ON SURJECTIVITY OF SMOOTH MAPS INTO EUCLIDEAN SPACES AND THE FUNDAMENTAL THEOREM OF ALGEBRA

PENG LIU AND SHIBO LIU

Department of Mathematics, Xiamen University Xiamen 361005, China

ABSTRACT. In this note we obtain the surjectivity of smooth maps into Euclidean spaces under mild conditions. As application we give a new proof of the Fundamental Theorem of Algebra. We also observe that any C^1 -map from a compact manifold into Euclidean space with dimension $n \ge 2$ has infinitely many critical points.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 -map such that for all $x \in \mathbb{R}^n$ we have det $Df(x) \neq 0$. Suppose moreover

$$\lim_{|x| \to \infty} |f(x)| = +\infty, \tag{1}$$

then it is well known that $f(\mathbb{R}^n) = \mathbb{R}^n$ (namely *f* is surjective), see e.g. [1, Page 24]. In this note, we will prove the following generalisation of this result. As we shall see, from this result, we can easily obtain the fundamental theorem of algebra.

Let *U* be an open subset of \mathbb{R}^m and $f : U \to \mathbb{R}^n$ be a C^1 -map, $a \in U$. If rank Df(a) < n, that is the differential map $Df(a) : \mathbb{R}^m \to \mathbb{R}^n$ is not surjective, then we say that *a* is a critical point of *f*.

Theorem 1. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a C^1 -map with $n \ge 2$. If f has only finitely many critical points and $f(\mathbb{R}^m)$ is a closed subset of \mathbb{R}^n , then $f(\mathbb{R}^m) = \mathbb{R}^n$.

Remark 2. If m = n, then det $Df(x) \neq 0$ is exactly that x is not a critical point of f. If the condition (1) is satisfied, then $f(\mathbb{R}^n)$ is closed. Therefore the classical result mentioned at the beginning is a corollary of our Theorem 1.

To prove this theorem, we need the following lemma.

Lemma 3. Let $n \ge 2$, A be a nonempty open subset of \mathbb{R}^n . If there exists k points p_1, \ldots, p_k such that $A \cup \{p_i\}_{i=1}^k$ is closed, then $\overline{A} = \mathbb{R}^n$.

Proof. Because $A \cup \{p_i\}_{i=1}^k$ is closed, we have

$$A \cup \{p_i\}_{i=1}^k = \overline{A \cup \{p_i\}_{i=1}^k} = \overline{A} \cup \{p_i\}_{i=1}^k = A \cup \partial A \cup \{p_i\}_{i=1}^k$$

Since *A* is open, $A \cap \partial A = \emptyset$. It follows that

$$\partial A \subset \{p_i\}_{i=1}^{\kappa}.\tag{2}$$

If $\bar{A} \neq \mathbb{R}^n$, we can choose $b \in \mathbb{R}^n \setminus \bar{A}$ and $a \in A$. Since both A and $\mathbb{R}^n \setminus \bar{A}$ are open, there exists $\varepsilon > 0$ such that

$$B_{\varepsilon}(a) \subset A$$
, $B_{\varepsilon}(b) \subset \mathbb{R}^n \setminus A$.

This means that *b* is not a boundary point of *A*.

Since $n \ge 2$, we can take a segment $\ell \subset B_{\epsilon/2}(a)$ such that ℓ is not parallel to the vector b - a. Of course we can choose k + 1 different points $\{x_i\}_{i=1}^{k+1} \subset \ell$.

Consider the k + 1 segments I_i with end points b and x_i . Because each such segment not only contains points (those near x_i) in A, but also contains points (those near b) in $\mathbb{R}^n \setminus A$, it is easy to see that for each i = 1, ..., k + 1, there exists at least one boundary point of A in I_i .

Emails: liupeng1729@qq.com (P. Liu), liusb@xmu.edu.cn (S.B. Liu).

Since for $i \neq j$ we have $I_i \cap I_j = \{b\}$ and $b \notin \partial A$, we conclude that ∂A contains at lease k + 1 points. This contradicts with (2).

Proof (Proof of Theorem 1). Let *K* be the set of critical points, then *K* is a finite set. Because $\mathbb{R}^m \setminus K$ is open and rank Df(x) = n for $x \in \mathbb{R}^m \setminus K$, using the Inverse Function Theorem it is well known that $A = f(\mathbb{R}^m \setminus K)$ is an open subset of \mathbb{R}^n . By the assumption,

$$A \cup f(K) = f(\mathbb{R}^m \setminus K) \cup f(K) = f(\mathbb{R}^m)$$

is closed. Since *K* is finite, its image f(K) is also finite. Applying Lemma 3, we conclude that $\overline{A} = \mathbb{R}^n$.

Using the assumption that $f(\mathbb{R}^m)$ is closed again, we deduce

$$f(\mathbb{R}^m) = \overline{f(\mathbb{R}^m)} \supset \overline{f(\mathbb{R}^m \setminus K)} = \overline{A} = \mathbb{R}^n.$$

Checking the above proof of Theorem 1, we find that the domain of our map f can be replaced by an *m*-dimensional C^1 -manifold M. Therefore we have the following corollary, whose proof is omitted.

Corollary 4. Let M be an m-dimensional C^1 -manifold (without boundary), $f : M \to \mathbb{R}^n$ be a C^1 -map with $n \ge 2$. If f has only finitely many critical points and f(M) is a closed subset of \mathbb{R}^n , then $f(M) = \mathbb{R}^n$.

From this corollary, we can further state the following result.

Corollary 5. Let M be a compact m-dimensional C¹-manifold (without boundary), $f : M \to \mathbb{R}^n$ be a C¹-map with $n \ge 2$. Then f has infinitely many critical points.

Now, we consider polynomials

$$p(z) = z^n + a_1 z^{n-1} + \dots + a_n$$

of degree $n \ge 1$, where the coefficients $a_i \in \mathbb{C}$. As corollary of our theorem we will give a new proof of the following classical theorem.

Theorem 6 (Fundamental Theorem of Algebra). If p(z) is a polynomial of degree $n \ge 1$, with complex coefficients a_i , then there exists $\xi \in \mathbb{C}$ such that $p(\xi) = 0$.

Proof. Write z = x + iy, and

$$p(z) = u(x, y) + \mathrm{i}v(x, y),$$

we know that the complex derivative of *p* is given by

$$p'(z) = u_x + \mathrm{i} v_x.$$

If we consider our polynomial as a map $p : \mathbb{R}^2 \to \mathbb{R}^2$,

$$v(x,y) = (u(x,y),v(x,y)),$$

then using the Cauchy-Riemann equations $(u_x = v_y, u_y = -v_x)$, we have

$$\det Dp(x,y) = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} = u_x^2 + v_x^2.$$

Therefore, p'(z) = 0 if and only if det Dp(x, y) = 0. Because p'(z) is a polynomial of degree n - 1, it has at most n - 1 zero points. Thus, the map $p : \mathbb{R}^2 \to \mathbb{R}^2$ has at most n - 1 critical points.

On the other hand, it is obvious that

$$\lim_{|(x,y)|\to\infty}|p(x,y)|=+\infty,$$

which implies that $p(\mathbb{R}^2)$ is closed. Applying Theorem 1 we deduce $p(\mathbb{R}^2) = \mathbb{R}^2$. In particular, $(0,0) \in p(\mathbb{R}^2)$, this is equivalent to the existence of $\xi \in \mathbb{C}$ such that $p(\xi) = 0$.

Remark 7. Although the Fundamental Theorem of Algebra has been proved for more than 200 years, new proofs of this theorem keep emerging even in the last decades, see e.g. [2, 3]. The spirit of our proof is somewhat similar to [3]. In [3], the concepts of open (and closed) subsets of a topological subspace of \mathbb{C} and the connectedness of such a subspace, are employed. In this sense, our proof is more elementary.

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