# A GRONWALL-TYPE TRIGONOMETRIC INEQUALITY 

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#### Abstract

We prove that the absolute value of the $n$th derivative of $\cos (\sqrt{x})$ does not exceed $n!/(2 n)$ ! for all $x>0$ and $n=0,1, \ldots$ and obtain a natural generalization of this inequality involving the analytic continuation of $\cos (\sqrt{x})$.


In 1913, Gronwall [5] proposed to prove that

$$
\begin{equation*}
\left|\frac{d^{n}}{d x^{n}} \frac{\sin x}{x}\right| \leq \frac{1}{n+1} \tag{1}
\end{equation*}
$$

for all real $x$ and all $n=0,1, \ldots$. A short and elegant proof of this inequality was given by Dunkel [2] in 1920. (Another proof in [2] provided by Uhler was extremely cumbersome.) In this note, another trigonometric inequality of a similar type is established: we shall show that

$$
\begin{equation*}
\left|\frac{d^{n}}{d x^{n}} \cos (\sqrt{x})\right| \leq \frac{n!}{(2 n)!}, \quad x>0 \tag{2}
\end{equation*}
$$

for all $n=0,1, \ldots$. While this statement can hardly be directly derived from (1), its demonstration given below bears a strong resemblance to Dunkel's proof of (1).

Our motivation for estimating the derivatives of $\cos (\sqrt{x})$ comes from the study of one-dimensional Schrödinger operators of the form

$$
\begin{equation*}
-\partial_{r}^{2}+\frac{\alpha}{r^{2}}, \quad r>0 \tag{3}
\end{equation*}
$$

where $\alpha$ is a real parameter. This system (which is the simplest one-particle variant of the well-known Calogero model [1]) exhibits a phase transition [3] at $\alpha=-1 / 4$ : the self-adjoint realizations of differential expression (3) have infinitely many eigenvalues for $\alpha<-1 / 4$ and at most one eigenvalue for $\alpha \geq-1 / 4$. In [8], we used the method of singular Titchmarsh-Weyl $m$-functions [6] to construct eigenfunction expansions for (3) in a neighborhood of the critical point $\alpha=-1 / 4$ and proved that the corresponding spectral measures depend continuously on $\alpha$. Inequality (2) (or, more precisely, its generalization (12) below) provides us with a tool for estimating the $\alpha$-derivatives of the $m$-functions for this model. This makes it possible to prove that the spectral measures are actually smooth in $\alpha$ near the critical point. (A detailed treatment of this problem will be given in a forthcoming paper.)

Let the entire analytic function Cos be defined by the relation

$$
\begin{equation*}
\operatorname{Cos} z=\sum_{k=0}^{\infty} \frac{(-1)^{k} z^{k}}{(2 k)!}, \quad z \in \mathbb{C} \tag{4}
\end{equation*}
$$

[^0]

Figure 1. Solid, dashed, and dotted lines correspond to the functions $n!^{-1}(2 n)!\operatorname{Cos}^{(n)} x$ for $n=0,1$, and 2 respectively. According to (6), these functions are equal to $\pm 1$ at $x=0$.

Then $\operatorname{Cos}\left(z^{2}\right)=\cos z$ for all $z \in \mathbb{C}$. Hence, $\operatorname{Cos} z$ is the analytic continuation of $\cos (\sqrt{x})$ from the positive half-axis to the entire complex plane, and we have

$$
\begin{equation*}
\operatorname{Cos} x=\operatorname{Cos}(-|x|)=\cos (i \sqrt{|x|})=\cosh (\sqrt{|x|}), \quad x \leq 0 \tag{5}
\end{equation*}
$$

(The graphs of $\operatorname{Cos} x$ and its first two derivatives are shown in Figure 1 )
It follows immediately from (4) that

$$
\begin{equation*}
\operatorname{Cos}^{(n)}(0)=(-1)^{n} \frac{n!}{(2 n)!}, \tag{6}
\end{equation*}
$$

where $\operatorname{Cos}^{(n)} z$ denotes the $n$th derivative of $\operatorname{Cos} z$. This equality implies that the constant in the right-hand side of (2) is optimal. For all $z \in \mathbb{C}$, we have

$$
\begin{equation*}
(\operatorname{Cos} z)^{2}+4 z\left(\operatorname{Cos}^{\prime} z\right)^{2}=1 \tag{7}
\end{equation*}
$$

Indeed, since

$$
\begin{equation*}
\operatorname{Cos}^{\prime} x=-\frac{\sin (\sqrt{x})}{2 \sqrt{x}}, \quad x>0 \tag{8}
\end{equation*}
$$

equation (7) holds on the positive half-axis. By the uniqueness theorem for analytic functions, (7) remains valid for all $z \in \mathbb{C}$. Differentiating (7), we obtain

$$
\operatorname{Cos} z+2 \operatorname{Cos}^{\prime} z+4 z \operatorname{Cos}^{\prime \prime} z=0, \quad z \in \mathbb{C}
$$

By induction on $n$, this implies that

$$
\begin{equation*}
\operatorname{Cos}^{(n-1)} z+(4 n-2) \operatorname{Cos}^{(n)} z+4 z \operatorname{Cos}^{(n+1)} z=0, \quad z \in \mathbb{C}, \tag{9}
\end{equation*}
$$

for all $n \geq 1$. It follows from (8) that both $\operatorname{Cos}^{\prime} x$ and $\operatorname{Cos}^{\prime \prime} x$ tend to zero as $x \rightarrow \infty$. Using induction on $n$, we deduce from (9) that

$$
\begin{equation*}
\lim _{x \rightarrow \infty} \operatorname{Cos}^{(n)} x=0 \tag{10}
\end{equation*}
$$

for all $n \geq 1$.
We now prove (2) by induction on $n$. Clearly, (2) is valid for $n=0$. Let $n \geq 1$ and suppose that (2) holds for all derivatives of orders less than $n$. By (10), the maximum value of $\left|\operatorname{Cos}^{(n)} x\right|$ on $[0, \infty)$ is attained at some $x_{0} \geq 0$. If $x_{0}=0$, then the required estimate is ensured by (6). If $x_{0}>0$, then we have $\operatorname{Cos}^{(n+1)} x_{0}=0$. By (9) and the induction hypothesis, it follows that

$$
\left|\operatorname{Cos}^{(n)} x_{0}\right|=\frac{\left|\operatorname{Cos}^{(n-1)} x_{0}\right|}{4 n-2} \leq \frac{(n-1)!}{(2 n-2)!(4 n-2)}=\frac{n!}{(2 n)!}
$$

This completes the proof of (21).
It should be noted that inequality (2) is strict for $n \geq 1$, i.e., $\left|\operatorname{Cos}^{(n)} x\right|<n!/(2 n)$ ! for all $x>0$ and $n \geq 1$. Indeed, by (8), this is true for $n=1$. Suppose $n>1$ and $\left|\operatorname{Cos}^{(n)} x_{0}\right|=n!/(2 n)!$ for some $x_{0}>0$. Then $\operatorname{Cos}^{(n+1)} x_{0}=0$ and it follows from (9) that $\left|\operatorname{Cos}^{(n-1)} x_{0}\right|=(n-1)!/(2 n-2)$ !. Iterating this argument, we find that $\left|\operatorname{Cos}^{(1)} x_{0}\right|=1 / 2$ in contradiction to (8), and our claim is proved.

Using (2) and the power series expansion for $\operatorname{Cos} z$, it is easy to estimate $\operatorname{Cos}^{(n)} x$ on every interval $[a, \infty)$ with $a \leq 0$. More precisely, we shall establish the inequality

$$
\begin{equation*}
\left|\operatorname{Cos}^{(n)} x\right| \leq \frac{n!(2 m)!}{(2 n)!m!}\left|\operatorname{Cos}^{(m)} a\right|, \quad x \geq a \tag{11}
\end{equation*}
$$

for every $a \leq 0$ and all nonnegative integer numbers $n$ and $m$ such that $m \leq n$. For $a=0$, this estimate follows immediately from (2) and (6). If $a=0$ and $m=0$, then (11) coincides with (21). In view of (5), substituting $m=0$ in (11) yields

$$
\begin{equation*}
\left|\operatorname{Cos}^{(n)} x\right| \leq \frac{n!}{(2 n)!} \cosh (\sqrt{|a|}), \quad x \geq a \tag{12}
\end{equation*}
$$

We now prove (11). By differentiating (4), we find that

$$
\operatorname{Cos}^{(n)} z=(-1)^{n} \sum_{k=0}^{\infty} c(n, k)(-z)^{k}, \quad c(n, k)=\frac{(k+n)!}{k!(2 k+2 n)!}
$$

for all $z \in \mathbb{C}$. This implies that

$$
\begin{equation*}
\left|\operatorname{Cos}^{(n)} x\right|=\sum_{k=0}^{\infty} c(n, k)|x|^{k}, \quad x \leq 0 \tag{13}
\end{equation*}
$$

It follows that $\left|\operatorname{Cos}^{(n)} x\right|$ is decreasing on the negative half-axis and, therefore,

$$
\begin{equation*}
\left|\operatorname{Cos}^{(n)} x\right| \leq\left|\operatorname{Cos}^{(n)} a\right| \tag{14}
\end{equation*}
$$

for every $x \in[a, 0]$. For $x>0$, we have $\left|\operatorname{Cos}^{(n)} x\right| \leq\left|\operatorname{Cos}^{(n)}(0)\right|$ by (21) and (6) and, hence, (14) holds for all $x \geq a$. It easily follows by induction on $n$ that

$$
c(n, k) \leq \frac{n!(2 m)!}{(2 n)!m!} c(m, k)
$$

for all $k=0,1, \ldots$ In view of (13), this implies that (11) holds for $x=a$. Combining (11) for $x=a$ with (14) yields inequality (11) for arbitrary $x \geq a$.

In conclusion, we return briefly to inequality (11). In fact, (1) is an easy consequence of the obvious formula

$$
\begin{equation*}
\frac{\sin x}{x}=\int_{0}^{1} \cos (t x) d t \tag{15}
\end{equation*}
$$

Indeed, differentiating (15) yields

$$
\frac{d^{n}}{d x^{n}} \frac{\sin x}{x}=\int_{0}^{1} t^{n} \cos \left(t x+\frac{n}{2} \pi\right) d t
$$

whence (11) follows immediately. Performing the change of variables $t \rightarrow y / x$, we obtain the identity

$$
\frac{d^{n}}{d x^{n}} \frac{\sin x}{x}=\frac{1}{x^{n+1}} \int_{0}^{x} y^{n} \sin \left(y+\frac{n+1}{2} \pi\right) d y
$$

which was proposed by Gronwall [4] and was mentioned to imply (1) in Section 3.4.24 of [7]. (It is interesting to note that Gronwall himself failed to link this identity to (11).) It is an open question whether inequality (2) can be derived from some similar integral representations for $\cos (\sqrt{x})$ and its derivatives.

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