GENERALIZATION OF A RAMANUJAN IDENTITY

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Abstract. The Euler product for the Landau–Ramanujan constant could have motivated a curious identity by Ramanujan that appears in his notebooks two times. This observation involves a square root and the first four prime numbers of the form $4n+3$, i.e., $3, 7, 11, 19$. Berndt asks whether Ramanujan's identity is an isolated result, or if there are other identities of this type. With this work we would like to give a possible answer to Berndt's question.

1. Introduction

Let $B(x)$ denote the number of positive integers not exceeding x that can be expressed as a sum of two squares. Landau [6], [7, pp. 641–669] in 1908 showed that

$$B(x) \sim K \frac{x}{\sqrt{\log x}} \quad \text{as } x \to \infty,$$

where K is a constant. Independently, Ramanujan in his first letter to Hardy [2, pp. 52 and 60–62], [3, p. 24], [5, p. xxiv] in 1913 stated the following. The number of numbers greater than A and less than x that can be expressed as a sum of two squares is

$$K \int_A^x \frac{dt}{\sqrt{\log t}} + \theta(x),$$

where $K = 0.764\ldots$ and $\theta(x)$ is very small when compared with the previous integral. This statement also appears in Ramanujan's second and third notebooks [8, pp. 307 and 363].

Since then the quantity K has been known as the Landau–Ramanujan constant [4, pp. 98–104]. An exact formula for K is given by its Euler product expansion

$$K = \frac{1}{\sqrt{2}} \prod_p \left(\frac{1}{1 - 1/p^2} \right)^{1/2},$$

where p runs through the primes of the form $4n+3$. This could have motivated the following observation that appears in Ramanujan's notebooks [8, pp. 309 and 363] two times:

$$\sqrt{2\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\left(1-\frac{1}{19^2}\right)} = \left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{19}\right). \tag{1}$$

"It may be somewhat interesting to note," Ramanujan wrote about (1) in one of his letters to Hardy [3, p. 177], referring to the fact that the squared numbers on the left-hand side are the first four prime numbers of the form $4n+3$. We examine the identity (1) that can be found in Berndt's book [2, p. 20] and also in the Andrews–Berndt book [1, pp. 410–411] with only a brief discussion.

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2. Generalization of Ramanujan's identity

Berndt [2, p. 20, Entry 6] asks whether Ramanujan's identity in (1) is an isolated result, or if there are other identities of this type. The result of this section is a possible answer to Berndt's question.

Lemma 2.1. *Let $n \geq m \geq 1$ be integers, and let (a_k) be a sequence of real numbers such that $a_k \neq 0, 1$ for all $k = m, \ldots, n$ and $-1 < a_\ell < 0$ for an even number of $\ell \in \{m, \ldots, n\}$ indices. Then*

$$\sqrt{\prod_{k=m}^{n} \frac{a_k + 1}{a_k - 1} \cdot \prod_{k=m}^{n} \left(1 - \frac{1}{a_k^2}\right)} = \prod_{k=m}^{n} \left(1 + \frac{1}{a_k}\right). \tag{2}$$

Proof. Because of the condition on the elements of (a_k), the expression under the square root and the right-hand side are nonnegative. Both sides of (2) are equal to zero if and only if $a_k = -1$ for some $k \in \{m, \ldots, n\}$. Suppose that $a_k \neq -1$ for all $k = m, \ldots, n$. Note that for a given real number $a \neq 0, \pm 1$, we have

$$\frac{a+1}{a-1} = \frac{1 + \frac{1}{a}}{1 - \frac{1}{a}} = \frac{\left(1 + \frac{1}{a}\right)^2}{\left(1 - \frac{1}{a}\right)\left(1 + \frac{1}{a}\right)} = \frac{\left(1 + \frac{1}{a}\right)^2}{1 - \frac{1}{a^2}}. \tag{3}$$

By using this observation, straightforward arithmetic gives the result. □

Henceforth we use Lemma 2.1 to deduce such identities of the form of (2), which have a closed-form expression for the product $\prod_{k=m}^{n} \frac{a_k+1}{a_k-1}$.

Theorem 2.2 (Generalization of Ramanujan's identity)**.** *Let $a \in (-\infty, -2/3) \cup (-1/2, -1/3] \cup ((-1/6, \infty) \setminus \{0, 1\})$ be a real number. Then*

$$\sqrt{\frac{a+1}{a-1}\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{(2a+1)^2}\right)\left(1 - \frac{1}{(3a+2)^2}\right)\left(1 - \frac{1}{(6a+1)^2}\right)}$$
$$= \left(1 + \frac{1}{2a+1}\right)\left(1 + \frac{1}{3a+2}\right)\left(1 + \frac{1}{6a+1}\right). \tag{4}$$

By substituting $a = 3$ into (4), we arrive at Ramanujan's identity (1).

Proof. Suppose that $a \neq -1/3$. We can use Lemma 2.1 with $(a_k) = (2a+1, 3a+2, 6a+1)$. For the first product of (2), we find that

$$\prod_{k=1}^{3} \frac{a_k + 1}{a_k - 1} = \frac{(2a+1)+1}{(2a+1)-1} \cdot \frac{(3a+2)+1}{(3a+2)-1} \cdot \frac{(6a+1)+1}{(6a+1)-1} = \frac{(a+1)^2}{a^2}.$$

On the other hand, if we suppose that $a \neq -1$, by using (3), we have

$$\frac{a+1}{a-1}\left(1 - \frac{1}{a^2}\right) = \frac{\left(1 + \frac{1}{a}\right)^2}{1 - \frac{1}{a^2}}\left(1 - \frac{1}{a^2}\right) = \frac{(a+1)^2}{a^2}.$$

Since both sides of (4) are equal to zero if and only if $a = -1$ or $a = -1/3$, the proof is complete. □

Remark 2.3 (Alternative form)**.** *It is clear from the proof of Theorem 2.2 that the following identity holds. Let* $a \in (-\infty, -2/3) \cup (-1/2, -1/3] \cup ((-1/6, \infty) \setminus \{0\})$ *be a real number. Then*

$$\frac{a+1}{a} \cdot \sqrt{\left(1 - \frac{1}{(2a+1)^2}\right)\left(1 - \frac{1}{(3a+2)^2}\right)\left(1 - \frac{1}{(6a+1)^2}\right)} = \left(1 + \frac{1}{2a+1}\right)\left(1 + \frac{1}{3a+2}\right)\left(1 + \frac{1}{6a+1}\right). \quad (5)$$

We can derive similar identities by using Lemma 2.1 with the sequence $(a_k) = (2a+1, 3a+1, 6a+5)$ or with $(a_k) = (2a+1, 4a+1, 4a+3)$ with a suitable condition on a. By substituting $a = 1$ into (5), we find that

$$2 \cdot \sqrt{\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)} = \left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{7}\right). \quad (6)$$

We generalize (6) for further odd denominators in (9) of Theorem 3.1.

3. Identities with telescoping products

In this section we use Lemma 2.1 with appropriate (a_k) sequences, for which the product $\prod_{k=m}^{n} \frac{a_k+1}{a_k-1}$ has a telescoping property.

Theorem 3.1. *Let* n *and* m *be integers. For* $n \geq m \geq 2$*, we have*

$$\sqrt{\frac{n(n+1)}{m(m-1)} \cdot \prod_{k=m}^{n} \left(1 - \frac{1}{k^2}\right)} = \prod_{k=m}^{n} \left(1 + \frac{1}{k}\right). \quad (7)$$

For $n \geq m \geq 1$*, we have*

$$\sqrt{\frac{2n+1}{2m-1} \cdot \prod_{k=m}^{n} \left(1 - \frac{1}{(2k)^2}\right)} = \prod_{k=m}^{n} \left(1 + \frac{1}{2k}\right) \quad (8)$$

and

$$\sqrt{\frac{n+1}{m} \cdot \prod_{k=m}^{n} \left(1 - \frac{1}{(2k+1)^2}\right)} = \prod_{k=m}^{n} \left(1 + \frac{1}{2k+1}\right). \quad (9)$$

Note that (7) gives $(n+1)/m$, which appears under the square root in (9).

Proof. In order to prove (7), according to Lemma 2.1, we have to show the closed-form of a telescoping product. We find that

$$\prod_{k=m}^{n} \frac{k+1}{k-1} = \frac{m+1}{m-1} \cdot \frac{m+2}{m} \cdot \frac{m+3}{m+1} \cdot \cdots \cdot \frac{n-1}{n-3} \cdot \frac{n}{n-2} \cdot \frac{n+1}{n-1} = \frac{n(n+1)}{m(m-1)}.$$

The proofs of (8) and (9) are analogous. □

Berndt's question may have other interesting answers. It would be worth examining Lemma 2.1 further with various (a_k) sequences. In the following theorem, we use $(a_k) = (k^3)$.

Theorem 3.2. *Let $n \geq m \geq 2$ be integers. Then*

$$\sqrt{\frac{m(m-1)+1}{m(m-1)} \cdot \frac{n(n+1)}{n(n+1)+1} \cdot \prod_{k=m}^{n} \left(1 - \frac{1}{k^6}\right)} = \prod_{k=m}^{n} \left(1 + \frac{1}{k^3}\right).$$

Proof. According to Lemma 2.1, we have to deduce the following.

$$\prod_{k=m}^{n} \frac{k^3+1}{k^3-1} = \frac{m^3+1}{(n+1)^3+1} \cdot \prod_{k=m}^{n} \frac{(k+1)^3+1}{k^3-1} = \frac{m^3+1}{(n+1)^3+1} \cdot \prod_{k=m}^{n} \frac{k+2}{k-1}$$
$$= \frac{m^3+1}{(n+1)^3+1} \cdot \frac{n(n+1)(n+2)}{(m-1)m(m+1)} = \frac{m(m-1)+1}{m(m-1)} \cdot \frac{n(n+1)}{n(n+1)+1}. \qquad \square$$

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