(X_t)$ can be re-written into

$$\gamma_t(X_t) = \hat{\gamma}_t(Y_t) = \begin{cases} \gamma_t^+ Y_t, & \text{if } Y_t \geq 0, \\ -\gamma_t^- Y_t, & \text{if } Y_t < 0. \end{cases}$$

Also, we have $\mathrm{Var}_t(X_T) = \mathrm{Var}_t(Y_T)$ according to the variance property. Hence, problem $(MV_t(\gamma_t(X_t)))$ in (2) can be reduced into the following equivalent problem,

$$\begin{aligned} \min \quad & \mathrm{Var}_t(Y_T) - \hat{\gamma}_t(Y_t)\mathbb{E}_t[Y_T] - \hat{\gamma}_t(Y_t) W, \\ \text{s.t.} \quad & Y_{j+1} = s_j Y_j + \mathbf{P}_j' \mathbf{u}_j, \quad j = t, t+1, \cdots, T-1, \end{aligned} \tag{15}$$

where $\mathrm{Var}_t(Y_T) = \mathrm{Var}(Y_T|Y_t)$ and $\mathbb{E}_t[Y_T] = \mathbb{E}[Y_T|Y_t]$.

At time t $(t = 0, 1, \cdots, T)$, the investor faces the following optimization problem,

$$\min_{\mathbf{u}_t} \quad J_t(Y_t; \mathbf{u}_t) = \Big(\mathbb{E}_t[Y_T^2] - (\mathbb{E}_t[Y_T])^2\Big) - \hat{\gamma}_t(Y_t)\mathbb{E}_t[Y_T] - \hat{\gamma}_t(Y_t) W, \tag{16}$$

where the conditional expectations $\mathbb{E}_t[Y_T] = \mathbb{E}[Y_T|Y_t]$ and $\mathbb{E}_t[Y_T^2] = \mathbb{E}[Y_T^2|Y_t]$ are computed along the policy $\{\mathbf{u}_t, \mathbf{u}_{t+1}^{TC}, \cdots, \mathbf{u}_{T-1}^{TC}\}$.

We now prove by induction that the following two expressions,

$$\mathbb{E}_t[Y_T] = \rho_t Y_t + a_t^+ Y_t 1_{\{Y_t \geq 0\}} + a_t^- Y_t 1_{\{Y_t < 0\}}, \tag{17}$$

$$\mathbb{E}_t[Y_T^2] = \rho_t^2 Y_t^2 + (2\rho_t a_t^+ + b_t^+) Y_t^2 1_{\{Y_t \geq 0\}} + (2\rho_t a_t^- + b_t^-) Y_t^2 1_{\{Y_t < 0\}}, \tag{18}$$

hold along the time consistent policy, $\{\mathbf{u}_t^{TC}, \mathbf{u}_{t+1}^{TC}, \cdots, \mathbf{u}_{T-1}^{TC}\}$, at time t.

At time T, we have

$$\mathbb{E}_T[Y_T] = Y_T, \quad \mathbb{E}_T[Y_T^2] = Y_T^2,$$

with $a_T^+ = a_T^- = 0$ and $b_T^+ = b_T^- = 0$. Assume that expressions of the first moment and the second moment in (17) and (18), respectively, hold at time $t+1$ along the time consistent policy $\{\mathbf{u}_{t+1}^{TC}, \cdots, \mathbf{u}_{T-1}^{TC}\}$. We will prove that these two expressions still hold at time t and the corresponding time consistent policy is given by (9).

As the dynamics of Y_t at period t is given by

$$Y_{t+1} = s_t Y_t + \mathbf{P}_t' \mathbf{u}_t.$$

It follows from the policy $\{\mathbf{u}_t, \mathbf{u}_{t+1}^{TC}, \cdots, \mathbf{u}_{T-1}^{TC}\}$ that we have

$$
\begin{aligned}
&\mathbb{E}_t[Y_T] = \mathbb{E}_t\big[\mathbb{E}_{t+1}[Y_T]\big] \\
&= \mathbb{E}_t\big[\rho_{t+1}Y_{t+1} + a_{t+1}^{+}Y_{t+1}1_{\{Y_{t+1}\geq 0\}} + a_{t+1}^{-}Y_{t+1}1_{\{Y_{t+1}<0\}}\big] \\
&= \mathbb{E}_t[\rho_{t+1}(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)] + \mathbb{E}_t\left[a_{t+1}^{+}(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)1_{\{s_tY_t+\mathbf{P}_t'\mathbf{u}_t\geq 0\}}\right] \\
&\quad + \mathbb{E}_t\Big[a_{t+1}^{-}(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)1_{\{s_tY_t+\mathbf{P}_t'\mathbf{u}_t<0\}}\Big]
\end{aligned} \tag{19}
$$

and

$$
\begin{aligned}
&\mathbb{E}_t[Y_T^2] = \mathbb{E}_t\big[\mathbb{E}_{t+1}[Y_T^2]\big] \\
&= \mathbb{E}_t\Big[\rho_{t+1}^2Y_{t+1}^2 + (2\rho_{t+1}a_{t+1}^{+} + b_{t+1}^{+})Y_{t+1}^21_{\{Y_{t+1}\geq 0\}} \\
&\quad + (2\rho_{t+1}a_{t+1}^{-} + b_{t+1}^{-})Y_{t+1}^21_{\{Y_{t+1}<0\}}\Big] \\
&= \mathbb{E}_t[\rho_{t+1}^2(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)^2] \\
&\quad + \mathbb{E}_t\Big[(2\rho_{t+1}a_{t+1}^{+} + b_{t+1}^{+})(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)^21_{\{s_tY_t+\mathbf{P}_t'\mathbf{u}_t\geq 0\}}\Big] \\
&\quad + \mathbb{E}_t\Big[(2\rho_{t+1}a_{t+1}^{-} + b_{t+1}^{-})(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)^21_{\{s_tY_t+\mathbf{P}_t'\mathbf{u}_t<0\}}\Big].
\end{aligned} \tag{20}
$$

For $Y_t > 0$, we denote any admissible policy as $\mathbf{u}_t = \mathbf{K}Y_t$ with $\mathbf{K} \in \mathbb{R}^n$. Then the cost functional can be expressed as

$$
\begin{aligned}
&J_t(Y_t;\mathbf{u}_t) = \Big(\mathbb{E}_t[Y_T^2] - (\mathbb{E}_t[Y_T])^2\Big) - \gamma_t^{+}Y_t\mathbb{E}_t[Y_T] - \gamma_t^{+}Y_tW \\
&= \mathbb{E}_t[\rho_{t+1}^2(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)^2] - \big(\mathbb{E}_t[\rho_{t+1}(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)]\big)^2 \\
&\quad + \mathbb{E}_t\Big[(2\rho_{t+1}a_{t+1}^{+} + b_{t+1}^{+})(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)^21_{\{s_tY_t+\mathbf{P}_t'\mathbf{u}_t\geq 0\}}\Big] \\
&\quad + \mathbb{E}_t\Big[(2\rho_{t+1}a_{t+1}^{-} + b_{t+1}^{-})(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)^21_{\{s_tY_t+\mathbf{P}_t'\mathbf{u}_t<0\}}\Big] \\
&\quad - \Big(\mathbb{E}_t\Big[a_{t+1}^{+}(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)1_{\{s_tY_t+\mathbf{P}_t'\mathbf{u}_t\geq 0\}}\Big] + \mathbb{E}_t\Big[a_{t+1}^{-}(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)1_{\{s_tY_t+\mathbf{P}_t'\mathbf{u}_t<0\}}\Big]\Big)^2 \\
&\quad - 2\rho_{t+1}\Big(\mathbb{E}_t\Big[a_{t+1}^{+}(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)1_{\{s_tY_t+\mathbf{P}_t'\mathbf{u}_t\geq 0\}}\Big] \\
&\quad + \mathbb{E}_t\Big[a_{t+1}^{-}(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)1_{\{s_tY_t+\mathbf{P}_t'\mathbf{u}_t<0\}}\Big]\Big)\mathbb{E}_t[s_tY_t + \mathbf{P}_t'\mathbf{u}_t] \\
&\quad - \gamma_t^{+}Y_t\Big(\mathbb{E}_t\left[a_{t+1}^{+}(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)1_{\{s_tY_t+\mathbf{P}_t'\mathbf{u}_t\geq 0\}}\right] + \mathbb{E}_t\Big[a_{t+1}^{-}(s_tY_t + \mathbf{P}_t'\mathbf{u}_t)1_{\{s_tY_t+\mathbf{P}_t'\mathbf{u}_t<0\}}\Big]\Big) \\
&\quad - \rho_{t+1}\gamma_t^{+}Y_t\mathbb{E}_t[s_tY_t + \mathbf{P}_t'\mathbf{u}_t] - \gamma_t^{+}Y_tW \\
&= Y_t^2F_t^{+}(\mathbf{K}) - \gamma_t^{+}Y_tW.
\end{aligned}
$$

Applying Proposition 3.1 yields the optimal time consistent policy at time t,

$$
\mathbf{u}_t^{TC} = \underset{\mathbf{u}_t\in\mathbb{R}^n}{\operatorname{argmin}}\, J_t(Y_t;\mathbf{u}_t) = \mathbf{K}_t^{+}Y_t.
$$

Then, substituting the above optimal time consistent policy back into (19) and (20) gives rise to

$$
\begin{aligned}
\mathbb{E}_t[Y_T] &= \rho_t Y_t + Y_t\Big(\rho_{t+1}\mathbb{E}[\mathbf{P}_t']\mathbf{K}_t^+ + \mathbb{E}\left[a_{t+1}^+(s_t+\mathbf{P}_t'\mathbf{K}_t^+)1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+\geq 0\}}\right] \\
&\quad + \mathbb{E}\left[a_{t+1}^-(s_t+\mathbf{P}_t'\mathbf{K}_t^+)1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+< 0\}}\right]\Big) \\
&= \rho_t Y_t + a_t^+ Y_t
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}_t[Y_T^2] &= \rho_t^2 Y_t^2 + 2\rho_t Y_t^2\Big(\rho_{t+1}\mathbb{E}[\mathbf{P}_t']\mathbf{K}_t^+ + \mathbb{E}\left[a_{t+1}^+(s_t+\mathbf{P}_t'\mathbf{K}_t^+)1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+\geq 0\}}\right] \\
&\quad + \mathbb{E}\left[a_{t+1}^-(s_t+\mathbf{P}_t'\mathbf{K}_t^+)1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+< 0\}}\right]\Big) \\
&\quad + \Big(\rho_{t+1}^2(\mathbf{K}_t^+)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\mathbf{K}_t^+ + 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^+(s_t+\mathbf{P}_t'\mathbf{K}_t^+)\mathbf{P}_t'\mathbf{K}_t^+1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+\geq 0\}}\right] \\
&\quad + 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^-(s_t+\mathbf{P}_t'\mathbf{K}_t^+)\mathbf{P}_t'\mathbf{K}_t^+1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+< 0\}}\right] \\
&\quad + \mathbb{E}\left[b_{t+1}^+(s_t+\mathbf{P}_t'\mathbf{K}_t^+)^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+\geq 0\}}\right] + \mathbb{E}\left[b_{t+1}^-(s_t+\mathbf{P}_t'\mathbf{K}_t^+)^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^+< 0\}}\right]\Big)Y_t^2 \\
&= \rho_t^2 Y_t^2 + (2\rho_t a_t^+ + b_t^+)Y_t^2.
\end{aligned}
$$

Furthermore,

$$
\mathrm{Var}_t(Y_T) = \mathbb{E}_t[Y_T^2] - (\mathbb{E}_t[Y_T])^2 = (b_t^+ - (a_t^+)^2)Y_t^2 \geq 0,
$$

implies $b_t^+ - (a_t^+)^2 \geq 0$.

For $Y_t < 0$, we denote any admissible policy as $\mathbf{u}_t = \mathbf{K}Y_t$ with $\mathbf{K} \in \mathbb{R}^n$. Then the cost functional can be expressed as

$$
\begin{aligned}
J_t(Y_t;\mathbf{u}_t) &= Y_t^2\Big\{\rho_{t+1}^2\mathbf{K}'(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t])\mathbf{K} \\
&\quad + \mathbb{E}\left[(2\rho_{t+1}a_{t+1}^+ + b_{t+1}^+)(s_t+\mathbf{P}_t'\mathbf{K})^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}\leq 0\}}\right] \\
&\quad + \mathbb{E}\left[(2\rho_{t+1}a_{t+1}^- + b_{t+1}^-)(s_t+\mathbf{P}_t'\mathbf{K})^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}> 0\}}\right] \\
&\quad - \Big(\mathbb{E}\left[a_{t+1}^+(s_t+\mathbf{P}_t'\mathbf{K})1_{\{s_t+\mathbf{P}_t'\mathbf{K}\leq 0\}}\right] + \mathbb{E}\left[a_{t+1}^-(s_t+\mathbf{P}_t'\mathbf{K})1_{\{s_t+\mathbf{P}_t'\mathbf{K}> 0\}}\right]\Big)^2 \\
&\quad - 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^+(s_t+\mathbf{P}_t'\mathbf{K})1_{\{s_t+\mathbf{P}_t'\mathbf{K}\leq 0\}}\right](s_t+\mathbb{E}[\mathbf{P}_t']\mathbf{K}) \\
&\quad - 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^-(s_t+\mathbf{P}_t'\mathbf{K})1_{\{s_t+\mathbf{P}_t'\mathbf{K}> 0\}}\right](s_t+\mathbb{E}[\mathbf{P}_t']\mathbf{K}) \\
&\quad + \gamma_t^-\Big(\mathbb{E}\left[a_{t+1}^+(s_t+\mathbf{P}_t'\mathbf{K})1_{\{s_t+\mathbf{P}_t'\mathbf{K}\leq 0\}}\right] + \mathbb{E}\left[a_{t+1}^-(s_t+\mathbf{P}_t'\mathbf{K})1_{\{s_t+\mathbf{P}_t'\mathbf{K}> 0\}}\right]\Big) \\
&\quad + \rho_{t+1}\gamma_t^-(s_t+\mathbb{E}[\mathbf{P}_t']\mathbf{K})\Big\} + \gamma_t^- Y_t W \\
&= Y_t^2 F_t^-(\mathbf{K}) + \gamma_t^- Y_t W.
\end{aligned}
$$

Applying Proposition 3.1 yields the optimal time consistent policy at time t,

$$
\mathbf{u}_t^{TC} = \underset{\mathbf{u}_t\in\mathbb{R}^n}{\operatorname{argmin}}\, J_t(Y_t;\mathbf{u}_t) = \mathbf{K}_t^- Y_t.
$$

Then, substituting the above optimal time consistent policy back into (19) and (20) gives rise to

$$
\begin{aligned}
\mathbb{E}_t[Y_T] &= \rho_t Y_t + Y_t\Big(\rho_{t+1}\mathbb{E}[\mathbf{P}_t']\mathbf{K}_t^- + \mathbb{E}\left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_t^-)1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^-\le 0\}}\right] \\
&\quad + \mathbb{E}\left[a_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}_t^-)1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^- > 0\}}\right]\Big) \\
&= \rho_t Y_t + a_t^- Y_t
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}_t[Y_T^2] &= \rho_t^2 Y_t^2 + 2\rho_t Y_t^2\Big(\rho_{t+1}\mathbb{E}[\mathbf{P}_t']\mathbf{K}_t^- + \mathbb{E}\left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_t^-)1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^-\le 0\}}\right] \\
&\quad + \mathbb{E}\left[a_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}_t^-)1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^- > 0\}}\right]\Big) \\
&\quad + \Big(\rho_{t+1}^2(\mathbf{K}_t^-)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\mathbf{K}_t^- + 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_t^-)\mathbf{P}_t'\mathbf{K}_t^- 1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^-\le 0\}}\right] \\
&\quad + 2\rho_{t+1}\mathbb{E}\left[a_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}_t^-)\mathbf{P}_t'\mathbf{K}_t^- 1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^- > 0\}}\right] \\
&\quad + \mathbb{E}\left[b_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_t^-)^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^-\le 0\}}\right] + \mathbb{E}\left[b_{t+1}^-(s_t + \mathbf{P}_t'\mathbf{K}_t^-)^2 1_{\{s_t+\mathbf{P}_t'\mathbf{K}_t^- > 0\}}\right]\Big)Y_t^2 \\
&= \rho_t^2 Y_t^2 + (2\rho_t a_t^- + b_t^-)Y_t^2.
\end{aligned}
$$

Furthermore,

$$
\mathrm{Var}_t(Y_T) = \mathbb{E}_t[Y_T^2] - (\mathbb{E}_t[Y_T])^2 = (b_t^- - (a_t^-)^2)Y_t^2 \ge 0,
$$

implies $b_t^- - (a_t^-)^2 \ge 0$.

For $Y_t = 0$, the cost functional reduces to the conditional variance of the terminal wealth along policy $\{\mathbf{u}_t, \mathbf{u}_{t+1}^{TC}, \cdots, \mathbf{u}_{T-1}^{TC}\}$, which can be expressed as

$$
\begin{aligned}
J_t(Y_t; \mathbf{u}_t) &= \rho_{t+1}^2\mathbf{u}_t'(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t])\mathbf{u}_t \\
&\quad + \mathbb{E}\left[b_{t+1}^+(\mathbf{P}_t'\mathbf{u}_t)^2 1_{\{\mathbf{P}_t'\mathbf{u}_t \ge 0\}}\right] + \mathbb{E}\left[b_{t+1}^-(\mathbf{P}_t'\mathbf{u}_t)^2 1_{\{\mathbf{P}_t'\mathbf{u}_t < 0\}}\right] \\
&\quad - \left(\mathbb{E}\left[a_{t+1}^+\mathbf{P}_t'\mathbf{u}_t 1_{\{\mathbf{P}_t'\mathbf{u}_t \ge 0\}}\right] + \mathbb{E}\left[a_{t+1}^-\mathbf{P}_t'\mathbf{u}_t 1_{\{\mathbf{P}_t'\mathbf{u}_t < 0\}}\right]\right)^2 \\
&\quad + 2\rho_{t+1}\left(\mathbb{E}\left[a_{t+1}^+(\mathbf{P}_t'\mathbf{u}_t)^2 1_{\{\mathbf{P}_t'\mathbf{u}_t \ge 0\}}\right] + \mathbb{E}\left[a_{t+1}^-(\mathbf{P}_t'\mathbf{u}_t)^2 1_{\{\mathbf{P}_t'\mathbf{u}_t < 0\}}\right]\right) \\
&\quad - 2\rho_{t+1}\Big(\mathbb{E}\left[a_{t+1}^+\mathbf{P}_t'\mathbf{u}_t 1_{\{\mathbf{P}_t'\mathbf{u}_t \ge 0\}}\right] + \mathbb{E}\left[a_{t+1}^-\mathbf{P}_t'\mathbf{u}_t 1_{\{\mathbf{P}_t'\mathbf{u}_t < 0\}}\right]\Big)\mathbb{E}[\mathbf{P}_t']\mathbf{u}_t \\
&\ge 0.
\end{aligned}
$$

It is not difficult to conclude that $\mathbf{u}_t^{TC} = \underset{\mathbf{u}_t \in \mathbb{R}^n}{\operatorname{argmin}} J_t(Y_t; \mathbf{u}_t) = \mathbf{0}$.

Therefore, along the time consistent policy $\{\mathbf{u}_t^{TC}, \mathbf{u}_{t+1}^{TC}, \cdots, \mathbf{u}_{T-1}^{TC}\}$, expressions (17) and (18) hold at time t, which completes our proof. □

Appendix C: The Proof of Theorem 4.1

Proof. Following the technique in the proof of Theorem 3.1, we can derive the main results directly with the following specifics.

i) For $X_t > \rho_t^{-1} W$, we denote any admissible policy as $\mathbf{u}_t = \mathbf{K}(X_t - \rho_t^{-1} W)$ with $\mathbf{K} \in \mathcal{A}_t$.
ii) For $X_t < \rho_t^{-1} W$, we denote any admissible policy as $\mathbf{u}_t = \mathbf{K}(X_t - \rho_t^{-1} W)$ with $\mathbf{K} \in -\mathcal{A}_t$, where $-\mathcal{A}_t$ is the negative cone of $\mathcal{A}_t$.
iii) For $X_t = \rho_t^{-1} W$, we can similarly prove $\mathbf{u}_t^{TC} = \mathbf{0}$.
Therefore, we have

$$\widetilde{\mathbf{K}}_t^{+} = \underset{\mathbf{K} \in \mathcal{A}_t}{\operatorname{argmin}} F_t^{+}(\mathbf{K}), \quad \widetilde{\mathbf{K}}_t^{-} = \underset{\mathbf{K} \in -\mathcal{A}_t}{\operatorname{argmin}} F_t^{-}(\mathbf{K}).$$

This completes the proof. □