MAXIMUM PRINCIPLES FOR MATRIX-VALUED ANALYTIC FUNCTIONS

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Abstract. To what extent is the maximum modulus principle for scalar-valued analytic functions valid for matrix-valued analytic functions? In response, we discuss some maximum *norm* principles for such functions that do not appear to be widely known, deduce maximum and minimum principles for their singular values, and make some observations concerning resolvents and matrix exponentials.

1. Introduction.

The maximum modulus principle (MMP) is a fundamental result in complex analysis. It is often used to deduce other important results such as the fundamental theorem of algebra, the open mapping theorem (i.e., analytic functions map open sets to open sets), Schwarz's lemma, the Phragmén–Lindelöff principle, etc. One of its various formulations states that if f is a *scalar-valued* function, analytic on a region Ω (i.e., a nonempty open connected subset) of the complex plane $\mathbb{C}$, whose *modulus* attains a local maximum in Ω, then f is constant on Ω. For a proof of the MMP, we refer the reader to [7, Chapter 10].

Many differential equations encountered in science and engineering lead to the consideration of *matrix-valued* functions, that is, functions with range in the set $\mathbb{M}_n$ of $n \times n$ matrices, $n > 1$, with entries in $\mathbb{C}$. For instance, the standard model of an RLC circuit in electrical engineering admits the formulation $x'(t) = A \cdot x(t)$, where $A \in \mathbb{M}_n$ and x is a function with values in $\mathbb{C}^n$. The vector-valued solutions $x(t) = \exp(tA)x_0$ to such an equation (with $x_0 \in \mathbb{C}^n$) depend on the matrix-valued function $t \mapsto \exp(tA) = \sum_{k=0}^{\infty} A^k t^k/k!$, and the decay of these solutions is controlled by its operator norm $\|\exp(tA)\|$. As usual, $\|T\| = \sup\{\|Tv\|_{\mathbb{C}^n} : \|v\|_{\mathbb{C}^n} = 1\}$ is the *operator norm* of $T \in \mathbb{M}_n$ induced by the Euclidean norm on $\mathbb{C}^n$, namely $\|v\|_{\mathbb{C}^n} = \left(|v_1|^2 + \cdots + |v_n|^2\right)^{1/2}$ when $v = (v_1, \ldots, v_n)$.

In linear algebra, too, matrix-valued functions arise (implicitly) in the study of eigenvalues, i.e., the spectrum $\sigma(A)$ of $A \in \mathbb{M}_n$. After all, $\lambda \in \mathbb{C}$ satisfies $Av = \lambda v$ for some nonzero vector $v \in \mathbb{C}^n$ if and only if the *resolvent* function $z \mapsto (A - zI)^{-1}$ has a singularity at $z = \lambda$, i.e., $A - \lambda I$ is not invertible. (Throughout, $I = I_n$ denotes the identity in $\mathbb{M}_n$.) Since the spectrum is often insufficient for the analysis of non-normal matrices (see [8]), focus has shifted to the study[1] of the norm of the resolvent $\|(A - zI)^{-1}\|$. For instance, the norm of the resolvent alone characterizes when A is a normal matrix [1].

[1]Equivalently, one may study the so-called "pseudospectra" of A. For an overview of that subject, see [9].

Thus, it is of interest to study the (operator) norms of the matrix-valued functions $\exp(zA)$ and $(A-zI)^{-1}$. As can be expected, these functions are analytic[2] in regions of $\mathbb{C}$, the entire plane $\mathbb{C}$, and $\mathbb{C}\backslash\sigma(A)$, respectively. The fact these functions are analytic leads one to question the extent to which the MMP for scalar-valued functions is valid for $\|F(z)\|$, where F is any matrix-valued analytic function. The purpose of this article is to find sufficient conditions, say involving the norm of a matrix-valued analytic function, that guarantee that the function is constant.

In Section 2, we state and discuss some maximum norm principles for matrix-valued analytic functions. Although it has been long known that a direct analog of the MMP fails in the context of matrix-valued functions in which the operator norm plays the role of the modulus, we find a suitable analog. Stated roughly, if $F:\Omega\to\mathbb{M}_n$ is such that $\|F(z)\|$ attains a maximum at some $z_0\in\Omega$, then there is a direction in which $F(z)$ is constant (although $F(z)$ need not be) namely that of any maximizing vector of $F(z_0)$ (see Theorem 3). We rediscovered this result originally noted by Brown and Douglas in [2] and use it to describe the structure of the function $F(z)$ (see Theorem 4). Since the result lends itself to iteration, we make natural assumptions on the function's singular values and explore the consequences further in Section 3. One of the section's main results (see Corollary 6) illuminates the equivalence of two apparently distinct statements to the single statement that the matrix function $F(z)$ is constant: the Frobenius norm of $F(z)$ attains a maximum, and every singular value of $F(z)$ attains a maximum (at possibly distinct points).

Once the maximum singular-values principle is established in Section 3, we proceed to prove a minimum singular-values principle in Section 4. That result (Theorem 9) is, in a sense, an analog of the well-known minimum modulus principle of complex analysis in the context of matrix-valued functions. Finally, in Section 5, we discuss the implications of our results in the context of the resolvent and the matrix exponential which involve their largest and smallest singular values.

It is worth mentioning that analytic matrix-valued functions appear in many other areas such as the harmonic analysis of operators on a Hilbert space (e.g., finite-rank perturbations of self-adjoint and unitary operators), and consequently in mathematical physics (e.g., Schrödinger operators); roughly, problems concerning spectral properties of an operator are often solved through the consideration of an analytic matrix-valued function defined on the upper-half plane, i.e. the so-called "characteristic function." Due to the scope of the paper, the reader is referred to the survey [6] and all references therein for further details.

We also remark that the results of this article could be written in the more general framework of operator-valued functions $F:\Omega\to\mathcal{B}(H)$, where H is a complex Hilbert space, or that of vector-valued functions $F:\Omega\to B$, where B is a complex Banach space. However, all statements in this article are kept in the context of matrix-valued functions so that the results are easier to read and appeal to a wider audience.

[2]Recall that a function $F:\Omega\to\mathbb{M}_n$ is analytic if, for each $z_0\in\Omega$, there is a member of $\mathbb{M}_n$, denoted by $F'(z_0)$, such that $\|(z-z_0)^{-1}\{F(z)-F(z_0)\}-F'(z_0)\|\to 0$ as $z\to z_0$. It can be shown that $F:\Omega\to\mathbb{M}_n$ is analytic if and only if F is "entry-wise analytic," i.e., every entry of $F(z)$ is an analytic function on Ω.

2. Maximum norm principles.

To find a suitable analog of the MMP for matrix-valued functions, it is reasonable to first test whether known proofs of the MMP can be easily adapted when replacing modulus with operator norm. One such proof of the MMP appears in [7, Chapter 10]. In it, the identity $|w|^2 = w\bar{w}$ ($w \in \mathbb{C}$) appears, and although the operator norm of $T \in \mathbb{M}_n$ does not readily provide a direct analog for $\|T\|^2$, the Frobenius norm does. In fact,[3]

$$\|T\|_{\mathcal{F}}^2 = \operatorname{trace}(T^*T) \quad \text{for } T \in \mathbb{M}_n, \tag{1}$$

when $\|T\|_{\mathcal{F}}$ is the Frobenius (Hilbert–Schmidt) norm of T, and an argument analogous to the proof of the MMP in [7] (that relies on (1)) gives the following result.

Theorem 1 (Maximum Frobenius Norm Principle)**.** *Let Ω be a region of $\mathbb{C}$ and let $F : \Omega \to \mathbb{M}_n$ be analytic. If $\|F(z)\|_{\mathcal{F}}$ assumes its maximum at some $z_0 \in \Omega$, then $F(z) = F(z_0)$ for all $z \in \Omega$.*

Despite its provision of a direct analog of the MMP for matrix-valued functions, in applications, it is the operator norm that is of interest, not the Frobenius norm. Unfortunately, the conclusion of Theorem 1 need not hold when the Frobenius norm is replaced by another matrix norm. For example, let $\mathbb{D}$ denote the open unit disk centered at the origin, let $g : \mathbb{D} \to \mathbb{D}$ be analytic (e.g., $g(z) = z$), and consider the 2×2 matrix-valued function

$$F(z) = \begin{bmatrix} 1 & 0 \\ 0 & g(z) \end{bmatrix}. \tag{2}$$

Notice that the operator norm of $F(z)$ satisfies

$$\|F(z)\|^2 = \max\{1, |g(z)|^2\} = 1 \quad \text{for all } z \in \mathbb{D},$$

even though $F(z)$ is not a constant function. Nevertheless, one can prove a weakened version for *any* norm.

Theorem 2 (Maximum Norm Principle)**.** *Let Ω be a region of $\mathbb{C}$ and let $F : \Omega \to \mathbb{M}_n$ be analytic. If $\|F(z)\|$ attains its maximum in Ω, then $\|F(z)\|$ is constant on Ω.*

Theorem 2 is well known and a proof can be found in [4, Section III.14]; we provide a different short proof based on a well-known consequence of the Hahn–Banach theorem on linear functionals, namely if X is any normed space and $x \in X$ is nonzero, then there is a bounded linear functional Λ on X such that $\|\Lambda\| = 1$ and $\Lambda(x) = \|x\|$. For further details and a simple proof of this fact, see [7, Chapter 5].

Proof of Theorem 2. Assume there is a $z_0 \in \Omega$ such that $\|F(z)\| \le \|F(z_0)\|$ for all $z \in \Omega$ and, without loss of generality, that $\|F(z_0)\| \ne 0$. Then we can choose a bounded linear functional $\Lambda : \mathbb{M}_n \to \mathbb{C}$ of norm 1 so that $\|F(z_0)\| = \Lambda(F(z_0))$. By continuity of Λ and analyticity of F, $\Lambda(F(z))$ defines an analytic function on Ω, and

$$|\Lambda(F(z))| \le \|F(z)\| \le \|F(z_0)\| = |\Lambda(F(z_0))|.$$

[3] As usual, T^* denotes the conjugate transpose of the matrix T.

It follows now from the usual MMP that $\Lambda(F(z))$ must be constant throughout Ω and

$$\|F(z)\| \geq \Lambda(F(z)) = \Lambda(F(z_0)) = \|F(z_0)\| \geq \|F(z)\| \quad \text{for all } z \in \Omega.$$

Thus, $\|F(z)\| = \|F(z_0)\|$ for all $z \in \Omega$. □

The conclusion of the maximum norm principle above may be seen as unsatisfactory because it gives limited information about the structure of $F(z)$ itself. This is not at all surprising; after all, the theorem holds for *any* norm. So, from now on we use the *operator norm* exclusively in an effort to gain more information about the function F.

A useful property of the operator norm of a matrix is that given any $A \in \mathbb{M}_n$, there is a unit vector $x_0 \in \mathbb{C}^n$, called a **maximizing vector** for A, so that $\|Ax_0\| = \|A\|$; in other words, matrices attain their operator norm at some vector in the unit ball of $\mathbb{C}^n$. This is a consequence of the compactness of the closed unit ball of $\mathbb{C}^n$.

Recently, we rediscovered a maximum operator norm principle due to Brown and Douglas. In [2, Theorem 4], the authors proved that if $F(z)$ is a nonconstant matrix-valued analytic function whose operator norm attains its maximum, then there is a direction x_0 in which $F(z)x_0$ is constant. Our version reads as follows.

Theorem 3 (Maximum Operator Norm Principle, cf. [2]). *Let Ω be a region of $\mathbb{C}$ and let $F : \Omega \to \mathbb{M}_n$ be analytic. If there is a $z_0 \in \Omega$ so that $\|F(z)\| \leq \|F(z_0)\|$ for all $z \in \Omega$ and x_0 is a maximizing vector for $F(z_0)$, then $F^{(k)}(z_0)x_0 = 0$ for every $k \geq 1$. In particular, $F(z)x_0$ is constant on Ω.*

The conclusion[4] of Theorem 3 here is, at first sight, a slight improvement to that in Theorem 4 (part (1)) of [2]; after all, using a series expansion of $F(z)$, the condition $F^{(k)}(z_0)x_0 = 0$ for $k \geq 1$ easily implies that $F(z)x_0$ is constant on Ω. In fact, the reverse implication is also true and a justification can be made using series, too. On the other hand, although our series proof of Theorem 3 below is not as short as that of Brown and Douglas, it elucidates the consideration of maximizing vectors x_0 (see (5) below).

Proof of Theorem 3. Let $R > 0$ be such that $D(z_0; R) \subseteq \Omega$. Then $F(z)$ admits a power series representation on $D(z_0; R)$, say

$$F(z) = \sum_{k=0}^{\infty} C_k (z - z_0)^k, \tag{3}$$

where $C_k \in \mathbb{M}_n$ for $k \geq 0$. For any vector x,

$$\|F(z)x\|^2 = \sum_{j,k \geq 0} (z - z_0)^j \overline{(z - z_0)}^k \langle C_j x, C_k x \rangle$$

by continuity of the inner product and so

$$\frac{1}{2\pi} \int_0^{2\pi} \|F(z_0 + re^{it})x\|^2 \, dt = \sum_{k=0}^{\infty} \|C_k x\|^2 r^{2k} \tag{4}$$

for any $r \in (0, R)$.

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[4] It is worth mentioning that our version of Theorem 3 also complements a result due to Daniluk in [3].

Now, since $\|F(z)\| \le \|F(z_0)\| = \|C_0\|$ for all $z \in \Omega$, it follows from (4) that

$$\sum_{k=0}^{\infty} \|C_k x\|^2 r^{2k} = \frac{1}{2\pi} \int_0^{2\pi} \|F(z_0 + re^{it})x\|^2 \, dt \le \|C_0\|^2 \|x\|^2 \tag{5}$$

for any vector x and $r \in (0, R)$. Let x_0 be a maximizing vector for C_0. Replace x by x_0 in (5), and conclude

$$\|C_0\|^2 + \sum_{k=1}^{\infty} \|C_k x_0\|^2 r^{2k} \le \|C_0\|^2 \|x_0\|^2 = \|C_0\|^2,$$

and $F^{(k)}(z_0)x_0 = C_k x_0 = 0$ for every $k \ge 1$. In particular, by (3), $F(z)x_0 = C_0 x_0 = F(z_0)x_0$ for all $z \in D(z_0; R)$ and so, by the identity theorem (e.g., [7, Theorem 10.18]), $F(z)x_0 = F(z_0)x_0$ for all $z \in \Omega$. □

Remark. Note that *the conclusion of Theorem 3 alone* implies that $\|F(z)\|$ has a *minimum* at z_0; after all, if $z \mapsto F(z)x_0$ is constant on Ω for some maximizing vector x_0 of $F(z_0)$, then

$$\|F(z_0)\| = \|F(z_0)x_0\| = \|F(z)x_0\| \le \|F(z)\| \text{ for all } z \in \Omega.$$

Hence, the conclusion of Theorem 3 is stronger than that of the maximum norm principle (when using the operator norm) because it implies that any maximizing vector x_0 for $F(z_0)$ is also a maximizing vector for $F(z)$, and $F(z)$ has constant norm equal to that of $F(z_0)$ for all $z \in \Omega$.

The observation made in the remark leads one to the following factorization.

Theorem 4. *Let Ω be a region of $\mathbb{C}$ and let $F : \Omega \to \mathbb{M}_n$ be analytic. If there is a $z_0 \in \Omega$ so that $\|F(z)\| \le \|F(z_0)\|$ for all $z \in \Omega$, then there are $n \times n$ (constant) unitary*[5] *matrices U and V, and an analytic function $G : \Omega \to \mathbb{M}_{n-1}$, such that*

$$F(z) = U \begin{bmatrix} \|F(z_0)\| & 0 \\ 0 & G(z) \end{bmatrix} V. \tag{6}$$

Roughly, in the case of 2×2 matrices, Theorem 4 states that when $F(z)$ is nonconstant, analytic, and achieves its maximum operator norm, say equal to 1, at a point of a region, then there is a nonconstant analytic function $g : \Omega \to \mathbb{D}$ such that

$$F(z) = \begin{bmatrix} 1 & 0 \\ 0 & g(z) \end{bmatrix}$$

up to multiplication by (constant) unitary matrices on the right and the left. Hence, in a sense, the example given in (2) is essentially the only example of a nonconstant 2×2 matrix function whose operator norm achieves a maximum value of 1.

Proof of Theorem 4. Without loss of generality, we assume $\|F(z_0)\| = 1$. By Theorem 3, if x_0 is a maximizing vector for $F(z_0)$, then the vector function $z \mapsto F(z)x_0$ is constant on Ω. Recalling that $\|v\|_{\mathbb{C}^n}^2 = v^* v$ for any $v \in \mathbb{C}^n$ and choosing $y_0 = F(z_0)x_0$, we obtain

$$\|y_0\|^2 = \|x_0\|^2 = 1 \text{ and } y_0^* F(z)x_0 = y_0^* F(z_0)x_0 = 1 \text{ for all } z \in \Omega.$$

[5] Recall that $A \in \mathbb{M}_n$ is said to be **unitary** if $A^*A = AA^* = I$.

Let X_0 and Y_0 be (constant) $n \times n$ unitary matrices whose first columns are x_0 and y_0, respectively. Then, in matrix blocks,

$$Y_0^* F(z) X_0 = \begin{bmatrix} a_{1,1}(z) & a_{1,2}(z) \\ a_{2,1}(z) & a_{2,2}(z) \end{bmatrix},$$

where $a_{1,1}(z) = y_0^* F(z) x_0 = 1$. Furthermore, as X_0 and Y_0 are unitary, $\|Y_0^* F(z) X_0\| = \|F(z)\| = 1$ (or, alternatively, this follows by the remark following the proof of Theorem 3). This implies that

$$a_{1,2}(z) = 0 \text{ and } a_{2,1}(z) = 0 \text{ for all } z \in \Omega$$

because the operator norm of an $n \times n$ matrix is an upper bound on the Euclidean (vector) norm of its columns and rows. In other words, the assumptions on $F(z)$ imply the existence of $n \times n$ constant unitary matrices X_0 and Y_0 so that

$$F(z) = Y_0 \begin{bmatrix} 1 & 0 \\ 0 & a_{2,2}(z) \end{bmatrix} X_0^*$$

where $a_{2,2}(z)$ is an analytic $(n-1) \times (n-1)$ matrix-valued function. Thus, the desired conclusion follows with $U = Y_0$, $V = X_0^*$, and $G(z) = a_{2,2}(z)$. □

3. Maximum singular value principles.

An attractive feature of Theorem 4 is that it lends itself to iteration. Indeed, the lower right block $G(z)$ in (8) may very well satisfy the assumptions of Theorem 4 just as $F(z)$ did. In this section, we explore this situation and its consequences, but first review some basic terminology and results concerning singular values.

We begin with the observation that the maximizing vectors for a matrix A admit the characterization that x_0 is a maximizing vector for $A \in \mathbb{M}_n$ if and only if x_0 has norm 1 and $A^* A x_0 = \|A\|^2 x_0$. More generally, for a vector x (whether it has norm 1 or not),

$$\|Ax\| = \|A\| \cdot \|x\| \text{ if and only if } A^* A x = \|A\|^2 x. \tag{7}$$

A proof of (7) can be based on the fact that every positive semi-definite matrix has a unique positive semi-definite square root (e.g., see [5, Theorem 7.2.6]). To that end, first note that the inequality $\|Av\| \le \|A\| \cdot \|v\|$ valid for all vectors v is equivalent to stating that the matrix $\|A\|^2 I - A^* A$ is positive semi-definite. So, $\|Ax\| = \|A\| \cdot \|x\|$ holds if and only if $\|(\|A\|^2 I - A^* A)^{1/2} x\| = 0$, or equivalently, $(\|A\|^2 I - A^* A)x = 0$. Hence, x_0 is a maximizing vector of A if and only if it is an eigenvector of $A^* A$ of norm 1, i.e., (7) holds.

The role played in Theorem 3 by maximizing vectors for a matrix and their alternative characterization as eigenvectors lead directly to the consideration of singular values.

Recall that the singular values $s_k(A)$, $1 \le k \le n$, of an $n \times n$ matrix A are the nonnegative square roots of the eigenvalues of $A^* A$ ordered in the nonincreasing order, that is,

$$s_1(A) \ge s_2(A) \ge \cdots \ge s_n(A).$$

In particular, $s_1(A) = \|A\|$ (see (7)) and $s_1^2(A) + s_2^2(A) + \cdots + s_n^2(A) = \|A\|_{\mathcal{F}}^2$. The latter can be deduced using any singular value decomposition (SVD) of A (e.g., [5, Theorem 7.35]) and (1).

The following result is a simple consequence of Theorem 4.

Theorem 5. *Let Ω be a region of $\mathbb{C}$ and let $F : \Omega \to \mathbb{M}_n$ be analytic. Suppose that, for each $k = 1, \dots, n$, the function $z \mapsto s_k(F(z))$ attains its maximum value on Ω. Then $F(z)$ is constant on Ω.*

In Theorem 5, the assumption does *not* require that the functions $s_1(F(z)), \dots, s_n(F(z))$ attain their maximum values at the same point[6] of Ω; they may assume their respective maxima at distinct points $z_1, \dots, z_n \in \Omega$.

Proof of Theorem 5. Our proof is by induction on n. When $n = 1$, the desired conclusion holds by the MMP. So, suppose that the result holds for $n = 1, \dots, m-1$ with $m \in \mathbb{N}$. We now show that it also holds for $n = m$.

Suppose $F : \Omega \to \mathbb{M}_m$ is analytic, and the function $z \mapsto s_k(F(z))$ attains its maximum value on Ω for each $k = 1, \dots, m$. Let $z_1 \in \Omega$ be such that $s_1(F(z)) \le s_1(F(z_1))$ for all $z \in \Omega$. By Theorem 4, there are $m \times m$ (constant) unitary matrices U_1 and V_1, and an analytic function $F_1 : \Omega \to \mathbb{M}_{n-1}$, such that

$$F(z) = U_1 \begin{bmatrix} s_1(F(z_1)) & 0 \\ 0 & F_1(z) \end{bmatrix} V_1. \tag{8}$$

In particular, $s_k(F_1(z)) = s_{k+1}(F(z))$ attains its maximum value on Ω for each $k = 1, \dots, m-1$. By the inductive hypothesis, $F_1(z)$ must be constant on Ω and, consequently, $F(z)$ is also constant. □

At first sight, the assumption in Theorem 5 that every function $z \mapsto s_k(F(z))$ attains its maximum value on Ω for $k = 1, \dots, n$ appears to be different from saying that $\|F(z)\|_{\mathcal{F}}$ attains its maximum value on Ω in the maximum Frobenius norm principle above. Based upon the results above one may conclude that they are in fact equivalent!

Corollary 6. *Let Ω be a region of $\mathbb{C}$. The following statements are equivalent for an analytic function $F : \Omega \to \mathbb{M}_n$.*

1. *$F(z)$ is constant on Ω.*
2. *For every $k = 1, \dots, n$, $s_k(F(z))$ is constant on Ω.*
3. *For every $k = 1, \dots, n$, $s_k(F(z))$ attains its maximum value at some $z_k \in \Omega$.*
4. *$\|F(z)\|_{\mathcal{F}}$ is constant on Ω.*
5. *$\|F(z)\|_{\mathcal{F}}$ attains its maximum value at some $z_0 \in \Omega$.*

Proof. It is evident that (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), (1) $\Longrightarrow$ (4), and (4) $\Longrightarrow$ (5). The only nontrivial implications (5) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1) are consequences of the maximum Frobenius norm principle and Theorem 5, respectively. □

In view of Corollary 6 (or Theorem 5), if Ω is region of $\mathbb{C}$ and $F : \Omega \to \mathbb{M}_n$ is a *nonconstant* analytic function such that $s_1(F(z))$ attains its maximum on Ω, then there is a largest integer $r < n$ such that the functions $s_1(F(z)), \dots, s_r(F(z))$ attain their maximum values on Ω. In this case, up to multiplication by (constant) unitary matrices on the right and the left, $F(z)$ has the block form

$$\begin{bmatrix} s_1(F(z_1)) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & s_r(F(z_r)) & 0 \\ 0 & \dots & 0 & F_r(z) \end{bmatrix}$$

[6] In fact, if the functions $s_1(F(z)), \dots, s_n(F(z))$ attain their maximum values at the same point $z_0 \in \Omega$, it follows already from Theorem 1 that $F(z)$ must be constant on Ω.

for some (necessarily nonconstant) analytic function $F_r : \Omega \to \mathbb{M}_{n-r}$.

A closer look at the proof of Theorem 5 also reveals the following refinement of the maximum norm principle. We omit the details.

Corollary 7. *Let $1 \le m \le n$, let Ω be a region of $\mathbb{C}$, and let $F : \Omega \to \mathbb{M}_n$ be analytic. Suppose that, for each $k = 1, \dots, m$, the function $s_k(F(z))$ attains its maximum value on Ω. Then $s_k(F(z))$ is constant on Ω for each $k = 1, \dots, m$.*

Note that for an arbitrary $F(z)$, it may happen that $s_n(F(z))$ is constant while $s_k(F(z))$ is not when $1 \le k < n$. For example, the function $F : \mathbb{D} \setminus \{0\} \to \mathbb{M}_2$ defined by

$$F(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix},$$

has $s_1(F(z)) = |z|^{-1}$ and $s_2(F(z)) = 1$ for all $z \in \mathbb{D} \setminus \{0\}$.

As seen in its proof, the key to obtaining the conclusion of Theorem 4 relies on choosing a maximizing vector for $F(z_0)$. The following theorem is a refinement of Theorem 4 that relies on choosing instead "all maximizing vectors" for $F(z_0)$.

Theorem 8. *Let Ω be a region of $\mathbb{C}$ and let $F : \Omega \to \mathbb{M}_n$ be analytic. Suppose there is a $z_0 \in \Omega$ so that $\|F(z)\| \le \|F(z_0)\|$ for all $z \in \Omega$ and set*[7]

$$d = \dim\{x \in \mathbb{C}^n : \|F(z_0)x\| = \|F(z_0)\| \cdot \|x\|\}. \tag{9}$$

Then there are $n \times n$ unitary matrices U and V such that

$$F(z) = \|F(z_0)\| U \cdot V \quad \text{when } d = n,$$

or, for some analytic function $R : \Omega \to \mathbb{M}_{n-d}$,

$$F(z) = U \begin{bmatrix} \|F(z_0)\| \cdot I_d & 0 \\ 0 & R(z) \end{bmatrix} V \quad \text{when } d < n. \tag{10}$$

In particular, $z \mapsto F(z)x$ is constant and $F^{(k)}(z_0)x = 0$ for all $k \ge 1$ when x satisfies $\|F(z_0)x\| = \|F(z_0)\| \cdot \|x\|$.

Note that one could apply Theorem 8 again to the lower-right matrix-block function $R(z)$ appearing in (10). More definitively, if $s_1(F(z))$ attains its maximum at $z_1 \in \Omega$, then d_1 is the largest integer such that

$$s_1(F(z_1)) = s_{d_1}(F(z_1)),$$

$s_{d_1+1}(F(z))$ attains its maximum at $z_2 \in \Omega$, and d_2 is the largest integer such that

$$s_{d_1+1}(F(z_2)) = s_{d_2}(F(z_2)),$$

then up to multiplication by (constant) unitary matrices on the right and the left, $F(z)$ has the block form

$$\begin{bmatrix} s_{d_1}(F(z_1)) \cdot I_{d_1} & 0 & 0 \\ 0 & s_{d_2}(F(z_2)) \cdot I_{d_2} & 0 \\ 0 & 0 & * \end{bmatrix}.$$

[7]Equivalently, d is the dimension of the subspace spanned by the "right-singular vectors" associated with the largest singular value of $F(z_0)$.

Hence, if every function $z \mapsto s_k(F(z))$ attains its maximum at some point of Ω then, up to multiplication by (constant) unitary matrices on the right and the left, $F(z)$ admits the block form

$$\begin{bmatrix} s_{d_1}(F(z_1)) \cdot I_{d_1} & 0 & \cdots & 0 \\ 0 & s_{d_2}(F(z_2)) \cdot I_{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{d_\kappa}(F(z_\kappa)) \cdot I_{d_\kappa} \end{bmatrix},$$

and is hence a constant matrix, as expected by Theorem 5.

Likewise, a completely analogous argument reveals that the refinement of the maximum norm principle in Corollary 7 is also a consequence of Theorem 8, because $s_j(F(z)) = \|F(z_0)\|$ for $j = 1, \ldots, d$ and $s_{\ell+d}(F(z)) = s_\ell(R(z))$ for $\ell = 1, \ldots, (n-d)$. We leave the details to the reader.

Proof of Theorem 8. Let D_{z_0} be the diagonal matrix whose main diagonal entries are the singular values of $F(z_0)$ listed in nonincreasing order. Then we may let U_{z_0} and V_{z_0} be unitary matrices such that $F(z_0) = U_{z_0} D_{z_0} V_{z_0}$ (i.e., an SVD for $F(z_0)$). Let r denote the largest positive integer such that $s_r(F(z_0)) = \|F(z_0)\|$. Note that, by (7), a vector x satisfies $\|F(z_0)x\| = \|F(z_0)\| \cdot \|x\|$ if and only if $V_{z_0}^*(\|F(z_0)\|^2 I - D_{z_0}^2)V_{z_0} x = 0$, or equivalently, x belongs to the linear span of first r columns of $V_{z_0}^*$ because $V_{z_0}^*$ is unitary. Thus, $r = d$ with d as in (9).

Now, consider the function $G(z) = U_{z_0}^* F(z) V_{z_0}^*$. Clearly, G is analytic on Ω and satisfies

$$\|G(z)\| \leq \|F(z_0)\| \quad \text{for all } z \in \Omega.$$

Since the $\mathbb{C}^n$ norm of every column (and row) of a matrix is bounded by its operator norm, the modulus of every (analytic) entry $G_{i,j}(z)$ is also bounded by $\|F(z_0)\|$. Moreover, if $1 \leq i \leq r$, then $G_{i,i}(z_0) = \|F(z_0)\|$ and so $G_{i,i}(z) = \|F(z_0)\|$ for all $z \in \Omega$ by the (usual) MMP. In particular, the first r columns and r rows of $G(z)$ have $\mathbb{C}^n$ norm at least $\|F(z_0)\|$. Therefore, $G_{i,j}(z) = 0$ when $i \neq j$ and $1 \leq i, j \leq r$. In other words, using matrix blocks, this shows that

$$F(z) = U_{z_0} G(z) V_{z_0} = U_{z_0} \begin{bmatrix} \|F(z_0)\| \cdot I_r & 0 \\ 0 & R(z) \end{bmatrix} V_{z_0},$$

for some analytic function $R : \Omega \to \mathbb{M}_{n-r}$ when $r < n$, while $F(z) = \|F(z_0)\| U_{z_0} V_{z_0}$ when $r = n$. This completes the proof of (10).

Finally, if $e_1, \ldots, e_n$ denotes the standard basis for $\mathbb{C}^n$ and $k \geq 1$, then the jth column $V_{z_0}^* e_j$ of $V_{z_0}^*$ satisfies $F(z) V_{z_0}^* e_j = \|F(z_0)\| U_{z_0} e_j$ and

$$F^{(k)}(z) V_{z_0}^* e_j = U_{z_0} \begin{bmatrix} 0 \cdot I_r & 0 \\ 0 & R^{(k)}(z) \end{bmatrix} e_k = 0 \cdot e_k = 0$$

for $j = 1, \ldots, r$. Thus, $z \mapsto F(z)x$ is constant and $F^{(k)}(z)x = 0$ whenever x belongs to the linear span of first r columns of $V_{z_0}^*$, or equivalently, when x satisfies $\|F(z_0)x\| = \|F(z_0)\| \cdot \|x\|$. □

4. Minimum singular value principles.

In the case of nonconstant scalar-valued functions, the MMP tells us that the *minimum modulus* (of an analytic function on a region) can only be attained at a *zero* of the function. This conclusion is often called the *minimum modulus principle* in complex analysis. As a consequence of Theorem 5, we state and prove an analog of that minimum principle in the context of matrix-valued functions.

Theorem 9. *Let Ω be a region of $\mathbb{C}$ and let $F : \Omega \to \mathbb{M}_n$ be a nonconstant analytic function. Then no point $z_0 \in \Omega$ can be a minimum value for all of the functions $s_k(F(z))$, $1 \leq k \leq n$, unless $F(z_0)$ is not invertible.*

Proof. We prove that if there is a $z_0 \in \Omega$ such that $F(z_0)$ is invertible and the functions $z \mapsto s_k(F(z))$ attain their minimum at z_0 for $k = 1, \dots, n$, then $F(z)$ must be a constant function.

To begin, recall that the collection of invertible matrices is open. This implies $F(z)$ must be invertible for all z sufficiently close to z_0. So, $G(z) \stackrel{\text{def}}{=} F^{-1}(z)$ exists in some neighborhood Ω_0 of z_0, $\det F(z)$ is nonzero and analytic on Ω_0, and the adjugate (or transpose of the cofactor matrix) $\operatorname{adj}(F(z))$ of $F(z)$ is analytic on Ω_0. Thus, $G(z) = F(z)^{-1} = \det^{-1}(F(z)) \operatorname{adj}(F(z))$ is analytic on Ω_0 as well.

By the singular value decomposition, at each $z \in \Omega_0$, the singular values of $G(z)$ are the reciprocals of those of $F(z)$; more specifically,

$$s_k(G(z)) = 1/s_{n-k+1}(F(z)) \text{ for } k = 1, \dots, n, \text{ and } z \in \Omega_0.$$

Therefore, the assumption of the theorem is equivalent to stating that the functions $z \mapsto s_k(G(z))$ attain a maximum on Ω_0 at z_0. By Theorem 5, $G(z)$ and $F(z)$ must be constant on Ω_0. Finally, applying the identity theorem (e.g., [7, Theorem 10.18]) to each entry of $F(z)$ implies that $F(z)$ is constant throughout Ω, as desired. $\square$

Corollary 10. *Let Ω be a region of $\mathbb{C}$ and let $F : \Omega \to \mathbb{M}_n$ be a nonconstant analytic function. If every function $s_k(F(z))$, $1 \leq k \leq n$, attains a minimum value at $z_0 \in \Omega$, then* $\det(F(z_0)) = 0$.

Remark. Notice that $s_n(F(z_0)) = 0$ if and only if z_0 is a zero of $\det F(z)$; indeed, with an SVD of $A \in \mathbb{M}_n$, we see that

$$\prod_{k=1}^{n} s_k(A) = |\det(A)|. \tag{11}$$

Thus, Corollary 10 states that if every function $s_k(F(z))$, $1 \leq k \leq n$, attains a minimum value at $z_0 \in \Omega$, then $s_n(F(z_0)) = 0$.

To illustrate Theorem 9, it suffices to take $F : \mathbb{D} \to \mathbb{M}_2$ as in (2); indeed, the functions $s_1(F(z)) = 1$ and $s_2(F(z)) = |g(z)|$ attain their respective minimum values at any zero z_0 of g and $F(z_0)$ is certainly not invertible.

In light of Theorems 5 and 9, one may ask whether the singular values of a matrix-valued analytic function could attain minimum values at *distinct* points. The following result gives an affirmative answer.

Theorem 11. *If $F : \mathbb{C} \to \mathbb{M}_2$ denotes the function defined by*

$$F(z) = \begin{bmatrix} 1 & z \\ 0 & z-1 \end{bmatrix}, \tag{12}$$

then $s_1(F(z))$ has a minimum at $z_1 = 0$ and $s_2(F(z))$ has a minimum at $z_2 = 1$.

Proof. The remark following the proof of Theorem 3 shows that $s_1(F(z))$ has a minimum at $z_1 = 0$; indeed, $z \mapsto F(z)x_0$ is constant when $x_0 = [1, 0]^T$. On the

other hand, if $z_2 = 1$, then
$$F(z_2) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
satisfies $s_2(F(z_2)) = 0$ because $\det F(z_2) = 0$. In particular, $s_2(F(z))$ has a minimum at $z_2 = 1$. □

Finally, it is worth mentioning that a singular value of a matrix function may attain its minimum value at specified locations. For instance, when g and h are analytic, the function defined by
$$K(z) = \begin{bmatrix} g(z) & 1 \\ 0 & h(z) \end{bmatrix} \tag{13}$$
satisfies $s_2(K(z)) = 0$ at every zero of g and h.

5. Return to the resolvent and matrix exponential.

With the wisdom acquired about the norms and singular values of analytic matrix-valued functions, we now return to the resolvent and matrix exponential of a given matrix. To simplify our notation, let $R_A(z)$ denote the resolvent of $A \in \mathbb{M}_n$ at z, i.e.,
$$R_A(z) = (A - zI)^{-1} \text{ for } z \in \mathbb{C} \setminus \sigma(A).$$
Also, set $L_A(z) = A - zI$.

Let Ω be a region of $\mathbb{C} \setminus \sigma(A)$. By Theorem 5, the singular values $s_k(L_A(z))$ and likewise $s_k(R_A(z))$ cannot all attain a maximum value on Ω as functions. Recalling that
$$s_k(R_A(z)) = 1/s_{n-k+1}(L_A(z)) \text{ when } 1 \le k \le n, \tag{14}$$
it follows that the functions $s_k(R_A(z))$ cannot all attain a maximum nor a minimum on Ω; in fact, this holds for $k = 1$ and $k = n$, respectively, as shown below.[8]

Theorem 12. *If $A \in \mathbb{M}_n$ and Ω is any region of $\mathbb{C} \setminus \sigma(A)$, then*
$$s_1(R_A(z)) < \sup_{\zeta\in\Omega} s_1(R_A(\zeta)) \text{ and } s_n(R_A(\zeta)) > \inf_{\zeta\in\Omega} s_n(R_A(\zeta)) \text{ for all } z \in \Omega.$$
In particular, the functions $s_1(R_A(z))$ and $s_n(R_A(z))$ are nonconstant on Ω.

Proof. To obtain a contradiction, assume instead there are points $z_0, w_0 \in \Omega$ such that
$$s_1(R_A(z_0)) \ge s_1(R_A(\zeta)) \text{ or } s_1(L_A(w_0)) \ge s_1(L_A(\zeta)) \text{ for all } \zeta \in \Omega$$
(see (14)). By Theorem 3,
$$R_A'(z_0)x_0 = 0 \text{ or } L_A'(w_0)y_0 = 0 \tag{15}$$
when x_0 and y_0 are maximizing vectors for $R_A(z_0)$ and $L_A(w_0)$, respectively. On the other hand, as
$$R_A(z) - R_A(z_0) = R_A(z)[(A - z_0I) - (A - zI)]R_A(z_0)$$
for any $z \in \Omega$, we have
$$R_A'(z_0) = \lim_{z\to z_0} R_A(z)R_A(z_0) = R_A^2(z_0)$$

[8] The first inequality in Theorem 12 was observed by Daniluk [3] for resolvents of operators on a complex Hilbert space.

and clearly $L'_A(w_0) = -I$. However, these equations, together with (15), imply that

$$x_0 = L_A^2(z_0)R_A^2(z_0)x_0 = L_A^2(z_0)R'_A(z_0)x_0 = 0$$

or $y_0 = -L'_A(w_0)y_0 = 0$, which are impossible because $\|x_0\| = \|y_0\| = 1$. $\square$

We now turn to the matrix exponential. Recall that given $T \in \mathbb{M}_n$, the **matrix exponential** of T is the $n \times n$ matrix defined by

$$\exp(T) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{n!} T^n.$$

It is not difficult to verify that the series above converges for any $T \in \mathbb{M}_n$ (say, under the operator norm), and $\exp(T)$ is invertible in $\mathbb{M}_n$ with inverse $\exp(-T)$.

For $A \in \mathbb{M}_n$, we see that the map $z \mapsto \exp(zA)$ is a well-defined matrix-valued function, analytic on the entire complex plane $\mathbb{C}$, and

$$\frac{d}{dz}[\exp(zA)] = A\exp(zA) = \exp(zA)A.$$

Furthermore, a straightforward verification[9] reveals that

$$(zI - A)^{-1} = \int_0^\infty e^{-zt} \exp(tA)\, dt \quad \text{when } \operatorname{Re} z > \|A\|, \tag{16}$$

while term-by-term integration of the power series representations for the exponential and the resolvent gives

$$\exp(tA) = \frac{1}{2\pi i} \int_{\Gamma_r} e^{t\xi} (\xi I - A)^{-1}\, d\xi, \tag{17}$$

where Γ_r denotes any circle of radius $r > \|A\|$ centered at the origin.

In addition to the intimate relationship between the resolvent and the matrix exponential (as described in (16) and (17)), intuition from the case of scalar-valued functions may suggest that, in analogy to Theorem 12, the functions $s_1(\exp(zA))$ and $s_n(\exp(zA))$ should not attain their maximum and minimum values, respectively,[10] over any region Ω of $\mathbb{C}$. This is in fact *false*. Notice that

$$\exp(zA) = \begin{bmatrix} 1 & 0 \\ 0 & e^z \end{bmatrix} \quad \text{with} \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

provides a counterexample; indeed, computation reveals that

$$s_1(\exp(zA)) = \max\{1, e^{\operatorname{Re} z}\} \quad \text{and} \quad s_2(\exp(zA)) = \min\{1, e^{\operatorname{Re} z}\}.$$

Thus, $s_1(\exp(zA))$ and $s_2(\exp(zA))$ are constant when $\operatorname{Re} z < 0$ and $\operatorname{Re} z > 0$, respectively.

Finally, we would like to propose a question for further investigation. Given an analytic function $F : \Omega \to \mathbb{M}_n$ such that $\|F(z)\|$ attains its maximum in Ω, Theorem 8 not only describes the structure of F, it also implies that $\|F(z)\| = \|F(z_0)\|$ for all $z \in \Omega$. So, in a sense, it is rare for $\|F(z)\|$ to attain its maximum. Instead, what may be less rare is for $\|F(z)\|$ to attain a *minimum* value (see Theorem 11).

[9] Indeed, when $\operatorname{Re} z > \|A\|$, the function $t \mapsto \exp(t(A - zI))$ has operator norm equal to $e^{-\operatorname{Re}(zt)}\|\exp(At)\|$, which tends to zero as $t \to \infty$, and so the integral over $[0, \infty)$ of its derivative equals the identity matrix.

[10] After all, for fixed $a \in \mathbb{C}$, $|e^{za}|$ cannot attain its maximum nor minimum values over any region Ω.

In fact, the remark made after the proof of Theorem 3 already gives a sufficient condition for $\|F(z)\|$ to have a minimum at z_0, namely when $z \mapsto F(z)x_0$ is constant for some maximizing vector for $F(z_0)$. Furthermore, by completely analogous reasoning, a sufficient condition for $s_n(F(z))$ to attain a maximum at z_0 is that $z \mapsto F(z)x_0$ is constant for some minimizing[11] vector for $F(z_0)$. For example, in light of this, it may be verified that $s_1(F(z))$ has a minimum at $z = 0$ and $s_2(F(z))$ has a maximum at $z = 0$ when $F(z)$ is the function in (12). This leads one to wonder what necessary and sufficient conditions permit $s_1(F(z))$ to attain a minimum and $s_n(F(z))$ to attain a maximum over a region Ω? Is it more attainable to consider the special case $F(z) = (A - zI)^{-1}$? How about when $F(z) = \exp(zA)$?

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[9] Trefethen, L. N., Embree, M. (2005). *Spectra and Pseudospectra. The Behavior of Nonnormal Matrices and Operators.* Princeton, NJ: Princeton Univ. Press.

[11] A vector x_0 is said to be a **minimizing vector** for $A \in \mathbb{M}_n$ if $\|Ax_0\| = \min\{\|Ax\| : \|x\| = 1\}$.