# A Probabilistic Proof of a Wallis-type Formula for the Gamma Function 

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#### Abstract

We use well-known limit theorems in probability theory to derive a Wallis-type product formula for the gamma function. Our result immediately provides a probabilistic proof of Wallis's product formula for $\pi$, as well as the duplication formula for the gamma function.


In 1655, Wallis [4, Prop. 191] wrote down the following beautiful formula for $\pi$ :

$$
\begin{equation*}
\frac{\pi}{2}=\prod_{n=1}^{\infty}\left(\frac{2 n}{2 n-1} \cdot \frac{2 n}{2 n+1}\right)=\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdots \tag{1}
\end{equation*}
$$

Ever since the formula's discovery, various proofs of Wallis's product formula have been found, and each of them has its own merits. One of the more common proofs of the formula uses a recursion derived from integrating trigonometric functions. Another proof simply plugs in $x=\pi / 2$ into Euler's infinite product formula

$$
\begin{equation*}
\frac{\sin x}{x}=\prod_{n=1}^{\infty}\left(1-\frac{x^{2}}{n^{2} \pi^{2}}\right) \tag{2}
\end{equation*}
$$

Although this proof is perhaps the shortest one, proving the above product formula for sine requires some amount of work.

The purpose of this note is to use well-known limit theorems in probability theory to derive a Wallis-type product formula for the gamma function. A socalled duplication formula for the Gamma function will easily follow from the product formula. The Gamma function $\Gamma:(0, \infty) \rightarrow \mathbf{R}$, which we only define for positive real numbers for simplicity, is given by

$$
\Gamma(\alpha)=\int_{0}^{\infty} e^{-t} t^{\alpha-1} d t
$$

A direct computation shows $\Gamma(1 / 2)=\sqrt{\pi}$, and this will let us derive (1) from a more general product formula for the Gamma function. By integration by parts, one can easily check that $\Gamma(\alpha)=(\alpha-1) \Gamma(\alpha-1)$ for any $\alpha>1$. From this $\Gamma(n)=(n-1)$ ! for all $n \in \mathbf{N}$ follows.

The Gamma function is closely related to spheres and spherical coordinates. For any $n \in\{2,3,4, \ldots\}$, the surface area of the unit sphere

$$
S^{n-1}=\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbf{R}^{n} \mid x_{1}^{2}+\cdots+x_{n}^{2}=1\right\}
$$

embedded in $\mathbf{R}^{n}$ is $2 \pi^{n / 2} / \Gamma(n / 2)$. Also, for any continuous $f:[0, \infty) \rightarrow \mathbf{R}$ with $\int_{0}^{\infty}|f(r)| r^{n-1} d r<\infty$, we have

$$
\begin{equation*}
\int_{\mathbf{R}^{n}} f\left(\sqrt{x_{1}^{2}+\cdots+x_{n}^{2}}\right) d x_{1} \cdots d x_{n}=\frac{2 \pi^{n / 2}}{\Gamma(n / 2)} \int_{0}^{\infty} f(r) r^{n-1} d r \tag{3}
\end{equation*}
$$

For more details on the Gamma function, see [1, p. 58 and Section 2.7].
If we restrict our interest to just proving (11), then there already exist some probabilistic proofs. A proof by Miller [2] uses the fact that for any $\nu \in \mathbf{N}$, the function $f: \mathbf{R} \rightarrow[0, \infty)$ given by

$$
f(t)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi \nu} \Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{t^{2}}{\nu}\right)^{-\frac{\nu+1}{2}}
$$

is a probability density, i.e., it is nonnegative and has a total integral of one. The distribution with the density $f$ is called Student's $t$-distribution with $\nu$ degrees of freedom. Another proof by Wei, Li, and Zheng [5] derives (1) from a version of the central limit theorem applied to certain familiar discrete random variables.

Theorem 1. If $\alpha>0$, then

$$
\begin{equation*}
\lim _{k \rightarrow \infty} k^{\alpha} \frac{(k-1)(k-2) \cdots 1}{(k-1+\alpha)(k-2+\alpha) \cdots \alpha}=\Gamma(\alpha) \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \sqrt{\pi} k^{\alpha} \frac{\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right) \cdots \frac{1}{2}}{\left(k-\frac{1}{2}+\alpha\right)\left(k-\frac{3}{2}+\alpha\right) \cdots\left(\frac{1}{2}+\alpha\right)}=\Gamma\left(\alpha+\frac{1}{2}\right) \tag{5}
\end{equation*}
$$

where $k$ ranges over positive integers.
Proof. Consider a family $X_{1}, X_{2}, \ldots$ of independent standard normal random variables. The proof is established by investigating the value of

$$
\mathbf{E}\left(X_{1}^{2}+\cdots+X_{n}^{2}\right)^{\alpha}
$$

in two different ways: the first way uses well-known limit theorems while the second way is purely computational. In fact, this value is the moment of order $\alpha$ of a chi-squared distribution. However, we won't assume any prior knowledge of chi-squared distributions in this note.

Let us start with the approach using limit theorems. By the weak law of large numbers we have

$$
\frac{X_{1}^{2}+\cdots+X_{n}^{2}}{n} \rightarrow 1 \quad \text { in probability }
$$

Applying the continuous function $f(y)=|y|^{\alpha}$ to both sides above and using the continuous mapping theorem [3, Corollary 6.3.1 (ii)], which tells us that convergence in probability is preserved under continuous maps, we also get

$$
\left(\frac{X_{1}^{2}+\cdots+X_{n}^{2}}{n}\right)^{\alpha} \rightarrow 1 \quad \text { in probability. }
$$

Note that

$$
\mathbf{E}\left(\frac{X_{1}^{2}+\cdots+X_{n}^{2}}{n}\right)^{2}=\frac{n \mathbf{E} X_{1}^{4}+n(n-1)\left(\mathbf{E} X_{1}^{2}\right)^{2}}{n^{2}} \leq \max \left\{\mathbf{E} X_{1}^{4},\left(\mathbf{E} X_{1}^{2}\right)^{2}\right\}
$$

Similarly, for any integer $p>\alpha$ we have

$$
\mathbf{E}\left(\frac{X_{1}^{2}+\cdots+X_{n}^{2}}{n}\right)^{p} \leq \max \left\{\mathbf{E} X_{1}^{2 p}, \ldots,\left(\mathbf{E} X_{1}^{2}\right)^{p}\right\}<\infty
$$

This shows that

$$
\mathbf{E}\left(\left(\frac{X_{1}^{2}+\cdots+X_{n}^{2}}{n}\right)^{\alpha}\right)^{p / \alpha}=\mathbf{E}\left(\frac{X_{1}^{2}+\cdots+X_{n}^{2}}{n}\right)^{p}
$$

is bounded uniformly in $n$, and thus the family

$$
\left\{\left(\frac{X_{1}^{2}+\cdots+X_{n}^{2}}{n}\right)^{\alpha}\right\}_{n=1}^{\infty}
$$

is uniformly integrable. What we used here is sometimes called the "crystal ball condition"; see [3, p. 184]. Since any uniformly integrable sequence of random variables that converges in probability also converges in $L^{1}$, see [3, Theorem 6.6.1], we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbf{E}\left(\frac{X_{1}^{2}+\cdots+X_{n}^{2}}{n}\right)^{\alpha}=\mathbf{E} 1=1 \tag{6}
\end{equation*}
$$

Let us next directly compute $\mathbf{E}\left(X_{1}^{2}+\cdots+X_{n}^{2}\right)^{\alpha}$ by integration:

$$
\begin{aligned}
\mathbf{E}\left(X_{1}^{2}+\cdots+X_{n}^{2}\right)^{\alpha} & =\int_{\mathbf{R}^{n}}\left(x_{1}^{2}+\cdots+x_{n}^{2}\right)^{\alpha} \cdot \frac{1}{(2 \pi)^{n / 2}} e^{-\left(x_{1}^{2}+\cdots+x_{n}^{2}\right) / 2} d x_{1} \cdots d x_{n} \\
& =\frac{2 \cdot \pi^{n / 2}}{\Gamma(n / 2)} \int_{0}^{\infty} r^{2 \alpha} \cdot \frac{1}{(2 \pi)^{n / 2}} e^{-r^{2} / 2} \cdot r^{n-1} d r \\
& =\frac{1}{2^{(n / 2)-1} \Gamma(n / 2)} \int_{0}^{\infty} r^{n+2 \alpha-1} e^{-r^{2} / 2} d r .
\end{aligned}
$$

We used (3) in the second equality. Continuing our calculations, we observe that

$$
\begin{aligned}
& \frac{1}{2^{(n / 2)-1} \Gamma(n / 2)} \int_{0}^{\infty} r^{n+2 \alpha-1} e^{-r^{2} / 2} d r \\
& =\frac{1}{2^{(n / 2)-1} \Gamma(n / 2)} \int_{0}^{\infty}(2 u)^{(n / 2)+\alpha-1} e^{-u} d u \\
& =2^{\alpha} \cdot \frac{\Gamma((n / 2)+\alpha)}{\Gamma(n / 2)}
\end{aligned}
$$

Finally, we conflate the two approaches. By (6) and the previous computation, we have

$$
\lim _{n \rightarrow \infty}\left(\frac{n}{2}\right)^{-\alpha} \cdot \frac{\Gamma((n / 2)+\alpha)}{\Gamma(n / 2)}=1
$$

Plug in $n=2 k$ and $n=2 k+1$. Then, using $\Gamma(x)=(x-1) \Gamma(x-1)$ to expand both the numerator and denominator of the left side, and applying $\lim _{k \rightarrow \infty}\left[\left(k+\frac{1}{2}\right) / k\right]^{\alpha}=1$, we have

$$
\lim _{k \rightarrow \infty} k^{-\alpha} \frac{(k-1+\alpha)(k-2+\alpha) \cdots \alpha \Gamma(\alpha)}{(k-1)(k-2) \cdots 1}=1
$$

and

$$
\lim _{k \rightarrow \infty} k^{-\alpha} \frac{\left(k-\frac{1}{2}+\alpha\right)\left(k-\frac{3}{2}+\alpha\right) \cdots\left(\alpha+\frac{1}{2}\right) \Gamma\left(\alpha+\frac{1}{2}\right)}{\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right) \cdots \frac{1}{2} \Gamma\left(\frac{1}{2}\right)}=1 .
$$

Taking the reciprocal and using $\Gamma(1 / 2)=\sqrt{\pi}$ concludes the proof.
In case $\alpha$ is rational, we can estimate $\Gamma(\alpha)$ by a ratio of products of integers.
Corollary 1. For any positive integers $p$ and $q$, we have

$$
\lim _{k \rightarrow \infty} q \cdot \frac{k^{p / q}((k-1) q)((k-2) q) \cdots q}{((k-1) q+p)((k-2) q+p) \cdots p}=\Gamma\left(\frac{p}{q}\right)
$$

where $k$ ranges over positive integers.
Proof. By applying (4) with $\alpha=p / q$, we obtain

$$
\begin{aligned}
& \lim _{k \rightarrow \infty} q \cdot \frac{k^{p / q}((k-1) q)((k-2) q) \cdots q}{((k-1) q+p)((k-2) q+p) \cdots p} \\
&=\lim _{k \rightarrow \infty} k^{p / q} \cdot \frac{(k-1)(k-2) \cdots 1}{\left(k-1+\frac{p}{q}\right)\left(k-2+\frac{p}{q}\right) \cdots \frac{p}{q}}=\Gamma\left(\frac{p}{q}\right) .
\end{aligned}
$$

The formula for $\Gamma(1 / 2)$ leads us to Wallis's original formula.
Corollary 2 (Wallis).

$$
\frac{\pi}{2}=\prod_{n=1}^{\infty}\left(\frac{2 n}{2 n-1} \cdot \frac{2 n}{2 n+1}\right)=\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdots .
$$

Proof. Applying Corollary 1 with $p=1$ and $q=2$ gives

$$
\lim _{k \rightarrow \infty} 2 \cdot \frac{\sqrt{k} \cdot 2 \cdot 4 \cdots(2 k-4)(2 k-2)}{1 \cdot 3 \cdots(2 k-3)(2 k-1)}=\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi} .
$$

Dividing both sides by $\sqrt{2}$ and taking the square of both sides, we have

$$
\lim _{k \rightarrow \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2 k-2}{2 k-3} \cdot \frac{2 k-2}{2 k-1} \cdot \frac{2 k}{2 k-1}=\frac{\pi}{2}
$$

which implies the desired formula.

Combining (4) and (5), we can provide a proof of the following.
Corollary 3 (duplication formula). For any $\alpha>0$, we have

$$
\Gamma(\alpha) \Gamma\left(\alpha+\frac{1}{2}\right)=2^{1-2 \alpha} \sqrt{\pi} \Gamma(2 \alpha)
$$

Proof. Multiplying (4) and (5), we have

$$
\sqrt{\pi} k^{2 \alpha} \frac{\left(k-\frac{1}{2}\right)(k-1) \cdots 1 \cdot \frac{1}{2}}{\left(k-\frac{1}{2}+\alpha\right)(k-1+\alpha) \cdots\left(\alpha+\frac{1}{2}\right) \cdot \alpha} \rightarrow \Gamma(\alpha) \Gamma\left(\alpha+\frac{1}{2}\right) .
$$

By multiplying $2^{2 k}$ to both the numerator and the denominator, we have

$$
2^{1-2 \alpha} \sqrt{\pi}(2 k)^{2 \alpha} \frac{(2 k-1)(2 k-2) \cdots 1}{(2 k-1+2 \alpha)(2 k-2+2 \alpha) \cdots(2 \alpha)} \rightarrow \Gamma(\alpha) \Gamma\left(\alpha+\frac{1}{2}\right) .
$$

By noticing that the previous formula contains (4) with $\alpha$ and $k$ replaced by $2 \alpha$ and $2 k$, we obtain the desired formula.

## References

[1] Folland, G. B. (1999). Real Analysis. Modern Techniques and their Applications, 2nd ed. New York: John Wiley \& Sons, Inc.
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