A GRONWALL-TYPE TRIGONOMETRIC INEQUALITY

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ABSTRACT. We prove that the absolute value of the nth derivative of $\cos(\sqrt{x})$ does not exceed n!/(2n)! for all x>0 and $n=0,1,\ldots$ and obtain a natural generalization of this inequality involving the analytic continuation of $\cos(\sqrt{x})$.

In 1913, Gronwall [5] proposed to prove that

$$\left| \frac{d^n}{dx^n} \frac{\sin x}{x} \right| \le \frac{1}{n+1}$$

for all real x and all $n=0,1,\ldots$ A short and elegant proof of this inequality was given by Dunkel [2] in 1920. (Another proof in [2] provided by Uhler was extremely cumbersome.) In this note, another trigonometric inequality of a similar type is established: we shall show that

(2)
$$\left| \frac{d^n}{dx^n} \cos(\sqrt{x}) \right| \le \frac{n!}{(2n)!}, \quad x > 0,$$

for all $n = 0, 1, \ldots$ While this statement can hardly be directly derived from (1), its demonstration given below bears a strong resemblance to Dunkel's proof of (1).

Our motivation for estimating the derivatives of $\cos(\sqrt{x})$ comes from the study of one-dimensional Schrödinger operators of the form

$$-\partial_r^2 + \frac{\alpha}{r^2}, \quad r > 0,$$

where α is a real parameter. This system (which is the simplest one-particle variant of the well-known Calogero model [1]) exhibits a phase transition [3] at $\alpha=-1/4$: the self-adjoint realizations of differential expression (3) have infinitely many eigenvalues for $\alpha<-1/4$ and at most one eigenvalue for $\alpha\geq -1/4$. In [8], we used the method of singular Titchmarsh–Weyl m-functions [6] to construct eigenfunction expansions for (3) in a neighborhood of the critical point $\alpha=-1/4$ and proved that the corresponding spectral measures depend continuously on α . Inequality (2) (or, more precisely, its generalization (12) below) provides us with a tool for estimating the α -derivatives of the m-functions for this model. This makes it possible to prove that the spectral measures are actually smooth in α near the critical point. (A detailed treatment of this problem will be given in a forthcoming paper.)

Let the entire analytic function Cos be defined by the relation

(4)
$$\operatorname{Cos} z = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{(2k)!}, \quad z \in \mathbb{C}.$$

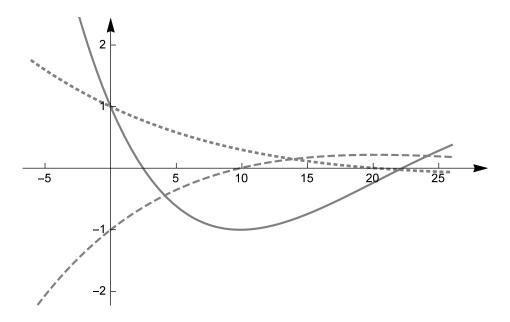


FIGURE 1. Solid, dashed, and dotted lines correspond to the functions $n!^{-1}(2n)! \cos^{(n)} x$ for n=0, 1, and 2 respectively. According to (6), these functions are equal to ± 1 at x=0.

Then $Cos(z^2) = cos z$ for all $z \in \mathbb{C}$. Hence, Cos z is the analytic continuation of $cos(\sqrt{x})$ from the positive half-axis to the entire complex plane, and we have

(5)
$$\operatorname{Cos} x = \operatorname{Cos}(-|x|) = \operatorname{cos}\left(i\sqrt{|x|}\right) = \operatorname{cosh}\left(\sqrt{|x|}\right), \quad x \le 0.$$

(The graphs of $\cos x$ and its first two derivatives are shown in Figure 1.) It follows immediately from (4) that

(6)
$$\operatorname{Cos}^{(n)}(0) = (-1)^n \frac{n!}{(2n)!},$$

where $\operatorname{Cos}^{(n)} z$ denotes the *n*th derivative of $\operatorname{Cos} z$. This equality implies that the constant in the right-hand side of (2) is optimal. For all $z \in \mathbb{C}$, we have

(7)
$$(\cos z)^2 + 4z(\cos' z)^2 = 1.$$

Indeed, since

(8)
$$\operatorname{Cos}' x = -\frac{\sin(\sqrt{x})}{2\sqrt{x}}, \quad x > 0,$$

equation (7) holds on the positive half-axis. By the uniqueness theorem for analytic functions, (7) remains valid for all $z \in \mathbb{C}$. Differentiating (7), we obtain

$$\cos z + 2 \cos' z + 4z \cos'' z = 0$$
, $z \in \mathbb{C}$.

By induction on n, this implies that

(9)
$$\operatorname{Cos}^{(n-1)} z + (4n-2)\operatorname{Cos}^{(n)} z + 4z\operatorname{Cos}^{(n+1)} z = 0, \quad z \in \mathbb{C},$$

for all $n \geq 1$. It follows from (8) that both $\cos' x$ and $\cos'' x$ tend to zero as $x \to \infty$. Using induction on n, we deduce from (9) that

$$\lim_{x \to \infty} \cos^{(n)} x = 0$$

for all $n \geq 1$.

We now prove (2) by induction on n. Clearly, (2) is valid for n = 0. Let $n \ge 1$ and suppose that (2) holds for all derivatives of orders less than n. By (10), the maximum value of $|\cos^{(n)} x|$ on $[0,\infty)$ is attained at some $x_0 \ge 0$. If $x_0 = 0$, then the required estimate is ensured by (6). If $x_0 > 0$, then we have $\cos^{(n+1)} x_0 = 0$. By (9) and the induction hypothesis, it follows that

$$|\cos^{(n)} x_0| = \frac{|\cos^{(n-1)} x_0|}{4n-2} \le \frac{(n-1)!}{(2n-2)!(4n-2)} = \frac{n!}{(2n)!}.$$

This completes the proof of (2).

It should be noted that inequality (2) is strict for $n \ge 1$, i.e., $|\cos^{(n)} x| < n!/(2n)!$ for all x > 0 and $n \ge 1$. Indeed, by (8), this is true for n = 1. Suppose n > 1 and $|\cos^{(n)} x_0| = n!/(2n)!$ for some $x_0 > 0$. Then $\cos^{(n+1)} x_0 = 0$ and it follows from (9) that $|\cos^{(n-1)} x_0| = (n-1)!/(2n-2)!$. Iterating this argument, we find that $|\cos^{(1)} x_0| = 1/2$ in contradiction to (8), and our claim is proved.

Using (2) and the power series expansion for $\cos z$, it is easy to estimate $\cos^{(n)} x$ on every interval $[a, \infty)$ with $a \leq 0$. More precisely, we shall establish the inequality

(11)
$$|\operatorname{Cos}^{(n)} x| \le \frac{n!(2m)!}{(2n)!m!} |\operatorname{Cos}^{(m)} a|, \quad x \ge a,$$

for every $a \le 0$ and all nonnegative integer numbers n and m such that $m \le n$. For a = 0, this estimate follows immediately from (2) and (6). If a = 0 and m = 0, then (11) coincides with (2). In view of (5), substituting m = 0 in (11) yields

(12)
$$|\operatorname{Cos}^{(n)} x| \le \frac{n!}{(2n)!} \cosh\left(\sqrt{|a|}\right), \quad x \ge a.$$

We now prove (11). By differentiating (4), we find that

$$\operatorname{Cos}^{(n)} z = (-1)^n \sum_{k=0}^{\infty} c(n,k)(-z)^k, \quad c(n,k) = \frac{(k+n)!}{k!(2k+2n)!},$$

for all $z \in \mathbb{C}$. This implies that

(13)
$$|\cos^{(n)} x| = \sum_{k=0}^{\infty} c(n,k)|x|^k, \quad x \le 0.$$

It follows that $|\cos^{(n)} x|$ is decreasing on the negative half-axis and, therefore,

$$|\cos^{(n)} x| \le |\cos^{(n)} a|$$

for every $x \in [a, 0]$. For x > 0, we have $|\cos^{(n)} x| \le |\cos^{(n)}(0)|$ by (2) and (6) and, hence, (14) holds for all $x \ge a$. It easily follows by induction on n that

$$c(n,k) \le \frac{n!(2m)!}{(2n)!m!}c(m,k)$$

for all $k = 0, 1, \ldots$ In view of (13), this implies that (11) holds for x = a. Combining (11) for x = a with (14) yields inequality (11) for arbitrary $x \ge a$.

In conclusion, we return briefly to inequality (1). In fact, (1) is an easy consequence of the obvious formula

$$\frac{\sin x}{x} = \int_0^1 \cos(tx) \ dt.$$

Indeed, differentiating (15) yields

$$\frac{d^n}{dx^n} \frac{\sin x}{x} = \int_0^1 t^n \cos\left(tx + \frac{n}{2}\pi\right) dt,$$

whence (1) follows immediately. Performing the change of variables $t \to y/x$, we obtain the identity

$$\frac{d^n}{dx^n} \frac{\sin x}{x} = \frac{1}{x^{n+1}} \int_0^x y^n \sin\left(y + \frac{n+1}{2}\pi\right) dy,$$

which was proposed by Gronwall [4] and was mentioned to imply (1) in Section 3.4.24 of [7]. (It is interesting to note that Gronwall himself failed to link this identity to (1).) It is an open question whether inequality (2) can be derived from some similar integral representations for $\cos(\sqrt{x})$ and its derivatives.

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