A Short Proof that Lebesgue Outer Measure of an Interval is Its Length

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MSC: Primary 28A12. The Lebesgue outer measure $m^*(E)$ of a subset E of real line is defined as $m^*(E) := \inf\{\sum_{k=1}^{\infty} \ell(I_k) \mid E \subseteq \bigcup_{k=1}^{\infty} I_k\}$, where each I_k is an open interval and $\ell(I_k)$ is its length. Establishing one of the inequalities in the standard proof of the fact in the title above turns out to be tedious in [1, p. 31]. Using the connectedness of the interval shortens the proof as follows.

Proof. Given two real numbers a and b with a < b, it is enough to prove that $m^*([a,b]) = b-a$. Clearly, $m^*([a,b]) \le b-a$. Now let $[a,b] \subset \bigcup_{k=1}^n I_k$ for some positive integer n, which is always possible since [a,b] is compact. Without loss of generality, assume that the set $[a,b] \cap I_k$ is nonempty for each k. Observe that the set $\bigcup_{k=1}^n I_k$ is connected. (Otherwise, if (P,Q) is its separation, then for each k, by connectedness of I_k , either $I_k \subset P$ or $I_k \subset Q$. Thus each of P and Q is equal to union of sets from the list $\{I_1,\ldots,I_n\}$. So the pair $(P \cap [a,b],Q\cap [a,b])$ determines a separation of [a,b], which contradicts connectedness of [a,b].) So $\bigcup_{k=1}^n I_k$ is an open interval containing [a,b]. Thus, $b-a \le \ell(\bigcup_{k=1}^n I_k) \le \sum_{k=1}^n \ell(I_k)$, where the last inequality holds since some intervals overlap[†]. Hence, $b-a \le m^*([a,b])$. \square

[†]The inequality $\ell(\bigcup_{k=1}^n I_k) \leq \sum_{k=1}^n \ell(I_k)$ can be justified as follows. Given a bounded interval I, by definition, $\ell(I) = \sup I - \inf I$. Observe that if two distinct bounded intervals I_1 and I_2 overlap, then $\ell(I_1 \cup I_2) \leq \ell(I_1) + \ell(I_2)$.

Given n>1 bounded open intervals $I_1,\ldots I_n$ with their union being connected implies that for each $k=1,\ldots,n-1$ we may choose I_k after re-indexing these intervals, such that $\cup_{j=1}^k I_j$ is connected and it overlaps at least one interval among rest of the (n-k) intervals, which we denote by I_{k+1} . So, we have $\ell(\cup_{k=1}^n I_k) = \ell(\cup_{k=1}^{n-1} I_k \cup I_n) \leq \ell(\cup_{k=1}^{n-1} I_k) + \ell(I_n) \leq \ell(\cup_{k=1}^{n-2} I_k) + \ell(I_{n-1}) + \ell(I_n) \leq \ldots \leq \ell(I_1) + \ldots + \ell(I_n)$.

References

[1] H. L. Royden, P. M. Fitzpatrick, Real Analysis. Fourth ed. Pearson, Boston, 2010.