# **RIESZ'S THEOREM FOR LUMER'S HARDY SPACES**

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ABSTRACT. In this note we obtain a version of the well-known Riesz's theorem on conjugate harmonic functions for Lumer's Hardy spaces  $(Lh)^2(\Omega)$  on arbitrary domains  $\Omega$ : If a real-valued harmonic function  $U \in (Lh)^2(\Omega)$  has a harmonic conjugate V on  $\Omega$  (i.e., a real-valued harmonic function such that U + iV is analytic on  $\Omega$ ), then U + iV also belongs to  $(Lh)^2(\Omega)$ , and for the normalized conjugate we have the norm estimate  $||U + iV||_{(Lh)^2(\Omega)} \leq \sqrt{2} ||U||_{(Lh)^2(\Omega)}$ , with the best possible constant.

# 1. ON LUMER'S HARDY SPACES

Let  $\mathbf{U} = \{z \in \mathbf{C} : |z| < 1\}$  be the unit disk and let  $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$  be the unit circle. For a function f on  $\mathbf{U}$  and  $r \in (0, 1)$  we denote by  $f_r$  the function  $f_r(\zeta) = f(r\zeta)$ ,  $\zeta \in \overline{\mathbf{U}}$ .

The harmonic Hardy space  $h^p$ , for  $p \in (1, \infty)$ , consists of all harmonic complex-valued functions U on U for which the integral mean

$$M_p(U,r) = \left\{ \int_{\mathbf{T}} |U_r(\zeta)|^p \frac{|d\zeta|}{2\pi} \right\}^{1/p}$$

remains bounded as r approaches 1. Since  $|U|^p$  is subharmonic on U, the integral mean  $M_p(U, r)$  is increasing in r. The norm on  $h^p$  is given by

$$||U||_p = \lim_{r \to 1} M_p(U, r).$$

The analytic Hardy space  $H^p$  is the subspace of  $h^p$  that contains all analytic functions. For the theory of Hardy spaces in the unit disk we refer to [1, 2, 3, 7].

There are generalizations of Hardy spaces for other domains in C. The generalizations we consider here are known as Lumer's Hardy spaces [1, 2, 4, 5, 6]. We mention below some facts regarding these spaces that we will need.

The harmonic Lumer's Hardy space  $(Lh)^p(\Omega)$  contains all harmonic complex-valued functions U on a domain  $\Omega \subseteq \mathbf{C}$  such that the subharmonic function  $|U|^p$  has a harmonic majorant on  $\Omega$ . In that case, the function  $|U|^p$  has the least harmonic majorant on  $\Omega$ . Let it be denoted by  $H_U$ . For  $\zeta_0 \in \Omega$  one introduces a norm on  $(Lh)^p(\Omega)$  in the following way:

(1.1) 
$$||U||_{p,\zeta_0} = H_U^{1/p}(\zeta_0).$$

The different norms on  $(Lh)^p(\Omega)$  that arise by selecting different elements of the domain  $\Omega$  are mutually equivalent. The analytic Lumer's Hardy space  $(LH)^p(\Omega)$  is the subspace of  $(Lh)^p(\Omega)$  that consists of all analytic functions. The two spaces  $(Lh)^p(\mathbf{U})$  and  $h^p$  coincide (as do  $(LH)^p(\mathbf{U})$  and  $H^p$ ). The norms on these spaces are equal, if we select  $\zeta_0 = 0$  for the Lumer case.

Lumer's Hardy spaces are conformally invariant in the following sense: If  $\Phi$  is a conformal mapping of a domain  $\tilde{\Omega}$  onto  $\Omega$ , then a function U belongs to  $(Lh)^p(\Omega)$  if and only if  $\tilde{U} = U \circ \Phi$  belongs to  $(Lh)^p(\tilde{\Omega})$ . The mapping  $\Phi$  induces an isometric isomorphism  $U \to \tilde{U}$  of the space  $(Lh)^p(\Omega)$  onto  $(Lh)^p(\tilde{\Omega})$ , since the equality for the least harmonic majorants  $H_U \circ \Phi = H_{\tilde{U}}$  implies that  $||U||_{p,\zeta_0} = ||\tilde{U}||_{p,\zeta_0}$ , where  $\tilde{\zeta_0} \in \tilde{\Omega}$  satisfies  $\zeta_0 = \Phi(\tilde{\zeta_0})$ .

## 2. RIESZ'S THEOREM FOR LUMER'S HARDY SPACES

The classical Riesz theorem on conjugate harmonic functions says that for every  $p \in (1, \infty)$  there exists a constant  $c_p$  such that

$$||U+iV||_p \le c_p ||U||_p$$

where U is a real-valued function in  $h^p$ , V is a harmonic conjugate to U on U, normalized such that V(0) = 0. See, for instance, [7, Theorem 17.26]. Verbitsky proved [9] that the best possible constant in the Riesz inequality is

(2.1) 
$$c_p = \begin{cases} \sec \frac{\pi}{2p}, & \text{if } 1$$

Note that, in particular, we have  $c_2 = \sqrt{2}$ .

Our aim in this section is to prove the Riesz theorem for real-valued harmonic functions in the Lumer's Hardy space  $(Lh)^2(\Omega)$  for which there exists a conjugate. We find that the constant  $\sqrt{2}$  is valid for all domains  $\Omega$ . This is the content of the following theorem.

**Theorem 2.1.** Let  $\Omega \subseteq \mathbf{C}$  be a domain and  $\zeta_0 \in \Omega$ . Assume that for real-valued  $U \in (Lh)^2(\Omega)$  there exists a harmonic conjugate of U on the domain  $\Omega$ , denoted by V, and let it be normalized such that  $V(\zeta_0) = 0$ . Then we have the Riesz inequality

(2.2) 
$$||U+iV||_{2,\zeta_0} \le \sqrt{2} ||U||_{2,\zeta_0}$$

with the best possible constant.

*Proof.* We will use the following elementary equality, which is easy to check:

(2.3) 
$$|z|^2 = 2(\Re z)^2 - \Re z^2, \quad z \in \mathbf{C}.$$

Indeed, since  $2\Re z = z + \overline{z}$ , we have

$$4(\Re z)^2 = (z + \overline{z})^2 = z^2 + \overline{z}^2 + 2z\overline{z} = 2\Re z^2 + 2|z|^2;$$

the equality mentioned above then follows.

Let the analytic function U + iV be denoted by F, and let  $H_U$  be the least harmonic majorant of the subharmonic function  $|U|^2$  on  $\Omega$ . By applying equation (2.3) for  $z = F(\zeta)$ ,  $\zeta \in \Omega$ , we obtain

$$|F(\zeta)|^{2} = 2(\Re F(\zeta))^{2} - \Re F^{2}(\zeta) = 2|U(\zeta)|^{2} - \Re F^{2}(\zeta)$$
  
$$\leq 2H_{U}(\zeta) - \Re F^{2}(\zeta),$$

which proves that  $2H_U - \Re F^2$  is a harmonic majorant of  $|F|^2$  on  $\Omega$ . It follows that  $F \in (LH)^2(\Omega)$ . Moreover, if  $H_F$  is the least harmonic majorant of  $|F|^2$  on  $\Omega$ , we have

$$H_F(\zeta) \le 2H_U(\zeta) - \Re F^2(\zeta).$$

Since  $F(\zeta_0) = U(\zeta_0)$  is a real number, we obtain

$$||F||_{2,\zeta_0}^2 = H_F(\zeta_0) \le 2H_U(\zeta_0) - \Re F^2(\zeta_0) = 2H_U(\zeta_0) - U^2(\zeta_0)$$
  
$$\le 2H_U(\zeta_0) = 2||U||_{2,\zeta_0}^2.$$

Finally, we conclude that

$$||F||_{2,\zeta_0} \le \sqrt{2} ||U||_{2,\zeta_0},$$

which is what we wanted to prove.

It is not hard to prove that  $\sqrt{2}$  is a sharp constant in the Riesz inequality (2.2). Indeed, consider the unit disk U as the domain  $\Omega$ . If we again use equation (2.3) for  $z = F(\zeta)$ , we have

$$|F(\zeta)|^{2} = 2U^{2}(\zeta) - \Re F^{2}(\zeta).$$

Since  $\Re F^2$  is a harmonic function on U, by applying the equality obtained above and the mean-value property for harmonic functions, it follows that

$$\begin{split} M_2^2(F,r) &= \int_{\mathbf{T}} |F_r(\zeta)|^2 \frac{|d\zeta|}{2\pi} = 2 \int_{\mathbf{T}} U_r^2(\zeta) \frac{|d\zeta|}{2\pi} - \int_{\mathbf{T}} \Re F_r^2(\zeta) \frac{|d\zeta|}{2\pi} \\ &= 2M_2^2(U,r) - \Re F^2(0) = 2M_2^2(U,r) - U^2(0). \end{split}$$

If we now let  $r \to 1$ , we obtain

$$||F||_2 = \sqrt{2} ||U||_2$$

provided that U(0) = 0.

Note that the constant  $\sqrt{2}$  in the Riesz inequality (2.2) does not depend on  $\zeta_0 \in \Omega$ , although the norm of a function in the Lumer's Hardy space  $(Lh)^2(\Omega)$  does. If  $\Omega$  is a simply connected domain with at least two boundary points, this is expected, since the group of all conformal automorphisms of the domain  $\Omega$  acts transitively on  $\Omega$ , i.e., for any  $\tilde{\zeta}_0 \in \Omega$  there exists a conformal automorphism  $\Phi$  of  $\Omega$  such that  $\Phi(\tilde{\zeta}_0) = \zeta_0$ . As we have already said, the mapping  $\Phi$  induces an isometric isomorphism of  $(Lh)^2(\Omega)$  onto itself. However, for multi-connected domains it is not true, in general, that the group of all conformal automorphisms acts transitively on a domain.

## 3. REMARKS ON THE HIGHER-DIMENSIONAL SETTING AND A CONJECTURE

Lumer's Hardy spaces  $(Lh)^p(\Omega)$  and  $(LH)^p(\Omega)$  on domains  $\Omega$  in  $\mathbb{C}^n$  are defined in a similar way as in the one-dimensional case [4]. However, instead of the harmonic majorant we have to use a pluriharmonic majorant, i.e., a function that is locally the real part of an analytic function on  $\Omega$ . Therefore, the Lumer's Hardy space  $(Lh)^p(\Omega)$  contains all pluriharmonic functions U on  $\Omega$  such that  $|U|^p$  has a pluriharmonic majorant on  $\Omega$ . The analytic Lumer's Hardy space  $(LH)^p(\Omega)$  is the subspace of  $(Lh)^p(\Omega)$  that consists of all analytic functions. The norm on  $(Lh)^p(\Omega)$  may be introduced with respect to any  $\zeta_0 \in \Omega$  using the least pluriharmonic majorant as in the ordinary case (1.1).

The proof of Riesz's theorem given in the preceding section may be adapted directly for Lumer's Hardy spaces on domains in  $\mathbb{C}^n$ . Therefore, the Riesz inequality (2.2) remains valid, with the same constant  $\sqrt{2}$ , in this setting.

We conjecture that for every  $p \in (1, \infty)$  and every domain  $\Omega \subseteq \mathbb{C}^n$  there holds the version of Riesz's theorem: Let F be an analytic function on  $\Omega$  such that  $\Im F(\zeta_0) = 0$ ; if  $\Re F \in (Lh)^p(\Omega)$ , then  $F \in (LH)^p(\Omega)$  and there is the Riesz inequality

$$||F||_{p,\zeta_0} \le c_p ||\Re F||_{p,\zeta_0},$$

where  $c_p$  is the Verbitsky constant (2.1).

In seventies, Stout [8, Theorem IV.1] proved Riesz's theorem for Lumer's Hardy spaces  $(LH)^p(\Omega)$  on  $\mathcal{C}^2$ -smooth domains  $\Omega \subseteq \mathbb{C}^n$  (without a precise constant in the Riesz inequality). In this case there exists an integral representation of the Lumer's norm of an analytic function that is used in order to obtain the result.

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