# ARITHMETIC PROGRESSIONS WITH RESTRICTED DIGITS

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ABSTRACT. For an integer  $b \ge 2$  and a set  $S \subset \{0, \dots, b-1\}$ , we define the *Kempner set*  $\mathcal{K}(S, b)$  to be the set of all non-negative integers whose base-b digital expansions contain only digits from S. These wellstudied sparse sets provide a rich setting for additive number theory, and in this paper we study various questions relating to the appearance of arithmetic progressions in these sets. In particular, for all  $b$  we determine exactly the maximal length of an arithmetic progression that omits a base-b digit.

### 1. INTRODUCTION

In 1914 Kempner [\[7\]](#page-10-0) introduced a variant of the harmonic series which excluded from its sum all those positive integers that contain the digit 9 in their base-10 expansions. Unlike the familiar harmonic series, Kempner's modified series converges (the limit later shown to be  $\approx 22.92$ , see [\[1\]](#page-10-1)). A simple generalisation of Kempner's original argument shows that convergence occurs as long as any non-empty set of digits is excluded, and that this result holds in any base (see [\[11\]](#page-10-2), for example).

Let us introduce some notation to describe these results in general. Fix an integer  $b \ge 2$  and a subset of integers  $S \subseteq [0, b-1]$ . Here and throughout the paper, for two integers x and y we use  $[x, y]$  to denote the set  $\{n \in$  $\mathbb{Z}: x \leqslant n \leqslant y$ . We then define the *Kempner set*  $\mathcal{K}(S, b)$  to be the set of non-negative integers that, when written in base  $b$ , contain only digits from S. Thus  $\mathcal{K}([0,8], 10)$  denotes the set originally studied by Kempner. We will assume throughout that  $0 \in S$ , to avoid the ambiguity of leading zeros, and require  $S \neq [0, b-1]$ , to preclude the trivial set  $\mathcal{K}([0, b-1], b)$  (which is nothing more than  $\mathbb{Z}_{\geqslant 0}$ . These sets S will be referred to as the *permitted* sets S, and the related Kempner sets  $K(S, b)$  as proper Kempner sets.

The arithmetic properties of proper Kempner sets have been the object of considerable study in recent years, beginning with the work of Erdős, Mauduit, and Sárközy, who studied the distribution of residues in  $\mathcal{K}(S, b)$ moduli small numbers [\[3\]](#page-10-3) and proved the existence of integers in  $\mathcal{K}(S, b)$  with many small prime factors [\[4\]](#page-10-4). Notable recent work includes Maynard's proof [\[8\]](#page-10-5) that the sets  $\mathcal{K}(S, b)$  contain infinitely many primes whenever  $b - |S|$  is at most  $b^{23/80}$ , provided b is sufficiently large.

In this paper we consider the additive structure of proper Kempner sets. In particular, we consider the following extremal question: what is the length of the longest arithmetic progression in a proper Kempner set with a fixed given base? Our methods will be combinatorial, rather than analytic (as in Maynard's work, [\[8\]](#page-10-5)).

A well known conjecture of Erdős-Turán (first given in [\[5\]](#page-10-6)) states that any set of positive integers with a divergent harmonic sum contains arithmetic progressions of arbitrary (finite) length. Since proper Kempner sets have convergent harmonic sums, this might suggest that the lengths of arithmetic progressions in a given proper Kempner set are uniformly bounded.

This is indeed the case. Let us say that a set  $T \subset \mathbb{Z}$  is k-free if T contains no arithmetic progression of length  $k$ . By a simple argument, given in Proposition [2.1,](#page-2-0) one may show that the proper Kempner set  $\mathcal{K}(S, b)$  is  $(b^2 - b + 1)$ -free for any  $b \ge 2$ .

The main purpose of this article is to understand how close this trivial upper bound is to the truth.

In our main theorem, we improve this bound for all  $b > 2$ , obtaining a tight result that expresses the length of the longest arithmetic progression in  $\mathcal{K}(S, b)$  in terms of the prime factorisation of b. To state this theorem, we need to introduce some arithmetic functions. If  $n$  and  $b$  are natural numbers, let  $\rho(n)$  denote the square-free radical of n (ie. the product of all distinct primes dividing n), and let  $\beta(b)$  denote the largest integer less than b such that  $\rho(\beta(b))|b$ . For example,  $\beta(10) = 8$ , and  $\beta(p^k) = p^{k-1}$  for any prime power  $p^k$ . In other words,  $\beta(b)$  is the greatest integer less than b that divides some power of b. Finally, let  $\ell(b)$  be the length of the longest arithmetic progression contained in some proper Kempner set of base b.

Our main theorem gives an exact evaluation of  $\ell(b)$ .

<span id="page-1-1"></span>**Theorem 1.1.** For all  $b \ge 2$ , one has  $\ell(b) = (b-1)\beta(b)$ .

For example,  $\ell(10) = 72$ . One particular set that achieves this bound is Kempner's original set,  $\mathcal{K}([0,8], 10)$ , which contains the 72-term arithmetic progression  $\{0, 125, 250, 375, \dots, 8875\}.$ 

The arithmetic functions  $\beta(b)$  and  $\ell(b)$  are of independent interest, but do not appear to have been considered seriously before.<sup>[1](#page-1-0)</sup> We establish average order results for  $\beta(b)$  which show that, for most b, the trivial upper bound on  $\ell(b)$  from Proposition [2.1](#page-2-0) is asymptotically correct.

<span id="page-1-2"></span>**Theorem 1.2.** There is a set of integers  $A \subset \mathbb{Z}$  with natural density 1, i.e. with

$$
\lim_{N \to \infty} \frac{1}{N} |A \cap [1, N]| = 1,
$$

such that  $\ell(b) \sim b^2$  as  $b \to \infty$  in A.

*Notation:* For  $x \in \mathbb{R}$ , let  $\{x\}$  denote the fractional part of x and let  $|x|$ denote the greatest integer that is at most  $x$ . For a natural number  $n$ , we

<span id="page-1-0"></span><sup>&</sup>lt;sup>1</sup>The sequence  $\beta(b)$  is entry A079277 on the Online Encyclopedia of integer Sequences.

let [n] denote the set of integers  $\{1, \dots, n\}$ . As mentioned previously, for two integers x and y we use  $[x, y]$  to denote the set  $\{n \in \mathbb{Z} : x \leqslant n \leqslant y\}$ . We use the notation  $\log_a p$  to denote the logarithm of p to base q (as opposed to any iterations of logarithms).

### 2. Progressions of Maximal Length in Kempner Sets

In this section we give our proof of Theorem [1.1,](#page-1-1) which is an exact evaluation of  $\ell(b)$  and the main result of this paper. This will be done in two parts: a constructive lower bound and a proof that this lower bound is sharp. Before that, as promised, we give a simple proof that the function  $\ell(b)$  is at least well-defined, i.e. that Kempner sets do not contain arbitrarily long arithmetic progressions.

## <span id="page-2-0"></span>**Proposition 2.1.** For all  $b \ge 2$ , we have  $\ell(b) \le (b - 1)b$ .

*Proof.* Suppose that  $A \subset \mathcal{K}(S, b)$  is a finite arithmetic progression of |A| terms with common difference  $\Delta$ . Choose  $k \geqslant 0$  such that  $b^{\overline{k}} \leqslant \Delta < b^{k+1}$ . If I denotes the shortest interval of integers containing A, then  $|I| = (|A| 1)\Delta + 1$ , hence  $|A| = 1 + (|I| - 1)/\Delta$ .

If A excludes the digit d, the upper bound  $\Delta < b^{k+1}$  confines A within the interval  $[0, db^{k+1} - 1]$  or within an interval of the form

$$
[b^{k+2}m + (d+1)b^{k+1}, b^{k+2}(m+1) + db^{k+1} - 1],
$$

for some  $m \in \mathbb{Z}_{\geqslant 0}$ . Thus  $|I| \leqslant b^{k+2} - b^{k+1}$ , which yields

$$
|A| \leq 1 + \frac{b^{k+2} - b^{k+1} - 1}{\Delta} < 1 + \frac{b^{k+2} - b^{k+1}}{b^k} \leq b^2 - b + 1,
$$

hence  $|A| \leq b^2 - b$  as claimed.

The bound in the previous proposition is simple and  $-$  as a consequence – occasionally weak. In particular, it neglects the potentially compounding effects of digit exclusion at different orders of magnitude, and the arithmetic properties of orbits in the group  $\mathbb{Z}/b\mathbb{Z}$ . This structure can affect the bounds dramatically, as seen most clearly in the case when the base b is prime.

## <span id="page-2-1"></span>**Proposition 2.2.** Let p be prime. Then  $\ell(p) \leq p - 1$ .

*Proof.* Suppose that  $\mathcal{K}(S, p)$  contains the progression  $A = \{k + j\Delta : j \in [p]\}$ with  $\Delta \neq 0$ . By the pigeonhole principle, there exist distinct  $i, j \in [p]$  with  $k + j\Delta \equiv k + i\Delta \mod p$  for some  $i \neq j$ , hence  $p | \Delta$  (since p is prime). By deleting the rightmost digits of the elements of A we obtain a new progression in  $\mathcal{K}(S, p)$  with common difference  $\Delta/p$ ; in particular, the progression

$$
\left\{ \left\lfloor \frac{k}{p} \right\rfloor + j\frac{\Delta}{p} : j \in [p] \right\}.
$$

The new common difference is strictly smaller, and we obtain a contradiction by infinite descent.

With a little more bookkeeping this proof generalizes to prime powers, and implies that  $\ell(p^k) \leqslant p^{k-1}(p^k-1)$ . So certainly  $\ell(b)$  is not asymptotic to  $b<sup>2</sup>$  as b ranges over all integers; some restriction in Theorem [1.2](#page-1-2) is required.

We now begin the proof of Theorem [1.1.](#page-1-1) Searching for long progressions in  $\mathcal{K}([0,8], 10)$ , one might happen across the example noted earlier, namely the first 71 multiples of 125, which – together with  $0$  – form an arithmetic progression of length 72, none of whose members contain the digit 9. This example succeeds due to properties of the prime factorisation of 1000/125, in relation to the base 10. These properties generalise, and one may use this to construct long digit-excluding arithmetic progressions in arbitrary bases.

**Proposition 2.3.** For all  $b \ge 2$ , the Kempner set  $\mathcal{K}([0, b-2], b)$  contains an arithmetic progression of length  $(b-1)\beta(b)$ . Hence  $\ell(b) \geq (b-1)\beta(b)$ .

*Proof.* Let  $K \geq 1$  be the smallest natural number such that  $\beta(b)|b^K$ . We claim that all the members of the arithmetic progression

$$
A = \frac{b^K}{\beta(b)}[0, (b-1)\beta(b) - 1]
$$

exclude the digit  $b-1$  from their base- $b$  expansions. To see this, let  $k$  satisfy  $0 \leq k \leq K-1$ . Then  $gcd(b^{k+1}\beta(b), b^K) \geq b^{k+1} > b^k\beta(b)$ , which implies that  $gcd(b^{k+1}, b^{K}/\beta(b)) > b^{k}$  (by dividing through by  $\beta(b)$ ). In particular, for all integers  $x$  and  $y$ , either

<span id="page-3-0"></span>
$$
\left| x \frac{b^K}{\beta(b)} - y b^{k+1} \right| > b^k \quad \text{or} \quad x \frac{b^K}{\beta(b)} = y b^{k+1}.
$$
 (2.1)

This observation implies that none of the  $K$  rightmost digits of any integer of the form  $xb^K/\beta(b)$  can be equal to  $b-1$ . Indeed, in base b, the  $b^k$  digit of  $xb^K/\beta(b)$  is the unique integer d in the range  $0 \leq d \leq b-1$  such that

$$
\left\{\frac{xb^K/\beta(b)}{b^{k+1}}\right\} \in \left[\frac{d}{b}, \frac{d+1}{b}\right).
$$

Yet [\(2.1\)](#page-3-0) implies that  $\frac{xb^{K}}{bk+1a}$  $\frac{xb^{K}}{b^{k+1}\beta(b)}\}\in\{0\}\cup(\frac{1}{b})$  $\frac{1}{b}$ ,  $\frac{b-1}{b}$  $\frac{-1}{b}$ ) for each  $0 \leq k \leq K - 1$ . Since this is disjoint from  $\left[\frac{b-1}{b}, 1\right)$ , we conclude that none of the K rightmost digits of any integer of the form  $xb^K/\beta(b)$  can be equal to  $b-1$ .

We now fix  $x \in [0,(b-1)\beta(b)-1]$  and consider the leftmost digits of  $xb^K/\beta(b)$ . Certainly  $xb^K/\beta(b) < (b-1)b^K$ . From this upper bound we see that the  $b^K$  digit of  $xb^K/\beta(b)$  lies in  $[0, b-2]$  and that the digits associated to larger powers of  $b$  are all  $0$ . Combining this with our previous observations, we conclude that  $xb^K/\beta(b)$  omits the digit  $(b-1)$  for all  $x \in [0,(b-1)\beta(b)-1]$ , so  $A \subset \mathcal{K}([0, b-2], b)$  as claimed. Since  $|A| = (b-1)\beta(b)$ , we have  $\ell(b) \geq$  $(b-1)\beta(b).$ 

We now proceed with the second half of our evaluation of  $\ell(b)$ , the verification that this lower bound is exact. This requires a more technical argument.

<span id="page-4-0"></span>**Proposition 2.4.** For all  $b \ge 2$ , we have  $\ell(b) \le (b-1)\beta(b)$ .

*Proof.* Without loss of generality, let  $S \subset [0, b-1]$  be any set of  $b-1$  digits (containing 0), and let  $A = \{x + j\Delta : j \in [0, \ell(b) - 1]\}$  be an arithmetic progression in  $\mathcal{K}(S, b)$  of maximal length, in which  $\Delta > 0$  is taken minimally over all arithmetic progressions of length  $\ell(b)$ .

Let  $\Delta = d_K b^K + \ldots + d_1 b + d_0$  denote the base b expansion of  $\Delta$ , where K is chosen such that  $d_K \neq 0$ . For notational convenience, let  $\Delta_k :=$  $d_k b^k + \ldots + d_1 b + d_0$  for each  $k \geq 0$ . (Note that  $\Delta_k = \Delta$  for  $k \geq K$ .) We may assume without loss of generality that  $d_0 \neq 0$ , else by removing the rightmost digit from all elements of A one constructs an arithmetic progression contained in  $\mathcal{K}(S, b)$  of common difference  $\Delta/b$ , contradicting our minimality assumption on  $\Delta$ . (This is the same device as we used in the proof of Proposition [2.2\)](#page-2-1).

Our proof of Proposition [2.4](#page-4-0) rests on the following claim, whose peculiar statement arises naturally from an inductive argument.

Claim 2.5. Consider the following statements:

C1:  $\ell(b) \leqslant (b-1)\beta(b);$ C2(k): there exist coprime integers  $\lambda_k, \mu_k \in [1, b-1]$  satisfying  $\lambda_k \Delta_k = \mu_k b^{k+1}.$ 

Then either C1 holds or  $C_2(k)$  holds for all  $k \geq 0$ .

This claim immediately settles the theorem, since the statement  $C_2(k)$ cannot possibly hold for all  $k \geq 0$ . Indeed, we have  $\lambda_k \Delta_k < b \Delta$ , while  $\mu_k b^{k+1}$  grows in k without bound.

Proof of Claim. We prove this claim by induction, showing that for every  $k \geq 0$ , either C1 holds or C2(k') holds for all  $k' \leq k$ . For the base case  $k = 0$ , note that  $\Delta_k = d_0$ . If  $(d_0, b) = 1$ , then  $d_0$  generates the additive group  $\mathbb{Z}/b\mathbb{Z}$  and the elements  $\{x + j\Delta : j \in [0, b-1]\}$  have b distinct units digits. Thus  $\ell(b) \leq (b-1) \leq (b-1)\beta(b)$ , so C1 holds.

Otherwise,  $(d_0, b) > 1$ , which implies that there exists  $\lambda \in [1, b-1]$  for which  $\lambda d_0 \equiv 0 \text{ mod } b$ . Thus  $\lambda d_0 = \mu b$  for some  $\mu$ , and we may assume that  $(\lambda, \mu) = 1$  by dividing through by common factors. This concludes the base case.

Proceeding to the inductive step, let  $k \geq 1$  and assume that the inductive hypothesis C2(k') holds for all smaller k'. In particular,  $\Delta_{k-1} =$  $(\mu_{k-1}/\lambda_{k-1})b^k$  for some coprime integers  $\lambda_{k-1}, \mu_{k-1} \in [1, b-1]$ , and hence  $\Delta_k = d_k b^k + (\mu_{k-1}/\lambda_{k-1})b^k.$ 

Let  $\lambda_k$  denote the order of  $\Delta_k/b^{k+1}$  in the additive group  $\mathbb{R}/\mathbb{Z}$ , and let  $\mu_k$  denote the integer  $\lambda_k(\Delta_k/b^{k+1})$ . We see that  $(\lambda_k, \mu_k) = 1$ , as one could divide through by any common factors of  $\lambda_k$  and  $\mu_k$  to contradict the fact that  $\lambda_k$  is the order of  $\Delta_k/b^{k+1}$  in  $\mathbb{R}/\mathbb{Z}$ . Now, if  $\lambda_k < b$ , then  $\mu_k < b$  as well, since  $\Delta_k/b_{k+1} < 1$  for any k. In this case,  $\lambda_k$  and  $\mu_k$  satisfy the conditions listed in C2(k). Therefore C2(k') holds for all  $k' \le k$ .

It remains to address the case  $\lambda_k \geq b$ . By usual facts about finite subgroups of  $\mathbb{R}/\mathbb{Z}$ , we note that the orbit of  $\Delta_k/b^{k+1}$  in  $\mathbb{R}/\mathbb{Z}$  is exactly the set of fractions with denominator dividing  $\lambda_k$ . In particular, the set of values

$$
T = \left\{ \frac{x}{b^{k+1}} + \frac{\Delta_k j}{b^{k+1}} \mod 1 : j \in [0, \lambda_k - 1] \right\}
$$

are equally spaced, with gaps of size  $1/\lambda_k$ . Since  $\lambda_k \geq 0$ , for any integer  $d \in [0, b-1]$  at least one member of T lies in the half-open interval  $\left[\frac{d}{b}, \frac{d+1}{b}\right]$  $\frac{+1}{b}$ ). In other words, at least one member of the progression  $x + \Delta_k[0, \lambda_k - 1]$  has  $b^k$  digit equal to d.

This information immediately implies that  $x+\Delta[0, \lambda_k-1]$  is not contained in any proper Kempner set  $\mathcal{K}(S, b)$ , and hence  $\ell(b) \leq \lambda_k - 1$ . However, more can be said with a slight refinement to our analysis. Equal spacing implies that at least  $\lfloor \lambda_k/b \rfloor$  members of T lie in the interval  $\left[ \frac{d}{b}, \frac{d+1}{b} \right]$  $\frac{+1}{b}$ ). We are left with the stronger bound  $\ell(b) \leq \lambda_k - |\lambda_k/b|$ .

We now establish an upper bound on the function  $\lambda_k - |\lambda_k/b|$ , given the known constraints on  $\lambda_k$ . For starters, the inductive hypothesis implies that  $\lambda_{k-1} \mid \mu_{k-1} b^k$ , hence  $\lambda_{k-1} \mid b^k$  (since  $\lambda_{k-1}$  and  $\mu_{k-1}$  are coprime). Since  $\lambda_{k-1} < b$  and  $\lambda_{k-1}$  divides a power of b, this implies that  $\lambda_{k-1} \leq \beta(b)$ . Secondly, the inductive hypothesis allows us to write

$$
\frac{\Delta_k}{b^{k+1}} = \frac{d_k \lambda_{k-1} + \mu_{k-1}}{b \lambda_{k-1}},
$$

which implies that  $b\lambda_{k-1}(\Delta_k/b^{k+1}) \equiv 0 \mod 1$ . This implies that  $b\lambda_{k-1}$  is a multiple of the order of  $(\Delta_k/b^{k+1})$  mod 1, ie.  $\lambda_k | b\lambda_{k-1}$ . We conclude that  $\lambda_k \leqslant b\lambda_{k-1} \leqslant b\beta(b).$ 

The function  $\lambda \mapsto \lambda - |\lambda/b|$  is non-decreasing as  $\lambda$  increases over integers, hence

$$
\ell(b) \leq \lambda_k - \left\lfloor \frac{\lambda_k}{b} \right\rfloor \leq b\beta(b) - \left\lfloor \frac{b\beta(b)}{b} \right\rfloor = b\beta(b) - \beta(b) = (b-1)\beta(b),
$$

which implies that C1 holds. This completes the inductive step, and so completes the proof of Theorem [1.1.](#page-1-1)  $\Box$ 

### 3. Asymptotic Analysis

In this section we analyse the function  $\beta(b)$ , with the ultimate goal of proving Theorem [1.2.](#page-1-2) We begin with the following simple observation.

### Proposition 3.1. We have

$$
\liminf_{n \to \infty} \frac{\beta(n)}{n} = 0 \quad and \quad \limsup_{n \to \infty} \frac{\beta(n)}{n} = 1.
$$

*Proof.* The first claim follows from the observation that  $\beta(p) = 1$  for all primes p. For the second, we note that  $\beta(2^k+2) = 2^k$  for all  $k > 1$ .

It is clear from this proposition that the behaviour of  $\beta(n)$  is erratic as n varies. However, its calculation may be understood as a certain integer programming problem, as illustrated by the following example.

<span id="page-6-1"></span>**Example 3.2.** In this example, we calculate  $\beta(24)$  using techniques from mixed integer programming. We may write  $\beta(24) = 2^a \cdot 3^b$ , with  $a, b \in \mathbb{N}$ . It follows that  $a \log 2 + b \log 3 < \log 24$ , and  $(a, b)$  may be visualized as a lattice point in the following figure (Figure [1\)](#page-6-0). The equation of the line is  $f(x) = \log_3 24 - x \log_3 2.$ 



<span id="page-6-0"></span>FIGURE 1. Lattice points  $(a, b)$  corresponding to  $\beta(24) = 2^a \cdot 3^b$ .

Let us restrict our attention to the set S of lattice points of the form  $(a, b)$ , in which b is taken maximally for fixed a. If  $(a, b) \in S$ , the vertical distance from  $(a, b)$  to the diagonal in Figure [1](#page-6-0) is then given by  $\{\log_3 24 - a \log_3 2\}.$ We also note that  $a \log 2 + b \log 3$  is maximized among the lattice points below the line when  $(a, b) \in S$  and  $\{\log_3 24 - a \log_3 2\}$  is minimized (as a function of a). In our example, minimization occurs at  $(a, b) = (1, 2)$ , and so we obtain  $\beta(24) = 2^1 \cdot 3^2 = 18$ .

The technique of Example [3.2](#page-6-1) generalizes easily: if  $n$  has  $k$  prime divisors  $p_1, \ldots, p_k$ , we may associate to n a set of lattice points in  $\mathbb{Z}^k$ , namely

$$
\{(a_1, \cdots, a_k) \in \mathbb{Z}_{\geq 0}^k : a_1 \log p_1 + \cdots + a_k \log p_k < \log n\}.
$$

The lattice point  $(a_1, \ldots, a_k)$  that minimizes distance to the the hyperplane

$$
x_1 \log p_1 + \dots x_k \log p_k = \log n
$$

determines  $\beta(n)$  by the formula  $\beta(n) = \prod_{i=1}^{k} p_i^{a_i}$ .

Combining this idea with well-known equidistribution results gives the following.

<span id="page-7-0"></span>**Lemma 3.3.** We have  $\beta(n) \sim n$  as  $n \to \infty$  within NZ if and only if N is not a prime power.

*Proof.* If  $N = p^k$  is a prime power, then  $N\mathbb{Z}$  contains the subsequence  ${p^{kn}}_{n\geqslant 1}$ . Since  $\beta(p^{kn})=p^{kn-1}$ , we cannot have  $\beta(n) \sim n$  within  $N\mathbb{Z}$ .

Otherwise, let  $p$  and  $q$  be distinct primes dividing  $N$ , and fix a positive constant  $\varepsilon$ . As  $\log_q p$  is irrational, the sequence  $(\{u_n\})_{n=0}^{\infty}$  given by  $u_n :=$  $n \cdot \log_a p$  is equidistributed mod 1 (by the Equidistribution Theorem: see Proposition 21.1 of [\[6\]](#page-10-7), say). In particular, there exists a positive parameter  $L_{\varepsilon}$  such that  $l > L_{\varepsilon}$  implies that the sequence  $(\{u_n\})_{n=0}^l$  contains at least one element in each interval mod 1 of length  $\varepsilon$ .

Now let m be a natural number and let  $l = \lfloor \log_n m \rfloor$ . From the above remarks, there exists a positive parameter  $M_{\varepsilon}$  such that, for each  $m > M_{\varepsilon}$ , the shifted sequence  $(\{\log_q m - u_n\})_{n=0}^l$  contains some element in the interval  $(0, \varepsilon)$ . In other words there exists  $n_0$  at most l (but dependent on l) such that

$$
0 < \{\log_q m - n_0 \cdot \log_q p\} < \varepsilon.
$$

Also note that  $\log_q m - n_0 \cdot \log_q p$  is positive.

Now, assume  $pq \mid m$  and consider  $(a, b) := (n_0, |\log_a m - n_0 \cdot \log_a p|)$ . We have  $\beta(m) \geqslant p^a \cdot q^b$  by construction. So

$$
\beta(m) \geqslant p^a \cdot q^b = q^{\log_q m - \{\log_q m - n_0 \cdot \log_q p\}} > q^{\log_q m - \varepsilon} = m \cdot q^{-\varepsilon}.
$$

Thus  $q^{-\varepsilon} < \beta(m)/m < 1$ , for all m satisfying  $m > M_{\varepsilon}$  and  $pq \mid m$ . Since  $\varepsilon$ was arbitrary, and q fixed, it follows that  $\beta(m) \sim m$  within  $pq\mathbb{Z}$ , and hence within  $N\mathbb{Z}$ .

By considering  $N = 6$ , for example, we obtain a set of density  $1/6$  (namely, 6Z) on which  $\ell(b) \sim b^2$  as b tends to infinity within that set. Any finite union of such sets  $N_i \mathbb{Z}$ , where  $N_i$  has two distinct prime factors  $p_i$  and  $q_i$ , will also have this property, and one may show with relative ease that such a union may be arranged to have natural density arbitrarily close to 1.

However, by quantifying estimates made in the previous lemma, we can do slightly better, and show the existence of a set with the desired property that has density 1, thereby proving Theorem [1.2.](#page-1-2)

*Proof of Theorem [1.2.](#page-1-2)* Let  $f(N)$  be a function that satisfies  $f(N) \rightarrow \infty$  as  $N \to \infty$  (to be further specified later). For integers  $j \geq 0$ , let  $D_j$  denote the set of  $n \in (2^{j-1}, 2^j]$  such that n has at least two distinct prime factors  $p, q \leqslant f(2^{j-1})$ . Let

$$
D:=\bigcup_{j\geqslant 0}D_j.
$$

The set D is our candidate set for use in Theorem [1.2.](#page-1-2)

**Lemma 3.4.** If f grows slowly enough, the set  $D$  has natural density 1.

*Proof.* We begin by fixing  $j \geq 0$  and bounding the size of  $D_j$  from below. For convenience, we write N for  $2^{j-1}$ .

To produce this lower bound, we find an upper bound for  $(N, 2N] \setminus D_j$ . Indeed, by a standard application of a small sieve (e.g. the Selberg sieve, in particular Theorem 9.3.10 of [\[10\]](#page-10-8)), one may show that the number of  $n \in (N, 2N]$  without any prime factor p less than  $f(N)$  is

$$
O\bigg(N\prod_{p
$$

provided  $f(N)$  grows slowly enough. By Mertens' Third Theorem, this quantity is  $O(N/\log f(N))$ .

By using a union bound and the sieve above, we bound the number of  $n \in (N, 2N]$  with exactly one prime factor  $p < f(N)$  by

$$
O\Bigg(\sum_{p
$$

This quantity is  $O(N \log \log f(N) / \log f(N))$  (again by Mertens' theorems), and we conclude by exclusion that

$$
|D_j| = N\left(1 - O\left(\frac{\log\log f(N)}{\log f(N)}\right)\right).
$$

This already establishes that D has full upper Banach density. To show that D has natural density 1, we fix  $\varepsilon > 0$  and note that, since  $f(N) \to \infty$ as  $j \to \infty$ , there exists  $j_0(\varepsilon)$  such that  $|D_j| \geq 2^{j-1}(1-\varepsilon)$  for all  $j \geq j_0(\varepsilon)$ . In particular,

$$
\sum_{\substack{n \leq X \\ n \in D}} 1 \geqslant \sum_{\substack{j_0(\varepsilon) \leqslant j \leqslant \lceil \log_2 X \rceil}} |D_j| - \sum_{X < n \leqslant 2^{\lceil \log_2 X \rceil}} 1.
$$
\n
$$
\geqslant (1 - \varepsilon) \left( 2^{\lceil \log_2 X \rceil} - 2^{j_0(\varepsilon) - 1} \right) + X - 2^{\lceil \log_2 X \rceil}
$$

Simplifying, we see that

$$
\liminf_{X \to \infty} \frac{|D \cap [1, X]|}{X} \geqslant \liminf_{X \to \infty} \frac{X - \varepsilon 2^{\lceil \log_2 X \rceil} - 2^{j_0(\varepsilon)}}{X} \geqslant 1 - 2\varepsilon,
$$

which implies that D has natural density 1, since  $\varepsilon$  was arbitrary.

.

Secondly, we prove that  $\beta(n)$  is asymptotically large within D.

<span id="page-8-0"></span>**Lemma 3.5.** If f grows slowly enough, then  $\beta(n) \sim n$  as  $n \to \infty$  within D.

Proof. Our proof presents a more quantitative adaptation of the argument used in Lemma [3.3.](#page-7-0) Let  $\varepsilon > 0$ , and fix  $n \in D$ . By the definition of D, there exist distinct primes  $p, q < f(n)$  for which  $p, q \mid n$ . We will show that,

provided n is large enough in terms of  $\varepsilon$ , there exist non-negative integers a and b for which

$$
e^{\varepsilon} \geqslant \frac{n}{p^a q^b} > 1.
$$

Since  $p^a q^b \leq \beta(n) < n$ , and  $\varepsilon$  is arbitrary, this will complete the proof.

Taking logarithms, it suffices to find non-negative integers  $a$  and  $b$  for which

$$
\frac{\varepsilon}{\log q} \geqslant \log_q n - a \log_q p - b > 0.
$$

Setting  $L = \lfloor \log_p n \rfloor$ , it will be enough to prove that the sequence of fractional parts  $\{\{a \log_a p\} : a \in [1, L]\}$  contains an element in every interval modulo 1 of length  $\varepsilon/\log q$ . Since  $p, q \leq f(n)$ , we reduce our theorem to the following claim:

Claim 3.6. Let  $L' = \lfloor \log n / \log f(n) \rfloor$ . Then  $S = \{\{a \log_q p\} : a \in [1, L']\}$ contains an element in every interval modulo 1 of length  $\varepsilon/\log f(n)$ , provided  $f(n)$  grows slowly enough.

The proof of this claim follows from the Erdős-Turán inequality (Corollary 1.1 of [\[9\]](#page-10-9)). Indeed, for any interval I modulo 1 of length  $\varepsilon / \log f(n)$ , we have

$$
\left| |S \cap I| - \frac{\varepsilon L'}{\log f(n)} \right| \ll \frac{L'}{K+1} + \sum_{k \leq K} \frac{1}{k} \left| \sum_{a=1}^{L'} e^{2\pi i a k \log_q p} \right| \tag{3.1}
$$

for any integer  $K \geq 1$ . It suffices to show that we may choose a K such that the right-hand side in [\(3.1\)](#page-9-0) is  $o(L'/\log f(n))$  as  $n \to \infty$ .

Choosing  $K = |\log^2 f(n)|$  ensures that  $L'/(K+1) = o(L'/\log f(n))$ . As for the second term in  $(3.1)$ , bounding the sum over a as a geometric series gives

<span id="page-9-0"></span>
$$
\sum_{k \leqslant K} \frac{1}{k} \left| \sum_{a=1}^{L'} e^{2\pi i a k \log_q p} \right| \leqslant G(K, p, q)
$$

for some function  $G$  that is independent of  $L'$ . We may assume without loss of generality that  $G$  is increasing in each variable. Then

$$
G(K, p, q) \ll G(\log^2 f(n), f(n), f(n)),
$$

so it suffices to show that

<span id="page-9-1"></span>
$$
G\left(\log^2 f(n), f(n), f(n)\right) = o\left(\frac{L'}{\log f(n)}\right). \tag{3.2}
$$

Recalling the definition of  $L'$ , this is equivalent to showing

$$
G\left(\log^2 f(n), f(n), f(n)\right) \cdot \log^2 f(n) = o\left(\log n\right).
$$

Yet  $G$  is simply some absolute function, so if  $f$  grows slowly enough then [\(3.2\)](#page-9-1) will hold. (If one so wished, one could quantify this growth condition using Baker's result [\[2\]](#page-10-10) on linear forms of logarithms of primes). This proves the claim, and hence the lemma. Combining Lemma [3.5](#page-8-0) with Theorem [1.1](#page-1-1) yields Theorem [1.2.](#page-1-2)  $\Box$ 

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