A GCD-weighted Trigonometric Sum

Proposed by Nikolai Osipov, Siberian Federal University, Krasnoyarsk, Russia. Given an odd positive integer N, compute

$$\sum_{k=1}^{N} \frac{\gcd(k,N)}{\cos^2(\pi k/N)}.$$

Solution by the proposer. For a positive integer N, let

$$\zeta = \cos(2\pi/N) + i\sin(2\pi/N)$$

be the fixed *primitive Nth root of unity* in the field of complex numbers $\mathbb{C}$. Below we use the following general statement.

Lemma. *Suppose a rational function $R \in \mathbb{C}(z)$ has no poles at $z = \zeta^k$ with $k \in \mathbb{Z}$. Then*

$$\frac{1}{N}\sum_{k=1}^{N} R(\zeta^k) = \sum_{0\neq a\in P_R} \operatorname*{res}_{z=a} F(z) + \operatorname*{res}_{z=0} F(z) + \operatorname*{res}_{z=\infty} F(z)$$

where P_R is the set of poles of R and $F \in \mathbb{C}(z)$ is defined by

$$F(z) = \frac{R(z)}{z(1-z^N)}.$$

The proof of lemma follows immediately from the well-known *residue theorem* in *complex analysis* applied to the rational function F (we recommend the reader calculate carefully the residue of $F(z)$ at $z = \zeta^k$ for any $k \in \mathbb{Z}$).

1. We begin by showing that

$$S(N) = \sum_{k=1}^{N} \frac{1}{\cos^2(\pi k/N)} = N^2$$

for any odd positive integer N. For this purpose, note that

$$\cos^2(\pi k/N) = \frac{1+\cos(2\pi k/N)}{2} = \frac{2+\zeta^k+\zeta^{-k}}{4} = \frac{(\zeta^k+1)^2}{4\zeta^k}.$$

Thus, we have

$$S(N) = \sum_{k=1}^{N} R(\zeta^k)$$

with the rational function

$$R(z) = \frac{4z}{(z+1)^2}.$$

Clearly, $P_R = \{-1\}$. For the corresponding rational function

$$F(z) = \frac{4}{(z+1)^2(1-z^N)}$$

we obtain the following:

$$\operatorname*{res}_{z=-1} F(z) = N, \quad \operatorname*{res}_{z=0} F(z) = \operatorname*{res}_{z=\infty} F(z) = 0$$

(an easy calculation of these residues is left to the reader as an exercise). Therefore, from the above lemma we find $S(N) = N^2$ as desired.

2. Let

$$N = \prod_{j=1}^{s} p_j^{\alpha_j}$$

be the *prime power decomposition* of N. For any odd positive integer N, we prove that

$$S^*(N) = \sum_k \frac{1}{\cos^2(\pi k/N)} = N^2 \prod_{j=1}^{s} \left(1 - \frac{1}{p_j^2}\right)$$

where the sum is taken over all k such that $1 \leqslant k \leqslant N$ and $\gcd(k, N) = 1$. Indeed, using the well-known *inclusion–exclusion principle*, we obtain

$$S^*(N) = S(N) - \sum_{j=1}^{s} S(N/p_j) + \ldots = N^2 - \sum_{j=1}^{s} \frac{N^2}{p_j^2} + \ldots =$$
$$= N^2 \prod_{j=1}^{s} \left(1 - \frac{1}{p_j^2}\right).$$

3. The required sum can be computed as follows. Firstly, we get

$$\sum_{k=1}^{N} \frac{\gcd(k, N)}{\cos^2(\pi k/N)} = \sum_{d|N} dS^*(N/d) = N \sum_{d'|N} \frac{S^*(d')}{d'}$$

where $d = \gcd(k, N)$ and $d' = N/d$. Secondly, we note that the function

$$f(N) = \sum_{d'|N} \frac{S^*(d')}{d'}$$

is *multiplicative*. Indeed, since the function

$$\frac{S^*(N)}{N} = N \prod_{j=1}^{s} \left(1 - \frac{1}{p_j^2}\right)$$

is obviously multiplicative, the function $f(N)$ is also multiplicative by the well-known theorem (see, e.g., [1, Sec. 1.4.1] for more details). So, we need only to calculate $f(N)$ for all prime powers $N = p^\alpha$. This is quite easy:

$$f(p^\alpha) = 1 + \sum_{\beta=1}^{\alpha} p^\beta \left(1 - \frac{1}{p^2}\right) = p^\alpha + p^{\alpha-1} - \frac{1}{p}.$$

Finally, we obtain

$$\sum_{k=1}^{N} \frac{\gcd(k, N)}{\cos^2(\pi k/N)} = N \prod_{j=1}^{s} \left(p_j^{\alpha_j} + p_j^{\alpha_j - 1} - \frac{1}{p_j} \right)$$

as the answer. ■

Remark. As a corollary, we see that the value of the sum in our problem is always an integer (it seems that this fact is not evident). One can find more information on the sums of such kind (including the well-known *Ramanujan's sums*) and their applications in the papers [2] and [3].

REFERENCES

1. S.Y. Yan, *Number theory for computing*, Springer-Verlag, New York, 2002.
2. S. Ramanujan, On certain trigonometrical sums and their applications in the theory of numbers, *Trans. Cambridge Philos. Soc.* **22** (1918) 259–276.
3. E. Cohen, Trigonometric sums in elementary number theory, *Amer. Math. Monthly* **66** (1959) 105–117.