A Probabilistic Proof of a Lemma that is not Burnside's

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If the group G acts on the set X, we define the *orbit* of the element $x \in X$ as

$$\mathrm{orb}(x) = \{gx \,:\, g \in G\}.$$

The orbits partition X, and the orbit-counting lemma (see Neumann [2] for the history of its name) shows how to compute the number of orbits. Bogart [1] gave a proof of this result using only multisets and the product rule. We give a probabilistic version of his proof.

The Orbit-Counting Lemma. *Suppose the finite group G acts on the finite set X and let* $\mathrm{fix}(g) = \{x \in X \,:\, gx = x\}$. *Then the number of orbits of X is*

$$\frac{1}{|G|}\sum_{g \in G} |\,\mathrm{fix}(g)|.$$

Proof. Choose, in order and each uniformly at random, an element $g \in G$, an orbit of X, and an element y of this orbit. Because $gy = x$ for some $x \in X$,

$$1 = \sum_{x \in X} \Pr\big[\, gy = x \,\big] = \sum_{x \in X} \Pr\big[\, y \in \mathrm{orb}(x) \,\big] \cdot \Pr\big[\, gy = x \,\big|\, y \in \mathrm{orb}(x) \,\big].$$

If $y \in \mathrm{orb}(x)$, then there is some $k \in G$ such that $kx = y$, so the mapping $h \mapsto k^{-1}h^{-1}$ is a bijection between $\{h \in G \,:\, hy = x\}$ and $\{h \in G \,:\, hx = x\}$. By applying this (measure-preserving) mapping, it follows that, for every $x \in X$,

$$\Pr\big[\, gy = x \,\big|\, y \in \mathrm{orb}(x) \,\big] = \Pr\big[\, gx = x \,\big|\, y \in \mathrm{orb}(x) \,\big] = \Pr\big[\, gx = x \,\big].$$

Also, since the orbit y was chosen from was itself chosen uniformly at random,

$$\frac{1}{\#\,\text{orbits}} \sum_{x \in X} \Pr\big[\, gx = x \,\big] = 1.$$

This implies that the number of orbits is equal to

$$\sum_{x \in X} \Pr\big[\, gx = x \,\big] = \frac{1}{|G|}\big|\big\{(g,x) \in G \times X \,:\, gx = x\big\}\big| = \frac{1}{|G|}\sum_{g \in G} |\,\mathrm{fix}(g)|,$$

as claimed. $\square$

References

[1] Bogart, K. (1991). An obvious proof of Burnside's lemma. *Amer. Math. Monthly.* **98**(10): 927–928.

[2] Neumann, P. (1979). A lemma that is not Burnside's. *Math. Sci.* **4**(2): 133–141.

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