## A RESOLVENT CRITERION FOR NORMALITY

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ABSTRACT. Given a normal matrix A and an arbitrary square matrix B (not necessarily of the same size), what relationships between A and B, if any, guarantee that B is also a normal matrix? We provide an answer to this question in terms of pseudospectra and norm behavior. In doing so, we prove that a certain distance formula, known to be a necessary condition for normality, is in fact sufficient and demonstrates that the spectrum of a matrix can be used to recover the spectral norm of its resolvent precisely when the matrix is normal. These results lead to new normality criteria and other interesting consequences.

## 1. NORMALITY, PSEUDOSPECTRA AND NORM BEHAVIOR.

Let n be a natural number and let  $\mathbb{M}_n$  denote the set of all  $n \times n$  matrices with entries in the complex plane  $\mathbb{C}$ . Denote by I and 0 the identity and zero matrices, respectively, whose sizes are understood in context.

For  $A \in \mathbb{M}_n$ , let  $A^*$  denote the conjugate transpose of A. The spectrum  $\sigma(A)$  of A is the set of all of its eigenvalues; that is,

$$\sigma(A) = \{ z \in \mathbb{C} : zI - A \text{ is not invertible} \} = \{ z \in \mathbb{C} : \det(zI - A) = 0 \}$$

In this article, we are interested in the notion of *normality*. Recall that a matrix  $A \in \mathbb{M}_n$  is said to be **normal** if it commutes with its conjugate transpose, i.e., if

$$A^*A = AA^*.$$

In the case that  $A^*A = AA^* = I$ , the matrix A is called **unitary**<sup>1</sup>. Equivalently, A is unitary if it is invertible and has  $A^*$  as its inverse. Two matrices A and B are said to be **unitarily similar** if there is a unitary matrix U so that  $A = UBU^*$ .

The simplest example of an  $n \times n$  normal matrix is a **diagonal** matrix, that is, a matrix of the form

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}.$$
 (1)

Notice that the eigenvalues of  $\Lambda$  are precisely the entries on its main diagonal and so  $\sigma(\Lambda) = \{\lambda_1, \ldots, \lambda_n\}.$ 

Normal matrices are essentially diagonal according to the spectral theorem (for normal matrices) which states that a matrix  $N \in \mathbb{M}_n$  is normal if and only N is unitarily similar to a diagonal matrix. (See Theorem 2.5.4 on page 101 in [5].)

So we ask, if given a normal matrix A and an arbitrary square matrix B, what relationships between A and B, if any, guarantee that B is also a normal matrix?

<sup>&</sup>lt;sup>1</sup>For matrices with real entries, unitary matrices are called orthogonal.

Certainly, by the Spectral Theorem stated above, a "non-metric" relationship that guarantees the normality of B from that of A is unitary similarity. This however requires that A and B also have the same size.

And so instead we ask,

Given a normal matrix A and an arbitrary square matrix B (not necessarily of the same size), what "metric" relationships between A and B, if any, guarantee that B is also a normal matrix?

Two such relationships that come to mind are "identical pseudospectra" and "same norm behavior." To describe these notions, we first need to choose a norm for matrices.

For  $A \in \mathbb{M}_n$ , we define the (spectral) norm ||A|| of A by

$$||A|| = \sup\{||Av||_2 : ||v||_2 = 1\}$$

where  $||v||_2$  denotes the Euclidean norm of the vector  $v \in \mathbb{C}^n$ , i.e.,

$$||v||_2 = \sqrt{|v_1|^2 + \dots + |v_n|^2}$$
 if  $v = (v_1, \dots, v_n)$ .

(The subscript allows one to differentiate between the norms.) A useful feature of the matrix norm chosen here is **unitary invariance**, that is,

$$\|VTU\| = \|T\| \tag{2}$$

for any  $T \in \mathbb{M}_n$  and  $n \times n$  unitary matrices U and V.

Two square matrices A and B (not necessarily of the same size) have **identical** pseudospectra<sup>2</sup> if

$$||(zI - A)^{-1}|| = ||(zI - B)^{-1}|| \quad \text{for all } z \in \mathbb{C}.$$
 (3)

In the case of a *normal* matrix  $A \in \mathbb{M}_n$ , the spectral theorem allows one to compute the norm of the **resolvent**  $(zI - A)^{-1}$  of A. By that theorem, there is a diagonal matrix  $\Lambda$  and a unitary matrix U so that  $A = U\Lambda U^*$  and so  $zI - A = U(zI - \Lambda)U^*$  for any  $z \in \mathbb{C}$ . It follows that

$$\|(zI - A)^{-1}\| = \|(zI - \Lambda)^{-1}\| = \max\{|z - \lambda|^{-1} : \lambda \in \sigma(A)\}$$

or equivalently<sup>3</sup>,

$$\|(zI - A)^{-1}\| = \frac{1}{\operatorname{dist}(z, \sigma(A))} \text{ for } z \notin \sigma(A).$$

$$\tag{4}$$

As the following theorem confirms, the condition of identical pseudospectra guarantees that B is normal whenever A is.

**Theorem 1.** Suppose A and B are square matrices<sup>4</sup>. If A is normal, and A and B have identical pseudospectra, then B is normal.

To prove Theorem 1, it is convenient to know whether the validity of the distance formula (4) for an "arbitrary" matrix B implies that B is normal. This turns out to be true.

**Theorem 2.** For a matrix  $T \in \mathbb{M}_n$  to be normal, it is necessary and sufficient that

$$\|(zI-T)^{-1}\| = \frac{1}{\operatorname{dist}(z,\sigma(T))} \quad \text{for } z \notin \sigma(T).$$
(5)

<sup>&</sup>lt;sup>2</sup>In view of inequality (14) below, we adopt the convention that  $||(zI-A)^{-1}|| = \infty$  for  $z \in \sigma(A)$ . <sup>3</sup>If  $z \in \mathbb{C}$  and  $E \subseteq \mathbb{C}$ , we define dist $(z, E) = \inf\{|z-w| : w \in E\}$ .

<sup>&</sup>lt;sup>4</sup>In Theorem 1, the matrices A and B are *not* assumed to have the same size.

The necessity of (5) was addressed in our remarks prior to the statement of Theorem 1 and is well known, e.g., see problem 6.42 on page 62 in [7]. In Section 2 below, we establish the sufficiency of (5) and some of its consequences.

Thus, in the remainder of this section, we prove Theorem 1 (assuming the validity of Theorem 2) and discuss its consequences; Theorem 2 is proved in the next section.

Proof of Theorem 1. Suppose that A is normal, and that A and B have identical pseudospectra. Not only does (3) imply that A and B have the same spectrum  $\sigma$ , it also implies that

$$||(zI - B)^{-1}|| = \frac{1}{\operatorname{dist}(z, \sigma)}$$
 for all  $z \notin \sigma$ 

by (4) because A is normal. Thus, the normality of B follows from Theorem 2.  $\Box$ 

Recall that if  $p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_m z^m$  is a polynomial with complex coefficients and  $T \in \mathbb{M}_n$ , then p(T) denotes the  $n \times n$  matrix defined by

$$p(T) = c_0 I + c_1 T + c_2 T^2 + \dots + c_m T^m$$

Using this definition, it can be verified that, for fixed T, the mapping  $p \mapsto p(T)$  is both linear and multiplicative.

Two square matrices A and B (*not* necessarily of the same size) have the **same** norm behavior if

$$||p(A)|| = ||p(B)|| \quad \text{for all polynomials } p.$$
(6)

It is worth mentioning that the norm of a polynomial and the norm of the resolvent of a matrix appear naturally in applications. For instance, the stability of a linear dynamical system is determined by the norm ||p(A)|| for suitable polynomials p. For example, given a discrete system  $v_{k+1} = Av_k$ ,  $k \ge 0$ , we have  $v_k = A^k v_0$  for all  $k \ge 0$  and so it suffices to use  $p(z) = z^k$ . Likewise, given a continuous system y' = Ay, there is a polynomial  $p_t$  with coefficients depending on t so that  $p_t(A) = \exp(tA)$  (see [8]) and so the behavior of y is determined by  $||p_t(A)||$  because  $y = \exp(tA)y(0)$ . Furthermore, if  $||(zI - A)^{-1}||$  is known, one can obtain (not necessarily sharp) upper bounds for ||p(A)|| using the Cauchy integral formula

$$p(A) = \frac{1}{2\pi i} \int_{\Gamma} (zI - A)^{-1} p(z) \, dz,$$

where  $\Gamma$  is any contour enclosing  $\sigma(A)$  (see page 46 in [7]).

In [3], Greenbaum and Trefethen showed that if two matrices have the same norm behavior, then they also have identical pseudospectra. Therefore, an application of their result and Theorem 1 give

**Corollary 3.** Suppose A and B are square matrices<sup>5</sup>. If A is normal, and A and B have the same norm behavior, then B is normal.

In the case that both matrices A and B have the same size, one obtains criteria for their unitary similarity. This is stated in the following corollary.

**Corollary 4.** Let  $A, B \in \mathbb{M}_n$  and suppose A is normal. The following statements are equivalent.

- (1) A and B are unitarily similar.
- (2) A and B have the same norm behavior and characteristic polynomials.

<sup>&</sup>lt;sup>5</sup>In Corollary 3, the matrices A and B are not assumed to have the same size.

(3) A and B have identical pseudospectra and characteristic polynomials.

*Proof.* Condition 1 and the normality of A imply that both A and B are unitarily similar to the same diagonal matrix, say D. In particular, if  $A = UDU^*$  for some unitary matrix U, then  $p(A) = Up(D)U^*$ . Therefore, ||p(A)|| = ||p(D)|| for any polynomial p by (2), and likewise ||p(B)|| = ||p(D)||. Moreover, if  $A = VBV^*$  for some unitary V, then factoring  $zI - A = V(zI - B)V^*$  implies that

$$\det(zI - A) = \det(zI - B)$$

because the determinant is a multiplicative map. Hence, condition 2 holds.

The fact that condition 2 implies condition 3 is an immediate consequence of the result mentioned from [3].

Finally, if condition 3 holds, Theorem 1 implies that both A and B are normal matrices of the same size. In addition, they also have the same eigenvalues (counting multiplicities) because they have the same characteristic polynomials. In other words, by the spectral theorem, A and B are unitarily similar to the same diagonal matrix, and so condition 1 holds, as desired.

Note that the implication "2  $\implies$  1" in Corollary 4 is also a consequence of Corollary 3 above; indeed, that corollary implies that A and B are normal matrices of the same size and so unitary similarity follows again by the spectral theorem. For yet another proof of the implication "2  $\implies$  1," we refer the reader to [2].

## 2. The distance formula, its consequences, and criteria for Normality.

We now give a straightforward and self-contained proof of Theorem 2 and state some of its own interesting consequences. We also mention two other approaches by which Theorem 2 can be proved.

*Proof of Theorem 2.* In view of the remarks preceding (4), we only need to show that if (5) holds, then T is normal. Instead of (5), let us assume the following equivalent formulation:

$$\|(zI - T)^{-1}\| = \max\left\{|z - \lambda|^{-1} : \lambda \in \sigma(T)\right\} \quad \text{for all } z \notin \sigma(T).$$
(7)

By Schur's Theorem (see Theorem 2.3.1 on page 79 and the Remark on page 80 in [5]), there is a unitary matrix U and a lower triangular matrix L such that  $T = ULU^*$ . In this case, the main diagonal of L must consist of the eigenvalues of T (in any desired order but counting multiplicities). So, zI - L is lower triangular and has  $(z - \lambda)$  with  $\lambda \in \sigma(T)$  as main diagonal entries. Consequently,  $(zI - L)^{-1}$  is also lower triangular with entries  $(z - \lambda)^{-1}$ ,  $\lambda \in \sigma(T)$ , on its main diagonal. Since  $(zI - T)^{-1} = U(zI - L)^{-1}U^*$ , (7) can be restated as

$$\|(zI - L)^{-1}\| = \max\left\{|z - \lambda|^{-1} : \lambda \in \sigma(L)\right\} \quad \text{for all } z \notin \sigma(L).$$
(8)

We show that L must be a diagonal matrix. To that end, we use the following well-known result (see Section 0.7.3 in [5]).

**Lemma 5.** Suppose  $M \in \mathbb{M}_n$  is invertible and has block form

$$M = \left[ \begin{array}{cc} A & 0\\ B & C \end{array} \right],\tag{9}$$

where A and C are square matrices. Then A and C are invertible, and the inverse of M has block form

$$\begin{bmatrix} A^{-1} & 0\\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}.$$
 (10)

To simplify notation, label the entries down the main diagonal in L as  $\lambda_n$ ,  $\lambda_{n-1}$ , ...,  $\lambda_2$ , and  $\lambda_1$ . For k = 1, ..., n, let  $L_k$  denote the principal submatrix of L obtained by removing the first n-k rows and columns of L. In particular, each  $L_k$  is a  $k \times k$  lower triangular matrix,  $L_n = L$ , and  $L_1 = \lambda_1$ .

For each  $1 \leq k \leq n-1$ ,

$$L_{k+1} = \left[ \begin{array}{cc} \lambda_{k+1} & 0\\ b_k & L_k \end{array} \right]$$

for some vector  $b_k \in \mathbb{C}^k$  and so, by Lemma 5,

$$(zI - L_{k+1})^{-1} = \begin{bmatrix} (z - \lambda_{k+1})^{-1} & 0\\ (zI - L_k)^{-1}b_k(z - \lambda_{k+1})^{-1} & (zI - L_k)^{-1} \end{bmatrix}$$
(11)

for every  $z \notin \sigma(L)$ . In particular, if  $v = (1, 0, \dots, 0) \in \mathbb{C}^{k+1}$ , then

$$||(zI - L_{k+1})^{-1}||^2 \ge ||(zI - L_{k+1})^{-1}v||_2^2,$$

or equivalently,

$$\|(zI - L_{k+1})^{-1}\|^2 \ge |z - \lambda_{k+1}|^{-2} + \|(zI - L_k)^{-1}b_k(z - \lambda_{k+1})^{-1}\|_2^2.$$
(12)

Similarly, we see that

$$\|(zI - L_n)^{-1}\| \ge \|(zI - L_{n-1})^{-1}\| \ge \dots \ge \|(zI - L_1)^{-1}\|.$$
 (13)

We now show that  $b_k = 0$ . Indeed, if  $z \in \mathbb{C}$  is chosen so that the maximum on the right-hand side of (8) equals  $|z - \lambda_{k+1}|^{-1}$ , then

$$|z - \lambda_{k+1}|^{-2} = \|(zI - L_n)^{-1}\|^2$$
  

$$\geq \|(zI - L_{k+1})^{-1}\|^2$$
  

$$\geq |z - \lambda_{k+1}|^{-2} + \|(zI - L_k)^{-1}b_k(z - \lambda_{k+1})^{-1}\|_2^2$$

by (8) and (12). Therefore,

$$(zI - L_k)^{-1}b_k(z - \lambda_{k+1})^{-1} = 0$$

and so  $b_k = 0$ . Since each column below a main diagonal entry of L is a zero vector, L is a diagonal matrix and so T is normal, as desired.

Even though the distance formula (5) does not hold for *non-normal*  $n \times n$  matrices, the following inequality does: for any  $T \in \mathbb{M}_n$ ,

$$\operatorname{dist}(z,\sigma(T)) \ge \|(zI-T)^{-1}\|^{-1} \quad \text{for all } z \notin \sigma(T).$$
(14)

This fact is verified using the Neumann (geometric) series; for if A and B are in  $\mathbb{M}_n$ , A is invertible, and  $||A - B|| < ||A^{-1}||^{-1}$  holds, then  $BA^{-1}$  is invertible and so B is also invertible. It follows that if (zI - T) is invertible and the inequality  $|z - w| = ||(zI - T) - (wI - T)|| < ||(zI - T)^{-1}||^{-1}$  holds, then (wI - T) is also invertible. In other words, if  $z \notin \sigma(T)$  and  $|z - w| < ||(zI - T)^{-1}||^{-1}$ , then  $w \notin \sigma(T)$  and so the inequality in (14) is obtained.

Our proof of Theorem 2 also reveals that equality (5) need not hold for all  $z \notin \sigma(T) = \{\lambda_1, \ldots, \lambda_n\}$ . Rather, normality of T follows from the weaker (equivalent) condition<sup>6</sup> that for each  $1 \leq k \leq n-1$ , there is a  $z_k \in \mathbb{C}$  so that

$$|(z_k I - T)^{-1}|| = |z_k - \lambda_k|^{-1}$$

We now state the following criteria for normality.

**Theorem 6.** The following statements are equivalent for a matrix  $T \in M_n$ .

(1) For all  $z \notin \sigma(T)$ ,

$$|(zI - T)^{-1}|| = \frac{1}{\operatorname{dist}(z, \sigma(T))}$$

(2) For each  $1 \leq k \leq n-1$ , there is a  $z_k \in \mathbb{C}$  so that

$$|(z_k I - T)^{-1}|| = |z_k - \lambda_k|^{-1},$$
(15)

where  $\lambda_1, \ldots, \lambda_n$  denote the eigenvalues of T (counting multiplicities).

- (3) T is normal.
- (4) For every polynomial p,

$$\|p(T)\| = \max\{|p(\lambda)| : \lambda \in \sigma(T)\}.$$
(16)

*Proof.* It is only left to prove that "4  $\implies$  1" as "3  $\implies$  4" is a straightforward consequence of the spectral theorem; for if T is normal and p is a polynomial, there is a unitary matrix U so that  $T = U\Lambda U^*$  with  $\Lambda$  as in (1). In this case,

$$|p(T)|| = ||Up(\Lambda)U^*|| = ||p(\Lambda)|| = \max\{|p(\lambda)| : \lambda \in \sigma(T)\}$$

and so condition 4 holds.

Now suppose condition 4 holds. Let  $z \notin \sigma(T)$  be fixed and define  $f(t) = (z-t)^{-1}$ . Let  $\lambda_1, \ldots, \lambda_k$  denote the distinct eigenvalues of T. For  $1 \leq i \leq k$ ,  $\lambda_i$  is a zero of the minimal polynomial m of T. It follows from Theorem 6.2.9 in [6] that  $(zI - T)^{-1} = q_z(T)$ , where  $q_z(t)$  is any polynomial of degree at most n - 1 that interpolates f(t) and its derivatives at the zeros of m. That is,  $f^{(u)}(\lambda_i) = q_z^{(u)}(\lambda_i)$  for  $u = 0, \ldots, (s_i - 1)$  and  $i = 1, \ldots, k$  (see page 390 in [6]). Therefore, (16) implies  $||q_z(T)|| = \max\{|q_z(\lambda)| : \lambda \in \sigma(T)\}$ , or equivalently,

$$||(zI-T)^{-1}|| = \max\{|z-\lambda|^{-1} : \lambda \in \sigma(T)\} = \frac{1}{\operatorname{dist}(z,\sigma(T))} \quad \text{for } z \notin \sigma(T).$$

Hence, condition 1 holds and the proof is now complete.

Surprisingly, the criteria for normality appearing in Theorem 6 are absent from the literature. Thus, these criteria may be considered addenda to the 89 other characterizations of normal matrices that appear in [1] and [4].

**Corollary 7.** If  $T \in \mathbb{M}_2$  and

$$||(zI - T)^{-1}|| = \frac{1}{\operatorname{dist}(z, \sigma(T))}$$

holds for one point  $z \notin \sigma(T)$ , then T is normal.

<sup>&</sup>lt;sup>6</sup>Even better, if d is the number of distinct eigenvalues of T and one eigenvalue is simple, then equality (15) need only hold for  $z_k$  near each of the other d-1 eigenvalues to conclude that T be normal.

Hence, for non-normal  $2 \times 2$  matrices T, the inequality in (14) must be *strict* for all  $z \notin \sigma(T)$ . For instance, if

$$N = \left[ \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right],$$

then  $\sigma(N) = \{0\}$  and so  $||(zI-N)^{-1}|| > \text{dist}^{-1}(z, \sigma(N)) = |z|^{-1}$  holds for all  $z \neq 0$  by (14) and Corollary 7.

**Remark.** The sufficient condition for normality in Theorem 2 may also be deduced two other ways.

(1) If  $\sigma_{\epsilon}(T)$  denotes the " $\epsilon$ -pseudospectrum" of T, that is,

$$\sigma_{\epsilon}(T) \stackrel{\text{def}}{=} \{ z : \| (zI - T)^{-1} \| > \epsilon^{-1} \},\$$

then Theorem 2.2 of [9] states that<sup>7</sup> if the equality

$$\sigma_{\epsilon}(T) = \{\zeta + \xi : \zeta \in \sigma(T) \text{ and } |\xi| < \epsilon\}$$
(17)

holds for all  $\epsilon > 0$ , then the matrix T is normal. In a sketch of a proof, the authors assert that one can deduce simultaneous diagonalizability of T and  $T^*$  from content in a later section in [9] on eigenvalue perturbation theory. Therefore, if equality (5) holds, then the equality of the sets in (17) follows, and so T is normal.

(2) The fact that (5) implies normality of T may also be deduced in the context of radial matrices. For  $z \notin \sigma(T)$ , equality (5) implies that  $(zI - T)^{-1}$  is radial and so unitarily similar to

$$||(zI - T)^{-1}|| (U \oplus B),$$
 (18)

where  $U \in \mathbb{M}_k$  is unitary,  $1 \leq k \leq n$ , and  $B \in \mathbb{M}_{n-k}$  has spectral radius less than 1 and  $||B|| \leq 1$  (see Problem 27(g) on page 45 in [6]). To conclude normality of T, one can obtain from (18) orthonormal eigenvectors of  $(zI - T)^{-1}$  (corresponding to the eigenvalue  $(z - \lambda)^{-1}$ ) and so orthonormal eigenvectors for T; thus, one can form a complete set of orthonormal eigenvectors of T through a choice of z near each eigenvalue.

Acknowledgments. The authors wish to thank Thomas Ransford and Roger Horn for comments on a preliminary version of this paper calling to our attention Theorem 2.2 of [9] and an alternative approach to obtaining Theorem 2 using unitary similarity to (18), respectively.

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<sup>&</sup>lt;sup>7</sup>The set on the right-hand side of (17) can be rewritten as  $\{z : \operatorname{dist}(z, \sigma(T)) < \epsilon\}$ .

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