An Alternative proof of Steinhaus Theorem

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Steinhaus's Theorem states that if A is a lebesgue measurable set on the real line such that the lebesgue measure of A is not zero then the difference set $A - A = \{a - b | a, b \in A\}$ contains an open neighbourhood of origin[1]. Since there is a compact set of positive measure inside any set of positive measure. It is enough to prove the result for compact sets.

Theorem For any compact set $C \subset \mathbb{R}$ of positive measure, the difference set $D = C - C = \{a - b | a, b \in C\}$ contains an interval containing 0.

Proof: Suppose the set D does not contain an interval around origin, then $\exists$ a sequence $x_n \to 0$ such that $x_n \notin D$, $\forall n \in \mathbb{N}$.
We proceed by induction to create arbitrarily large number of mutually disjoint sets $A_1, A_2, A_3 \ldots$such that A_i is of the form $x_{n_i} + C$ for all $i \geq 2$, $i \in \mathbb{N}$. Let $A_1 = x_1 + C$, clearly A_1 satisfies the above property. Now suppose $A_1, A_2 \ldots A_k$ is created in such a manner. We will now create A_{k+1}.
Suppose for every $n \in \mathbb{N}$, $\exists$ an $i \leq k$ such that the set $x_n + C$ intersects A_i, then there exists a $j \leq k$ such that $x_n + C$ intersects A_j for infinitely values of $n \in \mathbb{N}$, so $\exists$ a subsequence $\{y_m\}$ of $\{x_n\}$ such that $x_{n_j} - y_m \in D$ for all $m \in \mathbb{N}$. As $y_m \to 0$ and D is compact, x_{n_j} belongs to D (a contradiction). So there exists an $N \in \mathbb{N}$ such that the set $x_N + C$ does not intersect A_i for any $i \leq k$. Define $A_{k+1} = x_N + C$. So we have our desired set.
Now for any $n \in \mathbb{N}$, the set $x_n + C$ lies in some bounded set $[-R, R]$ for some $R \in \mathbb{N}$ and for any $p, q \in \mathbb{N}$ measure of $A_p > 0$, $A_q > 0$ and measure of A_p equals measure of A_q. So we have an arbitrary large number collection of mutually disjoint sets of the same positive measure in $[-R, R]$ (a contradiction). Hence the result follows.

1 References

[1]. T.Tao, *An Introduction to Measure Theory*, 1st edition, American Mathematical Society.

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